Math Explorations
Algebra I

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MATH EXPLORATIONS

Table of Contents

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Preface and Introduction

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. **A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together.** In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself. Some basic rules for discussion within a group include

1. **Encourage everyone to participate**, and value each person's opinions. Listening carefully to what someone else says can help clarify a question. The process helps the explainer often as much as the questioner.
2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely, and **never be afraid to ask questions**.
3. **Don't be afraid to make a mistake.** In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to discover difficulties in solving a problem. So rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.

4. **Finally, always share your ideas with one another**, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear, there are several things to try.

1. Look for simpler cases. Looking deeply at simple cases can help to you see a general pattern.
2. If an idea is unclear, ask your peers and teacher for help. Go beyond "Is this the right answer?"
3. Understand the question being asked. Understanding the question leads to to mathematical progress.
4. Focus on the process of obtaining an answer, not just the answer itself; in short become problem centered, not answer centered. One of the major goals of this book is to develop an understanding of ideas that can solve more difficult problems as well.
5. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Some hints to help in responding to oral questions in group and class discussion: As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times try exploring first by yourself and then discuss your ideas with others. When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.
2. In class, be recognized if you want to contribute or ask a question.
3. Be polite and listen when others in your group or class are talking.

This is one of the best ways to learn.

4. Finally, don't be shy. If you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Last, some general advice about reading and taking notes in math.

1. Reading math is a specific skill. When you read math, you need to read each word carefully. The first step is to know the mathematical meaning of all words.
2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you to relate what you are learning to real world situations.
3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.
4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues help you understand and remember definitions of new terms.

Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes. It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. To this end, they have made every effort to create explanations that are as transparent as possible. Parents are encouraged to read both the book and the accompanying materials.

The authors have written a 3 volume set of books that is designed to take all students from pre-algebra through Algebra I. This includes students who may not have understood the previous years' math. Students from 4^{th} through 8^{th} grade should enjoy the ingenuity and investigation problems at the end of each set of exercises. Math Explorations (ME) is intended to prepare all students for algebra, with algebraic concepts woven in throughout. In addition, ME Part 2, and ME Algebra I cover all of the 7^{th} and 8^{th} grade Texas Essential Knowledge and Skills (TEKS). The Teacher Guide has been written to make the textbook and its mathematical content as easy and intuitive for teachers as possible. Answers to the exercises are readily available and readable in the teacher edition. The teacher edition contains supplementary activities that the students might enjoy.

This text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 modeled after the Ross program at Ohio State, teaching students to "think deeply of simple things" (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with numerous students being named Siemens-Westinghouse semi-finalists, regional finalists, and national finalists. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also had significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Advisory Board.

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4 - 8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts the JSMC targeted gifted students; in other districts the program was delivered to a mixed group of students; and in other districts the program was used especially for ELL. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

The problem with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The 3 volume Math Explorations texts that we have written has taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills) for grades 6-8 while weaving in algebra throughout. In particular, this volume, Math Explorations 3, Algebra I, was developed especially for younger students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U.S. students on international tests.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation Foundation, and Kodosky Foundation. A special thanks to our Advisory Board, especially Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other

was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The basis for the book was our junior summer math camp curriculum, coauthored with my wife Hiroko, and friend, colleague, and coauthor Terry McCabe. We were very fortunate when we decided to extend that curriculum to cover all of Algebra I to find an extraordinarily talented co-author, Alex White. Alex has taken the lead on the algebra book in working with our team of students, faculty, and teachers, while also doing the amazing job of both making edits and doing typesetting. His specialty is math education and statistics, which are important and often neglected parts of an algebra book. Our team of authors has many lively discussions where we debate different approaches to introducing a new topic, talk about different ways to engage students to explore new ideas, and carefully go through each new idea and how it should be sequenced to best guide student learning.

Over the summers of 2005-2012, we have been assisted by an outstanding group of former Honors Summer Math Camp students, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge help and support, as well as describe the evolution of this three-volume series.

Briefly, in 2008, the Math Explorations Book was only one volume. This was piloted by a group of 6th and 7th grade teachers in McAllen, San Marcos, and New Braunfels. The results of these pilots have been extremely encouraging. We saw young 6th and 7th grade students reach (on average) 8th grade level and above as measured by the Orleans-Hanna test by the end of 7th grade. However, there was a consensus that it would be beneficial to split the Math Explorations book into a separate 6th grade and 7th grade book.

After meeting with the McAllen teachers in the summer, 2009, we carefully divided the Math Exploration text into two volumes. Hiroko

Warshauer led the team in developing this new book, Math Explorations Part 1, assisted by Terry McCabe and Max Warshauer. Alex White from Texas State provided valuable suggestions, and took over the leadership of the effort for the third volume, Algebra 1. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8, while also covering Algebra I.

Although there is naturally some overlap, the new books, Math Explorations Part 1 and Part 2, much more closely align with what the teachers felt would work best with their students. Math Explorations Part 1 should work for any 6th grade student, while Math Explorations Part 2 is suitable for either an advanced 6th grade student or any 7th or 8th grade student.

In this project, we have been incredibly fortunate to have the help of several talented teachers. Major contributors this past summer include Amanda Voigt and Ashley Beach from San Marcos, Patricia Serviere from McAllen, and Amy Warshauer from Austin. These teachers provided wonderful help in the development of a an accompanying workbook that provides a template for how to teach the book, with new explorations and supplemental problems. They also made numerous suggestions and edits, while checking that we covered all of the state mandated topics for Algebra I.

Sam Baethge and Michael Kellerman gave the entire book a careful reading, which provided amazing support for editing and revising the book. Michael focused primarily on readability edits. Sam continued to develop new challenge and ingenuity problems which should engage and excite young students in mathematics. Cody Patterson made key contributions to the original design, problems, and content of the text. Robert Perez developed additional resources for English Language Learners, including a translation of the glossary and key mathematical terms into Spanish.

As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help, the project could not have reached its present state.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with fabulous teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Patty Amdende and Andrew Hsiau have provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn't and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer
Director of Texas State Mathworks

VARIABLES, EXPRESSIONS, AND EQUATIONS

1

SECTION 1.1 CONSTRUCTING A NUMBER LINE

"What is Algebra?" Rather than give you an incomplete answer now, we hope that through learning, you will soon be able to answer this question yourself. As a preview, let's look at some questions that algebra can help us answer that we could not have answered before:

- If I drop a marble off a two-story building, how long will it take the marble to hit the ground?
- If I have $10 and go into a candy shop, where chocolate costs $.50 and licorice costs $.65, how many of each could I buy?

We begin by reviewing numbers and the ways you manipulate and represent them. We will develop collections of numbers in stages, building up from smaller groups of numbers until we have all numbers on the number line.

We first encounter numbers as children by counting, starting with one, two, and three. We call the numbers that we use in counting the *natural numbers*, or sometimes the counting numbers. They include the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$, where the "…" means that they go on forever in the same way. These numbers describe how many of something, for example, how many brothers or sisters you have, how many days in a week, the number of people in your town and even the number of grains of sand on a beach.

EXPLORATION 1

Make a number line on a large piece of paper. Put the number 1 in the middle of the line. Locate and label the first 20 natural numbers.

Now add 0 to your number line. Including the number 0 in this set of numbers gives us the *whole numbers*. The whole numbers take care of many situations, but if we want to talk about the temperature, there are places on Earth that routinely have temperatures below zero, like -5 °C or -20 °C. We must expand our idea of number to include the negatives of the natural numbers. This larger collection of numbers is called the *integers*, it is denoted by the symbol $\mathbb{Z}$ and includes the whole numbers. Notice that every integer is either positive, zero or negative. The natural numbers are positive integers and are sometimes denoted by $\mathbb{Z}^+$.

In this book we will try to be very precise in our wording, because we want our mathematics and words to be clear. Just as you learn new words in English class to express complicated concepts, we must learn new words and symbols in mathematics. We have discussed 3 collections of numbers so far: the integers, the whole numbers, and the natural numbers. In mathematics, we call collections of numbers (or other objects) *sets*. A set is defined by its members. We call these members *elements*. In order to write out what a set is, we want to describe its elements in *set notation*. This is done for example by listing the elements of a set inside braces. For example, the natural numbers are $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ and the integers could be written $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Not all sets go on forever, for example, the set of even natural numbers between 4 and 14 is $\{6, 8, 10, 12\}$.

Every natural number is also an integer. In our more precise language, this could be described as "every number in the set of natural numbers is also in the set of integers." For this reason, we call the natural numbers a *subset* of the integers. We will call a set a subset of another set when any element in the first set is also in the second. We use such precise wording in order to be clear in our discussion of mathematics. The use of precise language is more important in mathematics than it is in everyday life.

EXPLORATION 2

Continue to work on the number line from Exploration 1. Using a red marker, plot and label the negative integers from -1 to -20. What properties does the set of integers have that the set of whole numbers did not?

It does not take long to see the need for numbers that are not integers. You might hear in a weather report that it rained $2\frac{1}{2}$ inches or know that a person's normal body temperature is 98.6°Fahrenheit. Sometimes we need to talk about parts of whole numbers called fractions. This expanded set of numbers that includes fractions is called the set of *rational numbers*.

EXPLORATION 3

Using a different colored marker, plot and label 3 fractions between each of the following pairs of integers:

$$2 \text{ and } 3 \qquad 4 \text{ and } 5 \qquad -1 \text{ and } 0 \qquad -3 \text{ and } -2$$

A rational number is the quotient of 2 integers and the denominator cannot be zero. For example, both $\frac{3}{7}$ and $\frac{9}{4}$ are rational numbers. They are called rational numbers because they are the ratio of 2 integers. A rational number can be represented as quotient in more than one way. Also, every rational number can be written in decimal form. For example, $\frac{1}{2}$ is equivalent to $\frac{2}{4}$ and to 0.5, $\frac{3}{4}$ is the same as 0.75 and $2\frac{2}{5}$ is equal to 2.4 and $\frac{12}{5}$.

We asked you to find two fractions between 2 and 3. Could you find two fractions between the fractions you just found? How about two fractions between those two?

PROBLEM 1

How many fractions are there between 0 and 1? How many fractions are there between 2 and 3?

Notice that every integer is a rational number. There are, however, rational numbers that are not integers. This means that the set of integers is a subset of the set of rational numbers, but the set of rational numbers is not a subset of the set of integers.

PROBLEM 2

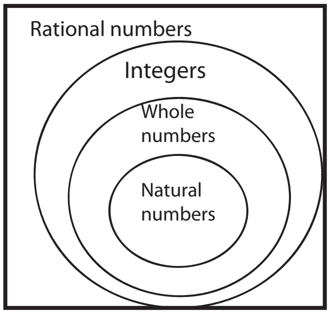
List 3 examples of rational numbers that are not integers and list 3 examples of integers that are not whole numbers. Locate these numbers on your number line.

EXAMPLE 1

Create a Venn Diagram to show the relationship between the following sets of numbers:

- rational numbers
- whole numbers
- integers
- natural numbers

SOLUTION The sets of numbers are nested. For example, every integer is a rational number, but not every rational number is an integer.



Operations on the Number Line

One advantage of representing numbers on the number line is that it shows a natural order among its members. The location of the points representing 2 numbers reflects a relationship between those 2 numbers that we define as greater than, equal to or less than. You can also examine distances between numbers and model the operations of addition, subtraction, multiplication and division on the number line. In fact, we will frequently use examples on the number line to illustrate algebraic ideas. Let's review how to do arithmetic with integers using a number line.

EXPLORATION 4

1. Use the number line to illustrate the sum $3+(-4)$ and the difference $3-4$. Explain how you arrived at your answer and location for each problem. Then, using the same pattern, explain how you compute the sum $38+(-63)$ and the difference $38-63$ without a detailed number line.
2. Use the number line to illustrate the difference $3-(-5)$ and the sum $3+5$. Then explain how you compute the difference $38-(-63)$ without a detailed number line.
3. Summarize the rules for addition and subtraction of integers.
4. Use the number line to illustrate the products $3(-4)$ and $-3(4)$. Explain how you arrived at your answer and location. Then using the same pattern, explain how you compute the products $18(-6)$ and

$-5(12)$ without a detailed number line.

5. Use the number line to illustrate the product $-3(-4)$. Explain how you arrived at your answer and location for each problem. Then using the same pattern, explain how you compute the product $-28(-3)$.
6. Summarize the rules for multiplication of integers.

The number line is also useful for thinking about operations with rational numbers and exploring the relationship between numbers.

EXPLORATION 5

1. Use the number line to illustrate the sums $1\frac{3}{4} + 2\frac{3}{4}$ and $\frac{4}{5} + \frac{3}{5}$.

2. Starting at the point representing 3, determine and locate on the number line the following numbers. Explain how you arrived at your answer.
   a. The number that is 5 more than 3.
   b. The number that is 5 less than 3.
   c. The number that is 3 times 3.
   d. The number that is half as big as 3.
3. Locate and label three numbers that are greater than -5. Locate and label three numbers that are less than -6.

Distance on the Number Line

Another important concept to study on the number line is the *distance* between points.

EXPLORATION 6

Make a new number line from -15 to 15, labeling all of the integers between them. Locate the points 6 and 13 on the new number line. Determine the distance between 6 and 13.

1. What is the distance from 12 to 4? Explain how did you got your answer.
2. What is the distance from -3 to -11? From -9 to -2? How did you get your answers?
3. What is the distance from -7 to 4? From 5 to -7? Explain.
4. Find the distance between $\frac{1}{2}$ and $3\frac{1}{2}$.
5. Find the distance between $\frac{1}{2}$ and $\frac{3}{4}$.
6. Find the distance between $\frac{3}{4}$ and $3\frac{1}{2}$.

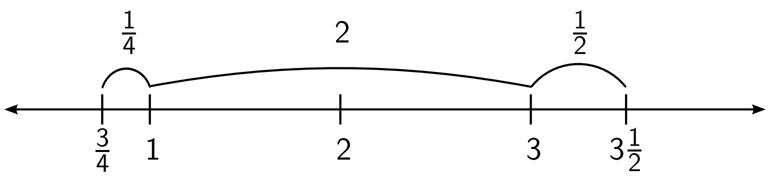
7. What is the distance from $-\frac{1}{2}$ to $\frac{7}{8}$?
8. What is the distance between $4\frac{2}{3}$ and $1\frac{1}{2}$?

One way you might have found the distance between two points representing integers on a number line is to "count up" from the leftmost number until you reach the one on the right or to "count down" from the rightmost number until you reach the one on the left. For example from 6 you might have counted up and noted that it took 7 units to arrive at 13 and so concluded that the distance between 6 and 13 is 7. Or in the second question asking for the distance between 12 and 4, you might have counted down from 12 until you reached 4 and noted that it took 8 units, to conclude that the distance between 12 and 4 is 8. However, you might also have noticed that $12 - 4 = 8$ and $13 - 6 = 7$. The distance between two numbers is the difference of the lesser from the greater.

In part 6, you might want to break the distance from $\frac{3}{4}$ to $3\frac{1}{2}$ into three parts:

- the distance from $\frac{3}{4}$ to 1 is $\frac{1}{4}$,
- the distance from 1 to 3 is 2,
- the distance from 3 to $3\frac{1}{2}$ is $\frac{1}{2}$.

These parts add up to $\frac{1}{4} + 2 + \frac{1}{2} = 2\frac{3}{4}$.



The absolute value of a number is the distance from 0. We have a special symbol to represent absolute value. For example, we write $|6|$ and read it as "the absolute value of 6". We write $|-6|$ and read it as absolute value of -6. Since 6 and -6 are both 6 units from 0, we see that $|6| = |-6| = 6$. Since the absolute value is a distance, it is never negative. We often use absolute value when computing or representing distances between numbers. For example, if we want to compute the distance between -5 and 3, we can either subtract the lesser number from the greater number $3-(-5)=8$. Or we can take the absolute value of the difference, $|-5-3|=|-8|=8$. The advantage of using the absolute value is that we can compute the difference in either order. Why is this true?

PROBLEM 3

Compute the distance between the following pairs of numbers.

1. -12 and 6
2. -52 and 27
3. -23 and -35
4. 1.75 and -1.25
5. $\frac{3}{4}$ and $-\frac{1}{3}$

EXERCISES

1. Compute the following sums or differences.

a. $45-64$

b. $42+(-36)$

c. $19-(-33)$

d. $17-(-25)$

e. $-13+26$

f. $\frac{2}{3}+\frac{1}{5}$

g. $\frac{3}{5}+\frac{2}{3}$

h. $\frac{4}{5}-\frac{2}{3}$

i. $\frac{5}{7}+\frac{1}{3}$

j. $2\frac{3}{4}+3\frac{1}{5}$

k. $5\frac{3}{4}-2\frac{2}{3}$

l. $5\frac{1}{4}-2\frac{2}{3}$

2. Compute the following products and quotients.

a. $-2 \cdot 7$

b. $5 \cdot (-5)$

c. $-11 \cdot (-6)$

d. $-24 \div 6$

e. $-33 \div (-5.5)$

f. $\frac{2}{3} \cdot \left(-\frac{4}{5}\right)$

g. $-\frac{5}{7} \div \left(-\frac{15}{16}\right)$

h. $-6 \div \frac{3}{5}$

i. $3\frac{1}{2} \cdot 2\frac{2}{5}$

3. Evaluate the following expressions.

a. $5 + 6 \cdot (-3)$

b. $6 \cdot 7 - (-3) \cdot 7$

c. $9 \cdot (-14 + 5)$

d. $-13 - (-6 - 29)$

e. $\frac{-2+20}{-3}$

f. $\frac{2 \cdot 8 - 21}{-3 \cdot 10}$

4. Compute the distance between each of the following pairs of numbers.

a. 8 and -3

b. 4 and -5

c. 1.1 and .9

d. 3.4 and 2.95

e. .26 and .3

f. $\frac{2}{3}$ and 2

g. $\frac{2}{3}$ and $\frac{1}{2}$

h. $\frac{3}{5}$ and $\frac{3}{10}$

i. 3.01 and 2.9

j. 3.01 and 2.99

k. 3.1 and 2.9

l. 3.1 and 2.99

5. Using words, describe 3 subsets of whole numbers that are each infinite. Describe another infinite subset of whole numbers that is a subset of one of your first 3 subsets.

6. Compute the distance between each pair in the following list of numbers. Explain which pair is closest and which pair is the greatest distance apart.

$$1.39 \text{ and } 2.4 \qquad 1.41 \text{ and } 3.1 \qquad 1\frac{5}{6} \text{ and } 3\frac{1}{3} \qquad \frac{7}{4} \text{ and } \frac{11}{3}$$

7. Copy the Venn Diagram from Example 1. For each condition given below, find a number that satisfies the condition and then place it on the Venn diagram.
   a. A whole number that is not a natural number.
   b. An integer that this not a whole number.
   c. A rational number that is not an integer.
   d. A rational number that is an integer, but not a whole number.

8. Find 3 numbers between each of the following pairs of numbers. Sketch a number line and plot the numbers on it.

   a. $\frac{2}{3}$ and 1
   b. $\frac{1}{4}$ and $\frac{1}{3}$
   c. $2\frac{3}{4}$ and $3\frac{1}{5}$
   d. $\frac{15}{7}$ and $\frac{5}{2}$

9. In each of the following problems, 3 numbers are given. Draw a number line and mark and label the 3 numbers. Pay attention to the distances between the numbers. Your picture should give an approximate sense of where the 3 numbers lie in relation to each other.

   a. 1, 4, and 7
   b. 5, 19, and 23
   c. 2, 4, and -5
   d. -2, -7, and -12
   e. -10, 20, and 30
   f. 6, 8, and -97

SECTION 1.2 VARIABLES ON THE NUMBER LINE

In mathematics, we try to solve problems. Sometimes we need to find or compute an unknown quantity. It is useful to have a name or symbol for this unknown value. Algebra is a language that uses symbols to describe problems mathematically. A symbol used to represent an unknown value is called a *variable*. A variable can denote a number that either changes or whose value is unknown in a given problem. We call them variables because the number they represent can vary from problem to problem.

In this section, we will use a letter, like a, b, x, S, etc., to represent an unknown number on the number line. We treat these variables just like numbers. We can do anything with variables that we can do with numbers. Combining variables with numbers and arithmetic operations, we can form an *algebraic expression*. For example, we could start with the variable a, and write the expressions $2 \cdot a$, $-a$, and $a+2$. These expressions have the same meaning they would have if a was a number instead of a letter, that is: $2a$ represents twice the number a, $-a$ represents the negative (or *opposite*) of a, and $a+2$ represents 2 more than the number a. You can use what you know about the number line to visualize what these expressions mean. In other problems, you will be given some extra information and you can use what you know about the number line to determine what value the variable must have. In the next section, we talk about how to create expressions in solving real problems.

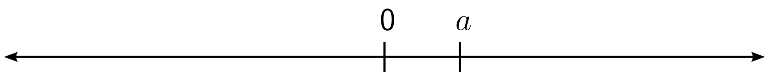
Variables and Expressions on a Number Line

On the number line, each point represents a number and every number is represented by exactly one point on the line. In the first exploration, assume the variable a represents the number located on the number line. The number 0 is also marked on the number line.

EXPLORATION 1

The number a is located on the number line below. Locate and label the points that represent the given algebraic expressions involving a. Use string or a ruler to help locate these points. Plot a point that represents each of the following:

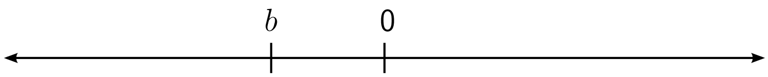
$$2a, \quad 3a, \quad -a, \quad -2a, \quad \frac{3a}{2}, \quad \frac{a}{3}.$$



EXPLORATION 2

Suppose b is a number that is located on the number line as seen below. Plot and label the points that represents each of the following:

$$2b, \quad 3b, \quad -b, \quad -2b, \quad \frac{3b}{2}, \quad \frac{b}{3}.$$



Is b a positive or negative number? Is $-b$ a positive or negative number? How can you tell? Compare the results from Explorations 1 and 2. How are the results similar? In what ways are they different?

On the number line, -5 is a point that is the same distance from 0 as $+5$, but on the opposite side of 0. The same is true of 3 and -3: both are a distance 3 from 0, but on opposite sides. The same is true for 234 and -234. Whatever number we take, the opposite of that number is the same distance from 0 and located on the other side of 0. With our language of algebra, we can write this more concisely as the *opposite* of n is $-n$.

Note that $3+(-3)=0$. Similarly, $5+(-5)=0$ and $234+(-234)=0$

and so on for every number. With the help of variables, we can state "and so on for every number" in a more precise way: for any number n, $n + (-n) = 0$. We call $-n$ the *additive inverse* of n.

ADDITIVE INVERSE

For any number n,

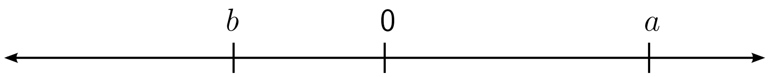
$$n + (-n) = 0.$$

For example, $3 + (-3) = 0$.

PROBLEM 1

On the number line below, the numbers a and b are marked. Locate and plot the point that represents each of the following:

$$-a, \quad -(-a), \quad -b, \quad -(-b).$$



Is b a positive or negative number? Is $-b$ a positive or negative number? Is $-(-b)$ a positive or negative number? How can you tell?

From the results of Problem 1, we discover the following rule:

THEOREM 1.1: DOUBLE OPPOSITE

For every number n, $-(-n) = n$.

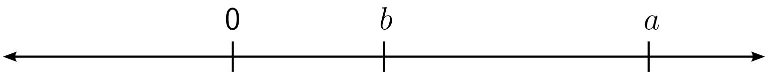
For example, $-(-3) = 3$.

Adding and Subtracting on the Number Line

We can use variables on the number line to develop general properties of addition and subtraction.

EXPLORATION 3

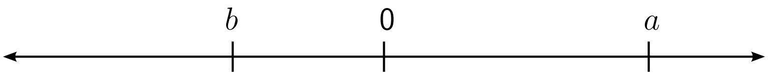
On the number line below, the numbers a and b are marked.



1. Use the number line model to locate and plot the point $a - b$.
2. Locate and plot the point $-b$.
3. Locate and plot the point $a + (-b)$. What do you notice?

PROBLEM 2

Consider the number line from Problem 1:



1. Use the number line model to locate and plot the point $a - b$.
2. Locate and plot the point $-b$.
3. Locate and plot the point $a + (-b)$.

The results of Problem 2 and Exploration 3 tell us how to rewrite any subtraction problem as an addition problem.

ADDITION OF THE OPPOSITE OF A NUMBER
For every pair of numbers a and b, $a - b = a + (-b)$.
For example, $3 - 5 = 3 + (-5) = -2$ and $3 - (-5) = 3 + 5 = 8$.

PROBLEM 3

Rewrite each of the following subtraction problems as an addition problem and then compute the sum.

1. $10 - 3$
2. $5 - (-2)$
3. $-2 - (-4)$
4. $-14 - 5$

Locating Expressions on a Number Line

In each of the following explorations, the numbers 0 and 1 are labeled on the number line. This means the length of 1 is marked. Use it to locate other points on the number line.

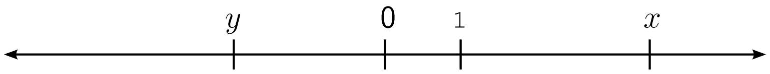
In the next exploration, you will be given the location of a variable like x and asked to locate related quantities that are expressions of x, like $2x - 1$ and $2x + 1$. Use the straight edge of a piece of paper as a ruler to locate these numbers. Be as accurate as you can be.

To use the straight edge, place the edge of your paper on the number line in such a way that you see the ticks and labels. Copy the ticks at numbers 0 and 1 from the number line to your paper. By moving one of the ticks to x, the other will show you where the number one less than x or the number one greater than x is located. You can repeat this to add or subtract larger integers. To divide a number you need to fold the paper. For example, to find $1 \div 2 = \frac{1}{2}$, copy the ticks at numbers 0 and 1 from the number line to your paper, and fold the paper in half to see where to place $\frac{1}{2}$ on the number line. The same idea can be used to divide $2x \div 2$.

EXPLORATION 4

Both x and y are numbers located on the number line below. Plot a point that represents each of the following expressions on the number line below:

$$x + 1, \quad x - 2, \quad y + 1, \quad y + 4, \quad y - 1.$$



It isn't possible to compute the exact value of x or y in the exploration. But it is possible to estimate the value of the variables x and y. We just need to compare their positions to where 0 and 1 are.

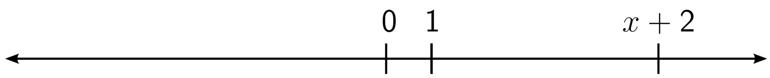
Recovering the Location of a Variable on a Number Line

In this exploration, you will discover how to find a variable on a number line given the position of an algebraic expression. This is the opposite of the process that used in the previous exploration.

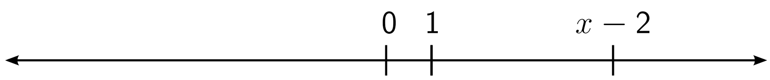
EXPLORATION 5

In each of the problems below, locate the point that represents x. Explain how you find your answer. Estimate the value of x from its location. In each of the following explorations, 0 and 1 are marked.

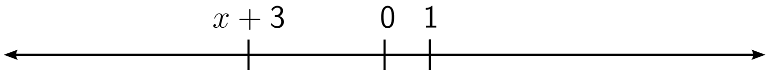
1.



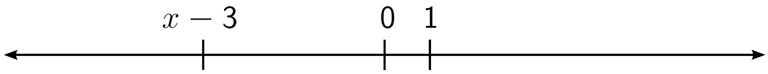
2.



3.



4.



In part 1, begin at point $x+2$ and move 2 units to the left to find the location of the point x. This has the effect of adding -2 to the number $x+2$. This is the same as subtracting 2 from $x+2$:

$$x+2+(-2)=x+0=x.$$

The result of this move on the number line and the calculation above is the number x.

In part 2, begin at $x-2$ and move 2 units to the right to find the location of the point x. This has the effect of adding 2 to the number $x-2$:

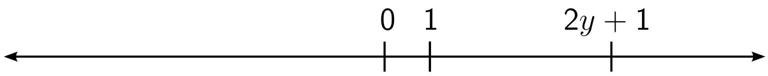
$$x-2+2=x+0=x.$$

Again, the result of this process is the position of x.

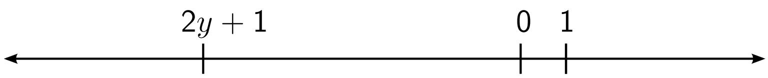
PROBLEM 4

For each number line below, locate the point that represents y. Estimate the value of y from its location.

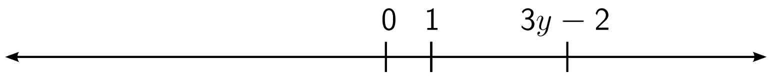
1.



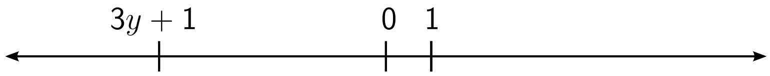
2.



3.



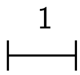
4.



We now explore another type of problem using the number line. Here we are given the location of an algebraic expression and its numerical value. For example, in the next problem, a point is labeled as $2x+1$ and 9.

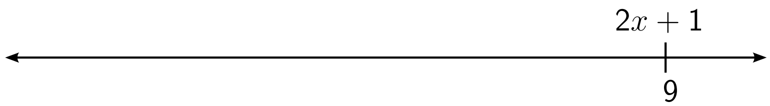
PROBLEM 5

In each of the following problems, assume that this line segment gives the length 1 unit:

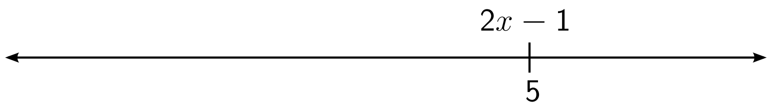


The given expression equals a numerical value. Locate the points that represent the numbers x and 0. Estimate the value of x by its location. Substitute the value of x in the expression to check your work.

1. $2x+1$ is the same as 9:



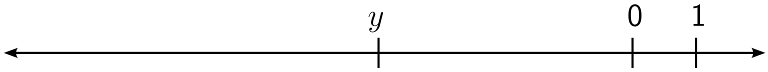
2. $2x-1$ is equal to 5:



EXERCISES

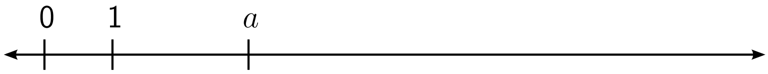
1. Plot a point that represents each expression and label it:

$$2y+2, \quad 2y-1, \quad 2y+5, \quad \frac{y}{2}.$$



2. Plot a point that represents each expression and label it:

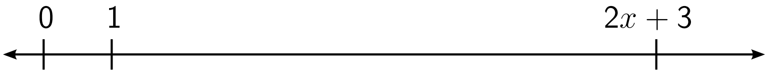
$$a-1, \quad a+2, \quad 2a, \quad 2a-2, \quad 2(a-1), \quad 2a+4, \quad 2(a+2).$$



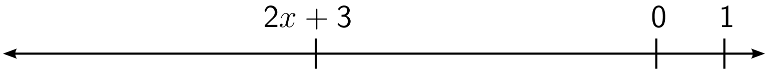
   Are any of the expressions located at the same point? Explain why.

3. Locate the point that represents x for each of the following:

   a.



   b.

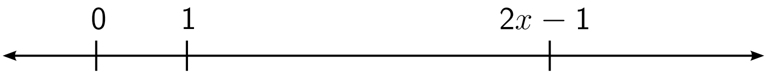


4. In each of the sections below, locate the point that represents x. Estimate the value of x by its location.

   a.

   b.

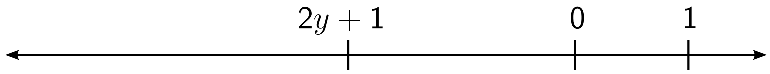
   c.



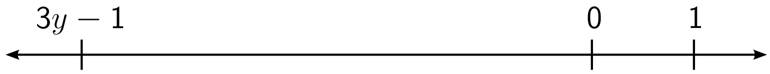
5. In each of the sections below, locate the point that represents the variable. Estimate the value of the variable by its location.

a.

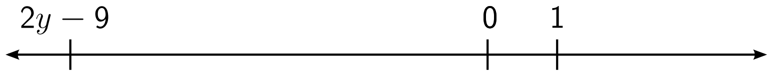
b.



c.

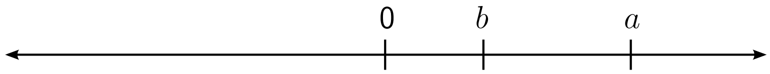


d.



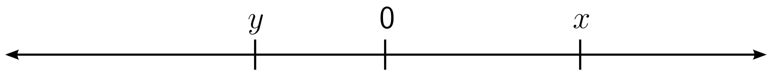
6. Plot a point that represents each expression and label it:

$$a - b, \quad a + b, \quad b - a.$$



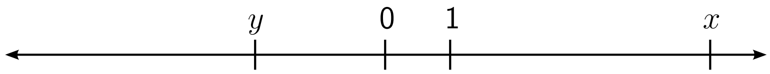
7. Plot a point that represents each expression and label it:

$$x - y, \quad x + y, \quad y - x.$$

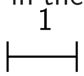


8. Plot a point that represents each expression and label it:

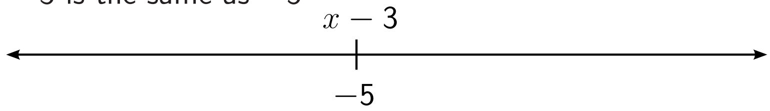
$$1 - x, \quad 2 - x, \quad 3 - x, \quad 1 - y, \quad 2 - y.$$



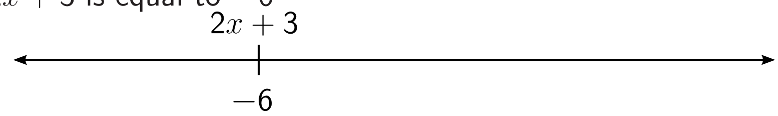
9. The line segment below is 1 unit long. Complete the following steps for each part:
    - Locate the point that represents the solution x.
    - Locate the point that represents the number 0.
    - Estimate the value of x by its location.
    - Substitute the value of x in the expression to check your work.



a. $x - 3$ is the same as -5

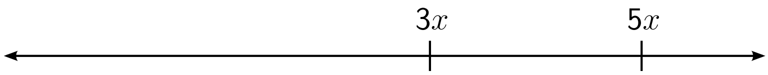


b. $2x + 3$ is equal to -6



10. For each of the following expressions and equivalent numbers, draw a number line to determine and locate the value of the variable. It is up to you to choose the length of the unit and the location of the given expression.
    a. $4x + 7$ is 19
    b. $3y + 1$ is 10
    c. $2a - 3$ is 7
    d. $5z - 9$ is -24

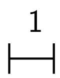
11. Given the location of the two expressions, find the location of x and 0:



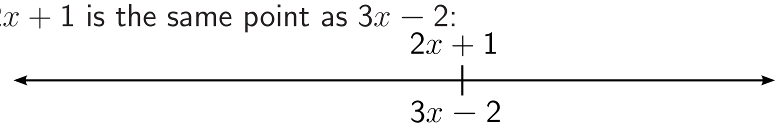
12. Given the location of the two expressions, find the location of x and 0:

13. **Investigation:**
Place the number a on the number line. Can we find $2a$ without locating 0 first? Can we find $a-1$ without finding 0 and 1 first? If the answer to these questions is yes, explain how to do so. If the answer is no, explain why not.

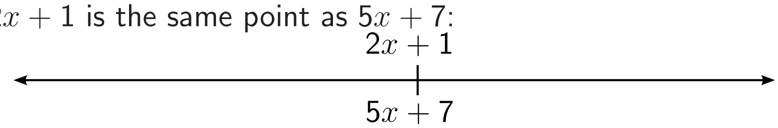
14. **Investigation:**
Locate the point that represents the number x on the number line. Also, locate the point that represents the number 0.



a. $2x+1$ is the same point as $3x-2$:



b. $2x+1$ is the same point as $5x+7$:



15. **Ingenuity:**
Consider the following sequence:

$$4, 7, 10, 13, 16, 19, 22, \ldots.$$

If the pattern continues:

a. What is the 20^{th} member of the sequence?
b. What is the 100^{th} member of the sequence?
c. The number n is a positive integer. What is the n^{th} member of the sequence?

SECTION 1.3 VARIABLES AND EXPRESSIONS

In Section 1.2 variables represented numbers on the number line whose value was unknown. In this section, we see how to use variables to create *algebraic expressions* and *equations*that describe a problem mathematically. An equation is a math sentence with an equality sign, $=$, between two expressions.

Variables and Expressions in Context

Here is an example that illustrates what variables are and how to use them.

EXAMPLE 1

Use the variable J to represent John's age in years. Though you don't know his age, you do know that John is 5 years older than Sue. Use the variable S to represent Sue's age and write an equation that expresses the relationship between J and S.

SOLUTION Define 2 variables in the following way:

Let $J =$ John's age in years.

Let $S =$ Sue's age in years.

Now translate the sentence "John is 5 years older than Sue" into an equation. Substituting variables for words, "J is 5 more than S", or $J = S + 5$. "Is" translates to "$=$". "5 more than" means add 5. Expressions involving algebraic symbols are called *algebraic expressions*. In this case, S, $S + 5$, and J are all algebraic expressions.

Once we have an algebraic equation, it is easy to make a table of some possible values of J and S that satisfy the conditions of this problem. For example, if Sue were 2 years old, then $S = 2$ and $J = S+5 = 2+5 = 7$. So John would be 7 years old. Fill in the table below with other possible values.

S	$J = S + 5$
2	7
3	____
10	____

EXPLORATION 1

We look at a family of 4 people and their ages. Today, Adam's age is 4 more than his sister Bonnie's age. Their mother Carmen is twice as old as Bonnie and their father Daniel's age is one year more than twice Adam's age. Let the variable A represent Adam's age right now.

1. Write an expression that represents Bonnie's age in terms of Adam's age.
2. Using what we learned about Bonnie's age, write Carmen's age in terms of Adam's age.
3. Write Daniel's age in terms of Adam's age.
4. Let the variable B represent Bonnie's age at this time. Write an expression that represents Adam, Carmen and Daniel's ages in terms of B. Explain in words and in symbols how you arrived at your expressions.

Some of these relationships and expressions take careful examination and thinking. Expressions give a simple way of looking at relationships between quantities. However, do not forget that there is meaning behind the symbols. Do not lose sight of what that meaning is while working with the expressions.

EXAMPLE 2

Mary takes a bike ride everyday at an average rate of 12 miles per hour. Use the variable t to represent the length of time for Mary's bike ride, measured in hours. Notice that t represents how long Mary rides her bike on the day in question. The reason t is a variable is that it can assume different values depending on other information that is given. In spite of the fact that t can vary, it can still give us interesting information about Mary's rides.

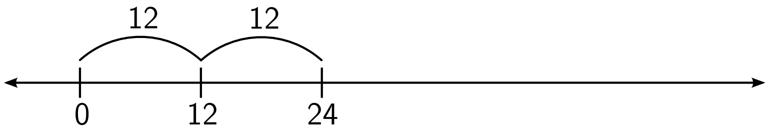
Find the distance that Mary travels if she rides for

1. 2 hours.
2. 5 hours.
3. t hours.

SOLUTION

1. The distance Mary travels in one hour is 12 miles, so the distance Mary travels in 2 hours should be twice as far as 12 miles, that is, 24 miles. Her speed is 12 $\frac{\text{miles}}{\text{hour}}$, so we can write this correspondence between time and distance as

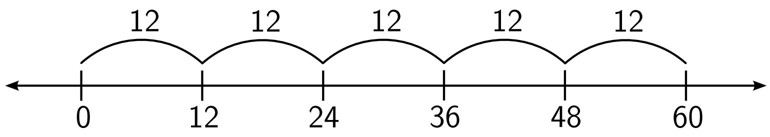
$$12\ \frac{\text{miles}}{\text{hour}} \cdot 2 \text{ hours} = 24 \text{ miles}.$$



2. When Mary rides for 5 hours, she travels

$$12\ \frac{\text{miles}}{\text{hour}} \cdot 5 \text{ hours} = 60 \text{ miles}$$

all together.



3. Since variables are just like numbers,

$$12\ \frac{\text{miles}}{\text{hour}} \cdot t \text{ hours} = 12 \cdot t \text{ miles.}$$

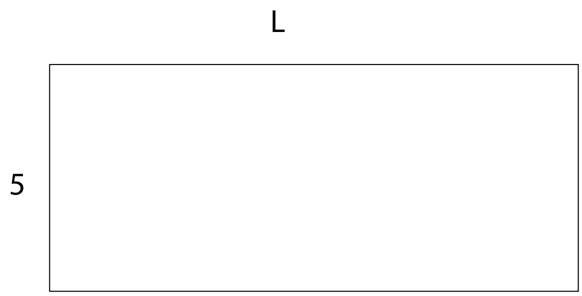
In t hours, Mary will average 12 miles for each hour, so she rides $12 \cdot t$ miles in total. The expression $12t$ represents the total number of miles Mary rides, depending on the value of t. This expression is very useful. For example, if $t = 2$, Mary rides $12(2) = 24$ miles, but if $t = 0.5$, Mary travels $12(.5) = 6$ miles. When Mary rides t hours, the distance she travels is $12t$ miles. Note that $12t$ is again an algebraic expression and it represents the distance Mary travels in t hours when she rides an average rate of 12 miles per hour.

Variables and Expressions in Geometry: The Area Model

EXAMPLE 3

Sketch a rectangle that has length L cm and width 5 cm. Express the area and perimeter of the rectangle in terms of L.

SOLUTION



Make a table with the area and perimeter of various rectangles that satisfy the given condition. That is, the width must be 5 cm.

Length in cm	Width in cm	Area in cm^2	Perimeter in cm
1	5	$5 \cdot 1 = 5$	$5 + 5 + 1 + 1 = 12$
2	5	$5 \cdot 2 = 10$	$5 + 5 + 2 + 2 = 14$
3	5	$5 \cdot 3 = 15$	$5 + 5 + 3 + 3 = 16$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
L	5	____ $=$	____ $=$

We can now write algebraic expressions that represent the patterns we see in the table. Let L be the length in cm. The area is $5L$. The perimeter is $5 + 5 + L + L = 10 + 2L$ cm.

PROBLEM 1

Sketch a rectangle whose length is twice its width. Let L represent the length and W the width. Write an algebraic expression for

1. the area in terms of W.
2. the area in terms of L.
3. the perimeter in terms of W.

PROBLEM 2

Sketch a rectangle whose length is 3 inches greater than its width. Let L represent the length and W the width. Write an algebraic expression for

1. the area in terms of W.
2. the area in terms of L.
3. the perimeter in terms of W.

Variables to Describe Sets: Set Notation

In Section 1.1, we constructed the number line and discussed many different sets of numbers. Sets are so important that we have many ways of describing them. We can describe sets using just words. For example, the whole numbers and the even numbers are 2 different sets of numbers. Sometimes we can graph sets on a number line. Also, variables can be used to describe elements in a set using *set notation*. We demonstrate all 3 methods in the following example.

EXAMPLE 4

Write the set of all numbers greater than or equal to 1 in set notation. Graph this set on the number line.

SOLUTION We represent this set using a variable and set notation. We use x to describe the numbers in the set, then translate the condition "x is greater than or equal to 1" to the language of algebra: $x \geq 1$.

Set Notation: $\{x | x \geq 1\}$

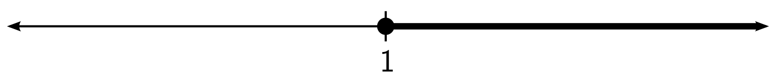
The brackets $\{ \}$ denote a set. The variable x denotes any element in the set. The vertical bar $|$ means "such that" and is followed by a condition to be satisfied by every element x in the set. So we read this notation as:

Words: "the set of all numbers x such that x is greater than or equal to 1."

In this case the variable x is used to describe an infinite set of numbers!

We can graph the set on a number line. First we locate 1, then indicate all the numbers greater than or equal to 1 by shading 1 as well as all the numbers to the right of 1 on the number line. We draw a small shaded circle around 1 to show that it is in the set.

Graph:



EXAMPLE 5

Write the set of all numbers greater than 2 in set notation. Indicate this set on a number line.

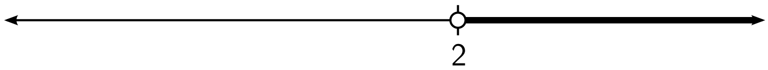
SOLUTION We can use x again to describe the numbers in this set. The condition "x is greater than 2" with algebraic symbols is written as $x > 2$. The set can be written

Set Notation: $\{x|x > 2\}$,

which reads as

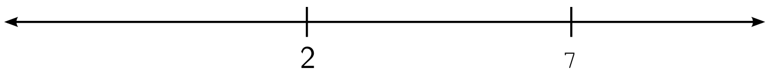
Words: "the set of all numbers x such that x is greater than 2."

On the number line, we shade in every number greater than 2, but not 2 itself. Again, 2 is only a single point, so it is not possible to shade it in such a way that can be seen clearly. To denote that it is excluded from the set, we draw an unshaded circle around it and only shade numbers to the right of the circle.



Every number greater than 2 is greater than or equal to 1. Therefore the second set is a subset of the first. On the other hand, 1.5 is in the first set but not in the second. So the first set is not a subset of the second.

PROBLEM 3



1. Express the statement that "x is greater than or equal to 2 and

smaller than 7" in the language of algebra.

2. Write the set of numbers greater than or equal to 2 and smaller than 7 using set notation.
3. On the number line show the set of numbers that are greater than or equal to 2 and smaller than 7.
4. Find 5 numbers x that satisfy this condition.

Summary

When working through problems in algebra, use variables and symbols to state things mathematically, just as you write sentences by combining letters and words. It is often possible to translate sentences into math using variables and expressions to form equations. Once we have translated the sentences into equations, we can solve these problems more easily than we could have when simply looking at the text of the problem. For many of the problems in this section, you may be able to find the answer without formally making equations. However, the approach of using equations becomes important as the problems become more complicated. Writing equations will organize our work and provide a clearer process for obtaining a solution.

Variables and algebra are part of the language of mathematics, and learning this language allows you to make mathematical statements using variables that you choose. When using variables, the important thing is to describe what the variable represents (in the examples above, variables represented units of time, lengths and numbers on the number line). Then translate real world problems into algebraic expressions, and operate on those expressions to solve the problems. In this way, algebra becomes a tool for modeling and solving problems.

EXERCISES

1. Draw a rectangle and call the lengths of the sides of the rectangle x units and y units. What is the perimeter of the rectangle? What is the area of the rectangle?
2. A rectangle has width W and a length that is three times as long as

the width.

a. Write an expression for the perimeter of this rectangle.
b. Write an expression for the area of this rectangle.

3. A rectangle has length L inches.
   a. The width is 6 inches longer than the length. Write an expression for the perimeter and an expression for the area of this rectangle.
   b. The width is 6 inches shorter than the length. Write an expression for the perimeter and an expression for the area of this rectangle.
4. Express each of the statements algebraically:
   a. x is two units less than y.
   b. x is three units more than y.
   c. x is three times the number y.
   d. x is one half the number y.
   e. x is y units less than z.
5. Express each of the statements algebraically:
   a. x is less than the number that is the sum of y and z.
   b. z is greater than 3.
   c. y is less than twice x.
6. Marisa is M years old, and Jenny is 5 years older than Marisa. Write an expression that gives Jenny's age.
7. Jessica is 10 years old today. How old will she be in 3 years? In 5 years? In x years?
8. Jacob is J years old today. How old will he be in y years?
9. Joe travels at a rate of 20 miles per hour for t hours. How far does he go?
10. Jeremy travels at a rate of J miles per hour for 3 hours. How far does he go?
11. Sam has marbles.
    a. He has 10 marbles and gives away x marbles. How many marbles does he have left?
    b. He has y marbles and gives away 10 marbles. How many marbles does he have left?
    c. He has u marbles, then Joe gives him 7 marbles and after that

he gives half of the marbles he has to Matt. How many marbles does he have left?

d. He has v marbles and gives half of them to Matt. Then Joe gives him 7 marbles. How many marbles does he have?

e. In each of the questions above, what are the possible values of the variable?

12. Juan is 5 years younger than Maria and twice as old as Pedro.
    a. Pedro's age is P. Write an equation that relates Maria's and Juan's ages.
    b. Call Maria's age M and write an equation that relates Pedro's and Juan's ages.

13. Translate each mathematical statement into an equation or expression:
    a. x is two greater than y.
    b. the number that is two greater than x.
    c. the number that is two less than x.
    d. x is less than y.
    e. x is five smaller than a number that is twice as large as y.

14. Express in words what each set represents.
    a. $\{x : x < 10\}$
    b. $\{x : x > 5\}$
    c. $\{x : 2 < x \leq 8\}$
    d. $\{x : -5 \leq x\}$

15. If x were 2 more, it would be 10 less than z. Express the relation between x and z.

16. A triangle has a base x cm long.
    a. When the height is 6 cm, write an expression for the area of the triangle.
    b. When the height is 3 times the base, write an expression for the area of the triangle.
    c. When the height is one half the base, write an expression for the area of this triangle.

17. Describe the set of numbers that are larger than 2 and less than 5 using set notation. Represent this set on the number line.

18. The low temperature near the Exit Glacier in Alaska on January 5 was −8 °F, and the high temperature was 22 °F.
    a. Describe in words the set of all the temperatures T in degrees Fahrenheit at this location on this day.
    b. Write this set using set notation.
    c. Represent this set on the number line.

19. Using set notation, describe 3 subsets of whole numbers that are each infinite. Then describe another infinite subset of whole numbers that is a subset of one of your first 3 subsets.
20. We can write the set of even natural numbers as $\{2, 4, 6, 8, \ldots\}$. It can also be described in set notation using a variable as $\{2n|\ n$ is a natural number$\}$. Write each of the following sets using a variable in set notation.
    a. $\{1, 3, 5, 7, 9, \ldots\}$
    b. the set of positive multiples of 5
    c. the set of multiples of 10
    d. $\{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$
21. What is the distance from −8 to 14? From −12 to −5?

SECTION 1.4 SOLVING LINEAR EQUATIONS

Solving real-world problems using mathematics is often a 3-step process. First, you translate the words in the problem to the language of mathematics. Then you use the tools that mathematics gives you to solve these problems. Finally, check the answer.

EXAMPLE 1

Sue sells sandwiches at some price we don't know and candy bars for \$2 each. Mark buys only one candy bar and 4 sandwiches for a total cost of \$14. How much do the sandwiches cost?

SOLUTION

First, notice that the total cost of \$14 is the sum of the cost of the 4 sandwiches and the cost of the candy bar. So:

$$\text{cost of 4 sandwiches} + \text{cost of one candy bar} = \text{total cost.}$$

How do we translate this into a mathematical sentence? Begin with the information given in the problem.

$$\begin{aligned} \$14 &= \text{the total cost} \\ \$2 &= \text{cost of the one candy bar Mark buys} \\ 4 &= \text{number of sandwiches Mark buys} \end{aligned}$$

We define the variable as the quantity that we want to compute:

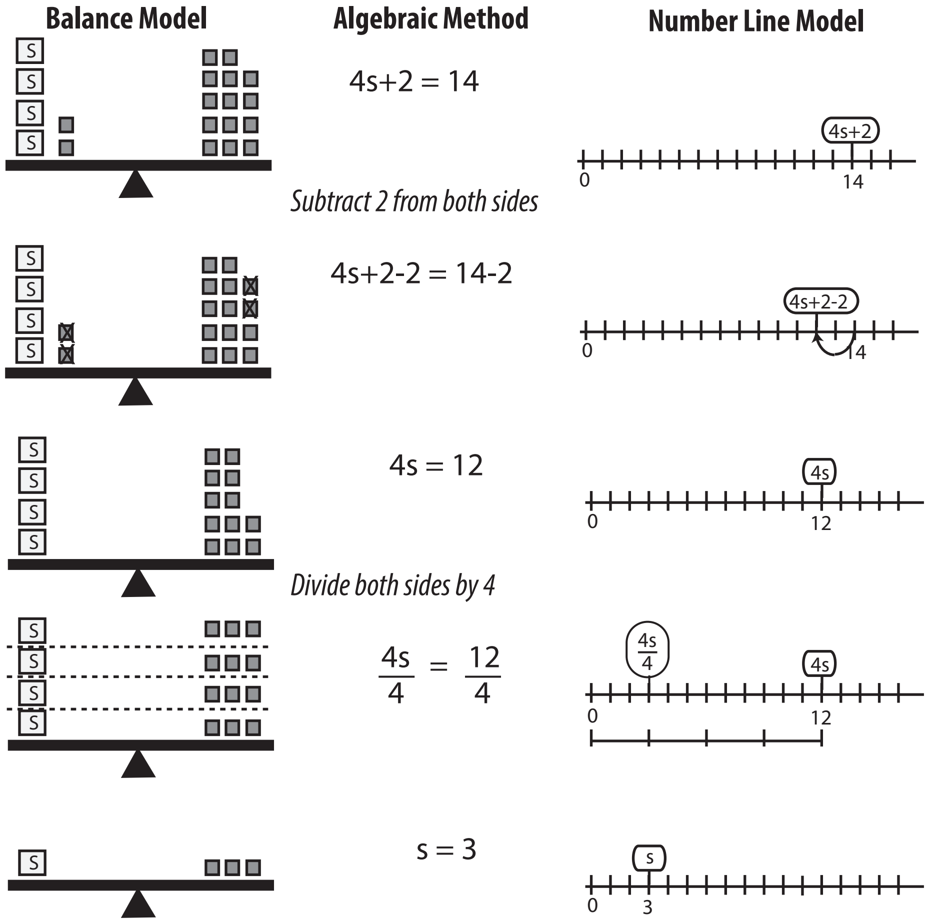
s = the price of a single sandwich in dollars.

Now combine the information to form an equation.

$$\begin{aligned} \text{cost of 4 sandwiches} + \text{cost of one candy bar} &= \text{total cost} \\ \text{number of sandwiches} \cdot \text{price of a sandwich} + \$2 &= \$14 \\ 4s + 2 &= 14 \end{aligned}$$

To demonstrate different ways we can think about solving equations, we will use three different approaches to solve this single problem. What do you notice about the three methods? How are they similar?

Solve $4s + 2 = 14$



Now we check our answer.

$$4 \cdot 3 + 2 = 14$$
$$14 = 14$$

EXPLORATION 1

In the previous example, we solved the equation $4s+2 = 4$ by performing the same operations on both sides of the equation: subtraction first and then division. This allowed us to *isolate the variable* s. Discuss which of these steps you should perform on each of the following equations. Some equations require two steps to solve for x. Does it matter in what order you perform these steps? Which order of the steps lets you isolate x the easiest. Explain.

1. $x - 4 = 10$
2. $x + 4 = 10$
3. $3x = 15$
4. $\frac{x}{4} = 3$
5. $15 = 2x + 7$
6. $4x + 3 = 17$
7. $2x - 4 = 10$

EXPLORATION 2

Think about the problems above. What steps were involved in translating the problem in Example 1 to the language of mathematics? What steps did you use to solve the equations in Exploration 1? How do you decide the order of the steps?

PROBLEM 1

Susan asks Fred to make some cookies for a party. Fred decides to make twice as many as Susan requested, and also 3 extra cookies for his neighbor. Fred makes 57 cookies. How many cookies did Susan request?

Now we can formally define the concepts and properties we have discussed so far.

SOLUTION

A *solution* to an equation with a variable x is a number that when substituted for x, makes the 2 sides of the equation equal. If the equation has more than one solution, then the collection of solutions is called the *solution set*.

EQUIVALENT EQUATIONS

Two equations are *equivalent* if they have the same solution or solution set.

The process of solving an equation for x is to find simpler equations that are equivalent. In each step, the equations are equivalent and will have the same solution. There are 4 basic ways to produce equivalent equations:

ADDITION PROPERTY OF EQUALITY

Starting with any equation, if you add the same amount to both sides of the equation you obtain an equivalent equation.

For example, the equations $x - 5 = 7$, $x - 5 + 5 = 7 + 5$, and $x = 12$ are equivalent.

SUBTRACTION PROPERTY OF EQUALITY

Starting with any equation, if you subtract the same amount from both sides of the equation you obtain an equivalent equation.

For example, the equations $x + 5 = 7$, $x + 5 - 5 = 7 - 5$, and $x = 2$ are equivalent.

MULTIPLICATION PROPERTY OF EQUALITY
Starting with any equation, if you multiply both sides of the equation by the same non-zero number you obtain an equivalent equation.
For example, the equations $\frac{x}{5} = 7$, $\frac{x}{5} \cdot 5 = 7 \cdot 5$, and $x = 35$ are equivalent.

DIVISION PROPERTY OF EQUALITY
Starting with any equation, if you divide both sides of the equation by the same non-zero number you obtain an equivalent equation.
For example, the equations $5x = 35$, $\frac{5x}{5} = \frac{35}{5}$, and $x = 7$ are equivalent.

In the next 3 examples, we use the properties of equality to solve equations. To solve an equation for x, we must find an equivalent equation giving us the value of x. In order to achieve this, we need to transform the original equation through a chain of equivalent equations into an equation that isolates x by itself on one side and a number on the other.

EXAMPLE 2

Solve the equation:

$$x - 3 = 20.$$

SOLUTION

To solve for x, we need to get rid of the -3. By the addition property of equality, the original equation is equivalent to:

$$x - 3 + 3 = 20 + 3.$$

Evaluating the sums leads to the solution $x = 23$.

EXAMPLE 3

Solve the equation:

$$4x = -8.$$

SOLUTION By the division property of equality, our original equation is equivalent to:

$$\frac{4x}{4} = \frac{-8}{4}.$$

Evaluating gives:

$$x = -2$$

as the solution.

EXAMPLE 4

Solve the equation:

$$3r + 5 = 17.$$

SOLUTION In order to solve this equation, we need to isolate r. However, in this case, we are unable to isolate r by performing a single operation. We observe that, in the expression on the left, r has been multiplied by 3, and a 5 has been added. This suggests that we should divide both sides by 3, and subtract 5 from each side. But in what order should we perform these operations? Let's think about how the expression on the left is constructed:

$$3r + 5 = \text{"Five more than three times } r\text{"}$$

The order of operations tells us that when we evaluate an expression like this, we do the multiplication first (“**three times** r”), and then the addition (“**five more than** three times r”). In Explorations 1 and 2you discovered that to isolate the variable, we can reverse these operations, beginning with the last operation we did. Using the Substraction Property of Equality, let's start by subtracting 5 from each side in order to reverse the addition:

$$3r + 5 - 5 = 17 - 5$$
$$3r = 12$$

Using the division property, solve for r by dividing both sides by 3:

$$\frac{3r}{3} = \frac{12}{3}$$
$$r = 4.$$

EXERCISES

1. For each equation below, solve for the unknown by using the Addition Property of Equality or the Subtraction Property of Equality. Name which one you use. Refer to Example 2.
   a. $x - 5 = 15$
   b. $y - 10 = -14$
   c. $z + 17 = 35$
   d. $a + 20 = 5$
   e. $b + 2 = -4$
2. For each equation below, solve for the unknown by using the Multiplication Property of Equality or the Division Property of Equality. Name which one you use. Refer to Example 3.
   a. $2a = 30$
   b. $3c = -12$
   c. $\frac{S}{3} = 2$
   d. $-4d = 2$
   e. $\frac{1}{8}d = -2$
   f. $-\frac{x}{5} = -7$

3. Combine the previous 2 methods to solve each equation using multiple steps. Name which properties of equivalence you use.
   a. $2n + 5 = 37$
   b. $2E - 8 = 20$
   c. $3g + 4 = 40$
   d. $\frac{X+3}{4} = 1$
   e. $-4a + 3 = 15$
   f. $5(C - 2) = 25$
   g. $-\frac{d}{3} + 1 = 5$
   h. $\frac{1}{6}x - 4 = 1$
4. For each equation below:
   - solve the equation for the unknown variable by using the Properties of Equality.
   - name which property or properties you have used.
   - check that the solution satisfies the original equation.

   a. $2z = 7$
   b. $\frac{b}{3} = 5$
   c. $3T = 5$
   d. $2x + 3 = 12$
   e. $3S = -8$
   f. $4l - 5 = 7$
   g. $2X + 5 = 16$
   h. $2a - 8 = 7$
   i. $3y + 4 = 20$
5. For each situation described below:
   - define a variable that represents the unknown quantity.
   - write a mathematical equation.
   - solve the equation to answer the question.

   a. Sam went on a bike ride averaging 10 miles per hour. He biked a total of 25 miles. How long did it take him?
   b. A square garden has a fence around it. The total length of the fence is 92 feet. What is the length of each side?
   c. A rectangular playing field is twice as long as it is wide. The perimeter of the field is 246 meters. What is its width?

6. A rectangular pool is 3 meters longer than it is wide. The perimeter is 38 meters.
   a. Define the width as W meters, then write an equation using W and solve it for W.
   b. Define the length as L meters, then write an equation using L and solve it for L.
   c. How do the answers to parts 6a and 6b compare?
7. Think about consecutive whole numbers. For example, 2 and 3 are consecutive whole numbers.
   a. Write three pairs of consecutive whole numbers.
   b. Let n represent a whole number. Write an expression for the next larger whole number.
   c. Write an expression for the sum of two consecutive whole numbers.
   d. The sum of two consecutive whole numbers is 37. What are the two numbers?
8. Sara has 82 inches of string candy. She wants to divide it equally and share with Sandra. She had already promised to give a 10-inch piece to Juanita. If Sara gives half to Sandra first and then gives Juanita 10 inches of candy from her part, how much does Sara have left? To find out, let S be the amount of candy Sara gets. Write an equation using S and solve for S.
9. Joe has 82 inches of string candy. He wants to divide it equally and share with Max. But he had promised to give a 10-inch piece to Jeremy. If Joe gives 10 inches of candy to Jeremy first, then divides the rest into to equal pieces for himself and Max, how much does he get? To find out, let J denote the amount of candy Joe gets. Write up an equation using J and solve.
10. Do you get the same or different result in the previous two questions? Explain.
11. **Investigation:**
    Return to the sandwich stand of Example 1. Suppose Isabel has $20 and she wants to spend all her money (no change) to buy sandwiches and candy bars for her friends. How many of each could she buy?

12. **Ingenuity:**
Consider the equation:

$$3x + 5y = 33.$$

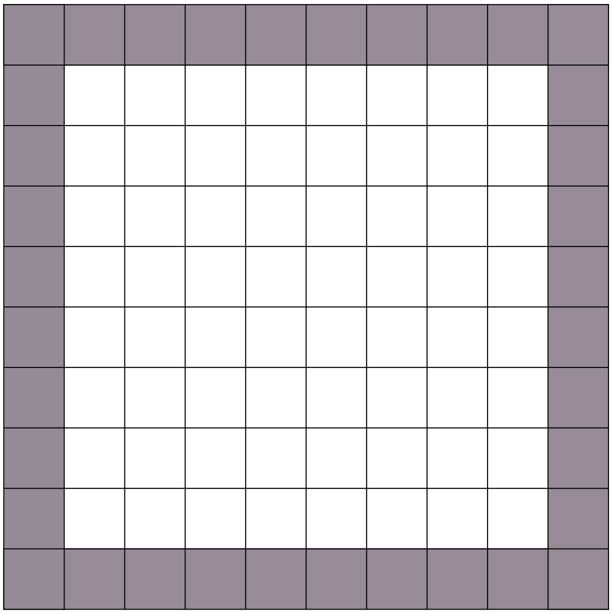
What would it mean to solve this equation for x?

13. Explain the difference between an expression and an equation. Give an example of each.

SECTION 1.5 EQUIVALENT EXPRESSIONS

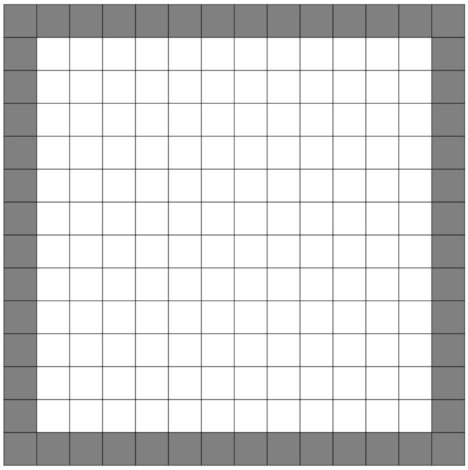
EXPLORATION 1

Consider the square grid below, representing an 8×8 swimming pool with a shaded border of width 1. How many squares are shaded in? Answer this without talking, without counting one by one, and without writing. When everyone is done, share your thinking with your classmates.



PROBLEM 1

Consider the grid below, representing a 12×12 pool with a shaded border of width 1. Calculate the number of shaded tiles without counting one by one.



EXPLORATION 2

Now look at a $n \times n$ square swimming pool with a border of width one. Determine the number of squares in the border using two of the methods your classmates described in Exploration 1. For each of the methods write out in words what the method is doing. Write an algebraic expression that explains each method you used. Make sure to say what the variable in your expression represents.

Equivalent Expressions

EXAMPLE 1

Write the following relationship mathematically in two different ways.

"Twice the amount of money that Jessica has in the bank now."

SOLUTION Let J = amount of money Jessica has in the bank now. One way we can write "twice this amount" is $2J$. Another way to write twice the amount is $J + J$.

The expressions $J + J$ and $2J$ are called equivalent expressions and we write $J + J = 2J$. This means that the two sides of the equation are equal no matter what value we choose for J.

PROBLEM 2

A rectangle has length L and width W. What is the perimeter of the rectangle? Write as many expressions for the perimeter as you can and explain how you arrived at the expressions. *Hint:* sketch the rectangle.

EXPLORATION 3

Determine which of the following number sentences are true. Try to justify your answer without calculating each side.

1. $5 + 7 = 7 + 5$
2. $56 + 89 = 89 + 56$
3. $457 + 684 = 684 + 457$
4. $578943 + 674321 = 674321 + 578943$
5. $7 - 5 = 5 - 7$
6. $56 - 89 = 89 - 56$
7. $457 - 684 = 684 - 457$
8. $578943 - 674321 = 674321 - 578943$
9. $1 + 2 - 3 = 2 + 1 - 3$
10. $4 + 5 - 6 = 5 - 6 + 4$
11. $11 - 14 + 21 = 11 - 21 + 14$
12. $11 + (-14) + 21 = 11 + 21 + (-14)$

Two equivalent expressions for the perimeter of the rectangle in problem 2 are $L + W + L + W$ and $W + L + W + L$. The order in which the sides of the rectangle are added does not affect the perimeter. This fact is an important property in mathematics. The property is called the *commutative property of addition*. It means that for any two numbers n and m, $n + m = m + n$. "Commute" here means to change the order of the terms.

COMMUTATIVE PROPERTY OF ADDITION

For any two numbers x and y,

$$x + y = y + x.$$

For example, $24 + 6 = 6 + 24$ and $8 + (-5) = (-5) + 8$.

PROBLEM 3

The number line is a good way to visualize properties of addition. Use a number line to illustrate that each of the following mathematical sentences is true.

1. $24 + 6 = 6 + 24$
2. $8 + (-5) = (-5) + 8$
3. $n + m = m + n$ where each of m and n are numbers

PROBLEM 4

Is it possible to change the order with other operations and maintain equivalence? Check to see if this works with subtraction, multiplication, and division. For each operation, if it is commutative, make a rule. If not, give an example of how it fails to be commutative.

The *commutative property of multiplication* is also very useful. The expression $50t$ is equivalent to $t50$. It is a convention in algebra to use the form $50x$ rather than $x50$. This is intended to make expressions with lots of variables and numbers simpler to read.

COMMUTATIVE PROPERTY OF MULTIPLICATION
For any two numbers x and y, $$xy = yx.$$ For example, $5 \cdot 6 = 6 \cdot 5 = 30$ and $3 \cdot -2 = -2 \cdot 3 = -6$.

Let's return to equivalent expressions for the perimeter. Another equivalent expression that you might have found is $2L+2W$. Consider $L+W+L+W$ and use the commutative property to write $L + L + W + W$, then group the first two and last two terms to get $2L + 2W$. Grouping is called the *associative property of addition*, because the numbers are changing their group or association. In general, for each number a, b, and c, $(a + b) + c = a + (b + c)$. Just as the commutative property of addition means that adding 2 numbers when placed in different orders does not change the sum, the associative property of addition means that the order of evaluating the addition of 3 or more numbers does not change the sum.

ASSOCIATIVE PROPERTY OF ADDITION

For any numbers x, y, and z,

$$(x + y) + z = x + (y + z).$$

For example, $3 + 4 + 5$ can be viewed as both:

$$(3 + 4) + 5 = 7 + 5 = 12$$

or

$$3 + (4 + 5) = 3 + 9 = 12.$$

PROBLEM 5

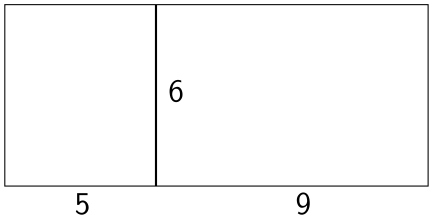
1. Does the same property work for subtraction? Is $(6 - 5) - 3 = 6 - (5 - 3)$?
2. Does multiplication have an associative property? Is $(8 \cdot 4) \cdot 2$ equal to $8 \cdot (4 \cdot 2)$? Is $(ab)c = a(bc)$?
3. Does division have an associative property? Check to see if $(8 \div 4) \div 2$ and $8 \div (4 \div 2)$ are equivalent.
4. Discuss why when carrying out any operation it is important to be particularly mindful of parentheses when subtracting and dividing are involved.

Let's explore another property that is useful when multiplying two expressions.

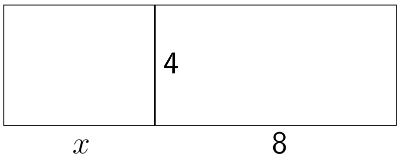
EXPLORATION 4

Compute the area of each of the large rectangles below in at least 2 ways:

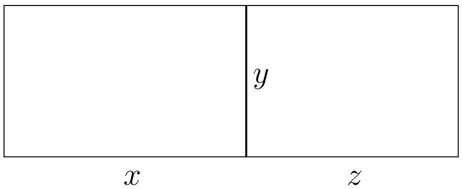
1.



2.



3.



You have just discovered the *distributive property*! For any numbers a, b, and c, $a(b+c) = ab + ac$. This is true even for negative numbers or zero. This property allows us to find one more equivalent expression for the perimeter, $2L + 2W = 2(L + W)$, and also to simplify or transform other expressions.

DISTRIBUTIVE PROPERTY

For any numbers x, y, and z,

$$x(y + z) = xy + xz.$$

For example, $3(2 + 4) = 3 \cdot 2 + 3 \cdot 4 = 6 + 12 = 18$.

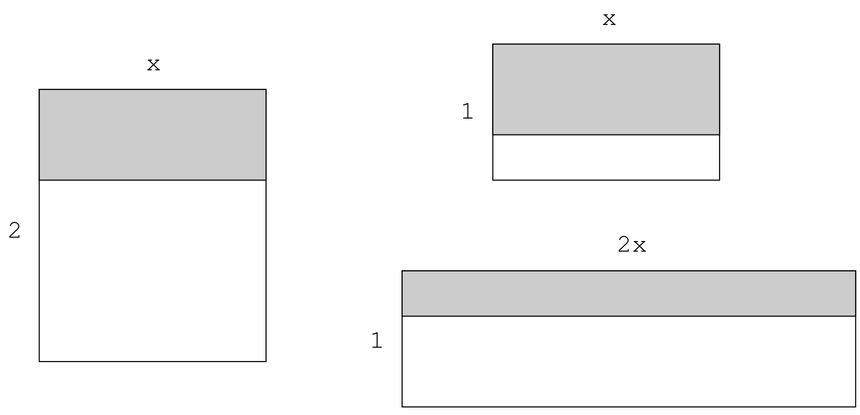
PROBLEM 6

Use the area model to show that $3(x + 4) = 3x + 3 \cdot 4$.

Let's explore equivalent expressions involving fractions.

EXAMPLE 2

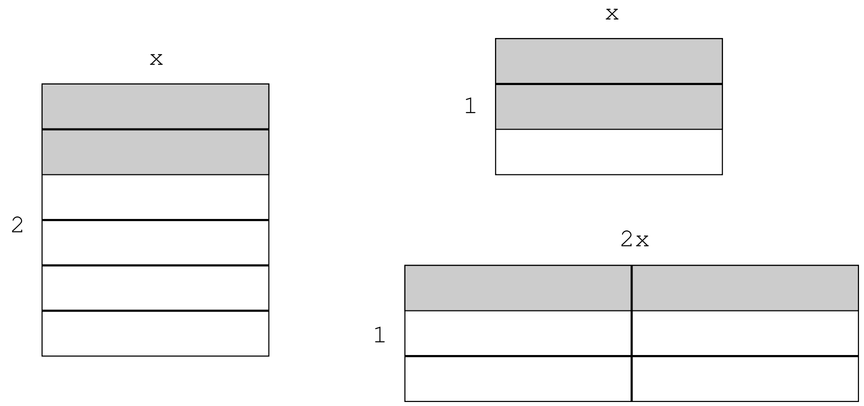
Use the three rectangles below to explain why the expressions $\frac{2x}{3}$, $\frac{2}{3}x$, and $\frac{1}{3}(2x)$ are equivalent.



SOLUTION To relate the rectangle to an algebraic expression, we need to imagine how the rectangle was drawn. On the left, we have shaded one third of the 2 by x rectangle which represents $\frac{2x}{3}$. On the top right, we have shaded two thirds of the 1 by x rectangle which represents

$\frac{2}{3}(1 \cdot x) = \frac{2}{3}x$ And on the bottom right, the shaded rectangle is $\frac{1}{3}$ by $2x$: $\frac{1}{3}(2x)$.

By cutting each shaded region into two equal pieces we can see the areas are all equal. So $\frac{2x}{3}$, $\frac{2}{3}x$, and $\frac{1}{3}(2x)$ are equivalent.



So we see that $\frac{2}{3}x = \frac{2x}{3} = \frac{1}{3}(2x)$.

To understand and manipulate fractions algebraically, we need to study the multiplicative inverse (reciprocal) of a number:

MULTIPLICATIVE INVERSE

For any non-zero number n,

$$n \cdot \frac{1}{n} = 1.$$

The expression $\frac{1}{n}$ is called the *multiplicative inverse* or *reciprocal* of n.

For example, $5 \cdot \frac{1}{5} = 1$. So $\frac{1}{5}$ is the multiplicative inverse of 5.

EXAMPLE 3

Using the properties of numbers, show that the following expressions are equivalent:

$$\frac{4y}{5} \quad \text{and} \quad \frac{4}{5}y.$$

SOLUTION It is easy to test for equivalence for a particular value of y. For example, if $y = 10$, then $\frac{4y}{5} = \frac{40}{5} = 8$, and $\frac{4}{5}y = \frac{4}{5} \cdot 10 = 4 \cdot 2 = 8$. But how do we know this is true for every value of y? There are infinitely many choices for y!

We will use the properties from this section. But first, what exactly is meant by $\frac{a}{b}$? We say that it is a divided by b. But what does this mean precisely? The number $\frac{a}{b}$ is the product $a \cdot \frac{1}{b} = \frac{1}{b} \cdot a$. Now we can use the properties of multiplication.

Now, return to the original question. Using the above notation for fractions and the associative property of multiplication,

$$\frac{4y}{5} = \frac{1}{5} \cdot (4 \cdot y) = (\frac{1}{5} \cdot 4) \cdot y = \frac{4}{5}y$$

Therefore, $\frac{4y}{5} = \frac{4}{5}y$ for every value of y. So $\frac{4y}{5}$ and $\frac{4}{5}y$ are equivalent.

PROBLEM 7

Show that the two expressions in Example 3are also equivalent to $4\left(\frac{y}{5}\right)$.

EXAMPLE 4

Show that $\frac{4x+6}{2}$ and $2x + 3$ are equivalent.

SOLUTION Using the distributive property and fraction notation, we can write

$$\frac{4x+6}{2} = \frac{1}{2} \cdot (4x+6) = \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 6 = 2x+3.$$

So $\frac{4x+6}{2} = 2x+3$ for all values of x. Fortunately it is not necessary to write all the steps when we work with algebraic expressions. In the future we can just show the distributive step:

$$\frac{4x+6}{2} = \frac{4x}{2} + \frac{6}{2} = 2x+3.$$

PROBLEM 8

Kate believes that $\frac{2x+4}{2} = x+2$. Jamie says, "No. It should be $\frac{2x+4}{2} = x+4$." Who is right? Draw an area model to support your answer.

Combining Like Terms

The algebraic expression $4y + 3x + 2x + 4 + 7$ contains five terms, two terms that involve the same variable x, two that involve only numbers and the term $4y$. In this case, $3x$ and $2x$ are called *like terms* because they each have same variable x. The two numbers 4 and 7 are also like terms. However, there are no terms that are "like" $4y$. It is often helpful to write an expression in a simpler form by combining like terms.

EXPLORATION 5

1. The number a is marked on the number line below. Locate $2a$, $3a$, and $5a$ on the number line. Explain why $2a + 3a = 5a$.

2. The numbers $2a$ and $3b$ are marked on the number line. Locate a,

b, and $2a+3b$ on the number line. In part 1, we "combined" $2a$ and $3a$ into $5a$. Do you think it is possible to write $2a+3b$ as one term? Explain.

Using the distributive property and combining like terms can often simply a complicated expression.

EXAMPLE 5

Find an expression that is equivalent to $3(x+4)-2(2x-3)+8x-1$ by:

1. Using the distributive property to remove the need for parentheses,
2. Combining like terms.

SOLUTION Use the distributive property and then combine like terms:

$$
\begin{aligned}
&3(x+4)-2(2x-3)+8x-1 \\
&\quad = 3x+12-4x+6+8x-1 \text{ [Dist. Prop.]} \\
&\quad = 3x-4x+8x+12+6-1 \text{ [Assoc. and Comm. Props.]} \\
&\quad = 7x+17 \text{ [Combining Like Terms]}
\end{aligned}
$$

Note that the resulting expression $7x+17$ is much simpler than the original expression $3(x+4)-2(2x-3)+8x-1$. However, in some cases a more complicated expression is preferred, because it is easier to interpret in the context of a problem.

PROBLEM 9

Match each expression on the left to an equivalent expression on the right. Explain.

1. $4(2x-3)+3x-2$	a. $-(x+2)$
2. $2(3-x)+3(x+5)-1$	b. $11x-14$
3. $5(2x-3)-2(3-2x)$	c. $8(x+3)$
4. $7(x+3)+3(x+3)-2x-6$	d. $x+20$
5. $4x-5(x+2)+8$	e. $14x-21$

Number Sense, Mental Math and Equivalent Expressions

Number sense is an understanding of numbers, their relationships, and how they are affected by operations. You have been developing your number sense since you began to count. Having a strong number sense gives you the ability to solve problems in many ways. Often schools host "number sense" competitions. In these competitions, students solve a long list of arithmetic problems in a short time without using paper and pencil. The winners usually have a bag of mathematical tricks that let them solve certain types of problems very quickly in their head. These mathematical tricks are just a set of useful equivalent expressions. Let's explore one of these tricks.

EXPLORATION 6

1. Compute each of the following.
   a. $54 \div 10$
   b. $54 \div 100$
   c. $54 \div 1000$
2. A fourth grader is learning how to divide and is given the problems above. How would you explain the "easy way" to find the answer?
3. Compute each of the following. What do you notice?

a. $100 \div 50$
b. $10 \div 50$
c. $27 \div 50$
d. $132 \div 50$

4. Use the properties in this chapter to show that $x \div 50 = 2x \div 100$.
5. Discuss with your neighbor how $x \div 50 = 2x \div 100$ can be used to compute $1234 \div 50$. Do you think this is easier than using long division?
6. Naveen is traveling in India to visit his grandparents. The exchange rate is 50.5 Indian Rupees to the dollar. He sees a shirt he wants to buy. The price is 332 rupees. Naveen wants to know if this is good price. Use the trick you explored here to estimate how much the shirt costs in dollars.

EXERCISES

1. A rectangle is twice as long as it is wide. Write expressions for the following in as many equivalent ways as you can
   - Perimeter
   - Area
2. Use the distributive property as the first step in solving the equations. Then solve the equation again without using the distributive property.

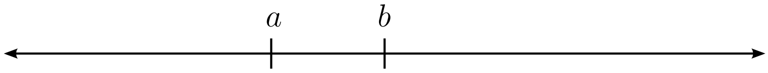
   a. $2(x - 3) = 4$
   b. $3(4 - y) = 6$
   c. $5(2z + 1) = 9$
   d. $2(5 - 3S) = 4$
3. Solve each of these equations by transforming them into equivalent, simpler equations using the properties of equality developed in the previous section. Name the properties used.
   a. $2A = 100$
   b. $b - 7 = 9$
   c. $6 + 5x = 21$

4. For each of the following, determine which of the three expressions are equivalent. Explain.
   a. $4a - 12$, $4(a-3)$, $4(a-8)$
   b. $5 \cdot \left(\frac{x}{3}\right)$, $\frac{5x}{15}$, $\frac{5x}{3}$
   c. $8 \cdot \left(\frac{x}{4}\right)$, $\frac{4x}{2}$, $2x$
   d. $\frac{6a+4}{2}$, $3a+2$, $6a+2$
   e. $-24b+12$, $-4(6b-3)$, $-4(6b+3)$
5. Match each expression on the left to an equivalent expression on the right.

1. $(2x+3)+x+7$	a. $x-4$
2. $(2x+3)-(x+7)$	b. $4x+44$
3. $5(2x+4)+3(8-2x)$	c. $3x-3$
4. $\frac{1}{2}(2x+6)+3(3x+4)$	d. $10x+15$
5. $\frac{2x+6}{2}+2(x-3)$	e. $3x+10$

6. Jack took a group of students on a river boat ride. He was charged the same amount for each student and \$8 for himself. Let x be the number of students. If each student was charged \$5 and the total bill was \$73, how many students were there? Set up an equation in terms of x and solve.
7. Jill took a group of students on a tour of a museum. She was charged the same amount for each student and \$6 for herself. Let c be the cost per student. If she took 22 students and the total bill was \$61, what is cost per student?
8. Jay and Sally are baking cakes using the same recipe. The recipe calls for x cups of flour per regular size cake. Jay triples the recipe, drops the bowl and spills half of the batter. He then bakes the remaining batter in a big pan. Sally mixes half the ingredients from the recipe and bakes a smaller cake. She does this 3 times. Which cook uses the most flour in the cakes they bake? Explain your answer.

9. **Investigation:**
Suppose a and b are numbers with $a < b$ as shown on the number line:



Notice the position of 0 and 1 on the number are both unknown.
 a. Write an expression for a number that is greater than b.
 b. Write an expression for a number greater than b that you can locate on the number line without knowing the length of the unit.
 c. Find a number less than a that you can locate precisely on the number line without knowing the length of the unit.
 d. Find a number between a and b that you can locate precisely on the number line without knowing the length of the unit or the distance between a and b.
 e. Find two more numbers between a and b that you can locate precisely.

10. Kiran is traveling in India. The exchange rate is 50.5 Rupees to the dollar. She wants to buy a painting for her house. The price is 42123 rupees. Use the method from Exploration 6 to estimate the price in dollars.

11. The Double-Half method is a number sense mathematics "trick". When multiplying two numbers, it is sometimes easier to first multiply the first number by 2 and divide the second by 2 and then find the product of the resulting numbers. For example, if you wanted to multiply 35×16, you could find:
 - $35 \times 2 = 70$
 - $16 \div 2 = 8$
 - $35 \times 16 = 70 \times 8$

 a. Explain why this method makes it easier to compute the product in your head.
 b. For what pairs of numbers do you think this method would work best?
 c. Express this "trick" using variables in an equation.

12. Explore on the internet for other number sense mathematics tricks. Find two tricks that make sense to you. For each trick:
    - Give an example of how it is used.
    - Explain for what kinds of numbers the trick seems the most useful.
    - Express the trick using variables in an equation.

13. **Ingenuity:**
    Andrew uses $1 \times 1 \times 1$ blocks to make larger cubes. For example, he uses 8 of the small blocks to make a $2 \times 2 \times 2$ cube. As he makes larger and larger cubes, some of the small blocks are hidden on the inside of the cube and some he can see on the surface of the cube.
    a. How many of the small blocks does Andrew need to make a $3 \times 3 \times 3$ cube? How many of the blocks are hidden in the inside and how many are on the surface?
    b. How many of the small blocks does Andrew need to make a $4 \times 4 \times 4$ cube? How many of the blocks are hidden in the inside and how many are on the surface?
    c. How many of the small blocks does Andrew need to make a $n \times n \times n$ cube? How many of the blocks are hidden in the inside and how many are on the surface?

14. **Ingenuity:**
    Joe invests P dollars in a fund. At the end of the year, his investment is worth one dollar more than twice his original investment. This pattern is repeated for the next 3 years. How much does he have in his investment at the end of the fourth year? Write an expression for how much the account will be worth at the end of 7 years.

SECTION 1.6 Equivalent Equations

In Section 1.4 we used the properties of equality to create equivalent equations and to find the solution of an equation. Recall that a *solution* is a value of the variable that makes both sides of the equation equal. In Section 1.5 we used the properties of arithmetic to manipulate expressions to find equivalent expressions.

The phrases equivalent expression and equivalent equation sound so similar it's easy to think they describe the exact same thing. But this is not true. Expressions are equivalent if they are equal for **all** possible values of the variable. Equations are equivalent if the set of **solutions** is the same.

EXPLORATION 1

1. Write two different expressions which are equivalent. Explain why they are equivalent.
2. Write two different equations which are equivalent. Explain why they are equivalent.

Now, we will combine the two ideas of equivalence to solve more complicated problems.

EXAMPLE 1

Solve the following equation, explaining each step and naming the property you use:

$$3x + 1 = x - 7.$$

SOLUTION To solve for x, we use the properties of equality and arithmetic to simplify things. The goal is to find an equivalent equation in which x is by itself on one side of the equation and the solution is on the other. Notice that the equation has x terms on both sides of the equal sign. So we begin by grouping the x terms on one side of the equation. Use the subtraction property to subtract x first:

$$\begin{aligned} 3x + 1 &= x - 7 \\ 3x + 1 - x &= x - 7 - x \\ 2x + 1 &= -7. \end{aligned}$$

Now we have an equation of the form studied in Section 1.4. We still have number terms on both sides of the equal sign. Use the subtraction property to subtract 1:

$$\begin{aligned} 2x + 1 &= -7 \\ 2x + 1 - 1 &= -7 - 1 \\ 2x &= -8. \end{aligned}$$

Finally, use the division property to divide by 2 to obtain the solution:

$$\begin{aligned} 2x &= -8 \\ \frac{2x}{2} &= \frac{-8}{2} \\ x &= -4. \end{aligned}$$

Note, we could have begun by grouping the number terms first and achieved the same result. Since we have 2 kinds of like terms (those with x and those with only numbers) and 2 sides of an equation, we can group the like terms separately on each side of the equation.

PROBLEM 1

Solve the following equations:

1. $2(2x + 2) + 3(x + 2) = x - 2.$
2. $3(3x + 5) - (x - 3) = 2(x - 3).$
3. $-2(x - 3) + 4x = 5x - 7.$

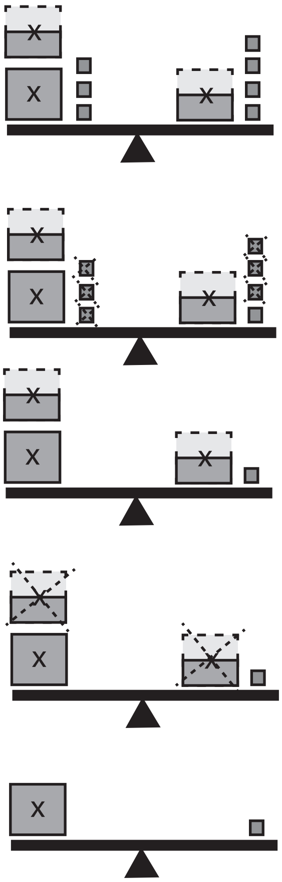
In Section 1.4 we used the balance model to visualize solving the equation $4s + 2 = 14$. Now, we use this model on a more complicated equation.

EXAMPLE 2

Use the balance model to solve the equation $\frac{3}{2}x + 3 = \frac{1}{2}x + 4$.

SOLUTION

Balance Model



Algebraic Method

$$\frac{3}{2}x + 3 = \frac{1}{2}x+4$$

Subtract 3 from both sides

$$\frac{3}{2}x + 3 - 3 = \frac{1}{2}x + 4 - 3$$

$$\frac{3}{2}x = \frac{1}{2}x + 1$$

Subtract $\frac{1}{2}x$ from both sides

$$\frac{3}{2}x - \frac{1}{2}x = \frac{1}{2}x - \frac{1}{2}x + 1$$

$$x = 1$$

Now we check our answer.

$$\begin{aligned}\frac{3}{2}\cdot 1+3&=\frac{1}{2}\cdot 1+4\\ \frac{3}{2}+3&=\frac{1}{2}+4\\ \frac{3}{2}+\frac{6}{2}&=\frac{1}{2}+\frac{8}{2}\\ \frac{9}{2}&=\frac{9}{2}\end{aligned}$$

EXPLORATION 2

1. Consider the equation $5(2x-3)-4x=6x-15$.
   a. Try to solve the equation. What happens?
   b. Substitute $x=1$ on both sides of the equation. What happens? Substitute $x=0$. Now what happens?
   c. Use the distributive property and combining like terms to show that $5(2x-3)-4x$ is equivalent to $6x-15$. What does this say about which values of x make the equation true?
2. Consider the equation $3(x-2)+2x=5x+4$.
   a. Try to solve the equation. What happens?
   b. Use the distributive property and combining like terms to show that $3(x-2)+2x$ is equivalent to $5x-6$.
   c. Is there any value of x so that $5x-6$ is the same as $5x+4$? Explain. What does this say about which values of x make the equation $3(x-2)+2x=5x+4$ true?

In part 1 of the exploration above, you found that an equation can have more than one solution. In this case, we talk about the set of solutions or *solution set*. Since $5(2x-3)-4x$ and $6x-15$ are equivalent, the solution set for $5(2x-3)-4x=6x-15$ is the set of all numbers. In part, we saw that no value of x makes the equation $3(x-2)+2x=5x+4$ true. So the equation has no solution and we say the solution set is empty.

EXAMPLE 3

Suppose n is a number so that if you triple it and subtract 4 you obtain the same number as if you decrease it by 3 and then double the result. What is n?

SOLUTION The problem describes two different procedures you can perform on the number n that give the same result. First we need to write each procedure as an expression involving n.

Procedure One: Triple it and subtract 4: $3n - 4$

Procedure Two: Decrease it by 3 and double the result: $2(n - 3)$

Second, we set the the two expressions equal to one another. Then we solve for n.

$$
\begin{aligned}
3n - 4 &= 2(n - 3) \\
3n - 4 &= 2n - 6 \\
3n - 2n - 4 &= 2n - 6 - 2n \\
n - 4 &= -6 \\
n - 4 + 4 &= -6 + 4 \\
n &= -2
\end{aligned}
$$

It is a good idea to check our answer. $3 \cdot (-2) - 4 = -10$ and $2(-2-3) = 2(-5) = -10$. So yes, it works.

PROBLEM 2

Montserrat has two job offers to deliver fliers around the neighborhood. The first offers to pay her \$50 per week plus $10\frac{1}{2}$ cents per flier. The second will pay only \$30 per week, but will give 20 cents per flier.

1. Set up an equation to find x, the number fliers she must deliver so that the two offers pay the same per week.
2. Solve for x. Which job would you take and why?

EXERCISES

1. In each of the following equations, the variable appears on both sides of the equal sign. Use the properties of equality to solve them for the unknown.
   a. $4x = 2x + 3$
   b. $16z + 7 = 3z - 4$
   c. $2A - 1 = 9 - 3A$
   d. $y + 4 = 4y - 8$
   e. $3n - 4 = \frac{3}{10}n - 1$
2. Use the distributive property as the first step in solving the following equations. Then solve the equation again without using the distributive property.
   a. $2(x - 3) = 4$
   b. $3(4 - y) = 6$
   c. $5(2z + 1) = 9$
   d. $2(5 - 3S) = 4$
3. Solve the following linear equations:
   a. $2(x - 3) = 4x + 7$
   b. $3x - 2(x - 4) = 5x - 13$
   c. $4(x + 3) = 2x + 2(x - 5)$
   d. $-3a + 7 = a - 5(3 - a)$
   e. $2a + 4.1 = 3(a - 1.3)$
   f. $a + \frac{1}{3} = 3a - \frac{1}{2}$
   g. $\frac{1}{3}(2c - 11) = c + 7$
   h. $9(x - \frac{2}{3}) = 5x$
   i. $3x - \frac{1}{2}(x - 1) = 2(x + 10)$
   j. $3x - 2(2x + 6) = 4(x + 1) - 5x - 16$
   k. $2.3(x + 7) = .4(2x - 3)$
   l. $17 = 3(x - 5) - 2(x + 1)$
   m. $11(x + 2) = 45$
   n. $9(x - 8) = 27$
   o. $\frac{2}{5}(2x + 3) = 7x - 1$
   p. $6x + 7 = 3(2x - 1)$
   q. $11 - 5(x + 6) = 3(x - 4)$
   r. $13x - [3(2x + 5)] = 6(x + 1)$

s. $\frac{2}{3} = 5(x - 1)$
t. $3(x + \frac{1}{2}) = 2(x - \frac{1}{2})$
u. $5x - \frac{1}{2}(2x - 4) = 2x + 2(x + 1)$
v. $4.7(3x - 2) = 2.3 + 4(x + .6)$
w. $-6(1 - 3x) = 4(2x - 1)$
x. $2(-3a + 2) = (a + 2) - 4(a - 2)$
y. $33 - 10(c + \frac{3}{5}) = \frac{2}{3}(6c - 9) + 1$
z. $13x - 17 - (-2x + 1) = 37 - 3(x - 11)$

4. Juan and Kate each plan to bake the same number of cookies. Juan tripled his planned number of cookies and then added 5 more. Kate decides to increase her planned number of cookies by 8 and then double this number of cookies. They then realized that they would still bake the same number of cookies. How many did they originally plan to bake?

5. Suppose n is a number so that if you take one third of it and add 12 you obtain the same number as if you double n and decrease it by 9. Write an equation and solve for n.

6. **Investigation:**
Recall two formulas for the circumference of a circle: $C = 2\pi r$ and $C = \pi d$.
   a. In your own words, write down the meaning of circumference.
   b. C represents the circumference. What do r, π, and d represent?
   c. In your house find 4 cylinder shaped objects. Good examples: cans of food or soda, rolls of toilet paper. For each object:
      i. Measure the diameter of the base (the circle on the bottom) in mm.
      ii. Measure the circumference. You can do this using a string. Or consider rolling the object on a piece of paper and marking when it completes a full circle.
      iii. Make a table of each pair of measurements.

7. Each cell phone text messaging plan charges a flat monthly fee for having text-messaging service plus an additional cost per text message.

Plan	Service Cost per Month	Cost per Msg
A	$15	$0.05
B	$10	$0.10
C	$25	unlimited texting

   a. Set up an equation to determine how many text messages you must make so that plan A and plan B cost the same.

   b. Set up an equation to determine how many text messages you must make so that plan A and plan C cost the same.

   c. Solve the two equations.
   d. Which plan would you choose and why?

8. Explain what it means for two equations to be equivalent. Give an example.
9. Explain what it means for two expressions to be equivalent. Give an example.

SECTION 1.7 FORMULAS AND LITERAL EQUATIONS

Mathematicians and scientists use formulas to describe relationships between different quantities. These formulas are often referred to as *literal equations*. Formulas usually involve two or more variables or letters and hence the term "literal" equation is used. If possible we choose the letter to remind us of the quantity we are describing. A familiar literal equation is the formula $P = 2L + 2W$ that relates the perimeter of a rectangle to its length and width.

EXPLORATION 1

What formulas do you know? Which have you worked with in other grades? Which formulas have you used in other classes besides math? Remember to define what each variable in a formula means.

When you work with a rectangle, you are often asked to find its area, $A = L \cdot W$ where the variable A represents the area of a rectangle, L, the length of the rectangle and W its width. The perimeter P of a rectangle with length L and width W is given by the formula $P = 2L + 2W$ or $P = 2(L + W)$. In both examples, A and P are expressed in terms of L and W. The formulas written in this form are particularly useful when we are looking for the area and perimeter. However, suppose you must find the length or the width. Because you are working with equations, you can use properties to create equivalent equations and rearrange the variables to solve for L or W.

EXPLORATION 2

Use $P = 2L + 2W$ and find equations equivalent to it. Find one that has W by itself on one side of the equal sign and the one that has L by itself on one side of the equal sign. Explain each of your steps.

PROBLEM 1

The perimeter of rectangle A is 45 meters and its width is 18 meters. Rectangle B has perimeter 78 meters and width 10 meters. Determine the lengths of each of the two rectangles. Try using the information from your exploration.

EXAMPLE 1

The area of a rectangle is 54 square meters. Its length is 3 meters. Determine its width.

SOLUTION One way to approach this problem is to use the formula for the area of a rectangle, $A = LW$, and substitute the known values, $A = 54$ and $L = 3$. The equation then becomes $54 = 3W$. What times 3 equals 54? From the definition of division, this is another way of asking what is 54 divided by 3. The width $W = 18$. An alternate way to solve the equation $54 = 3W$ is to multiply both sides by $\frac{1}{3}$. This means $\frac{1}{3} \cdot 54 = \frac{1}{3} \cdot 3W$. The equivalent equation is $\frac{54}{3} = W$ or $18 = W$.

Another approach to solving literal equations is to work with the given formula, $A = LW$ and first solve for the specified quantity in the problem. The problem is asking you to solve for the width, W. Now, because $A = LW$ is an equation, you can solve for W by dividing both sides by L. The equivalent equation is $W = \frac{A}{L}$. Now if we substitute the values for A and L then $W = \frac{54}{3}$ or $W = 18$. Using either approach, your answer is still $W = 18$. When might there be an advantage of one approach over the other?

PROBLEM 2

The area of Cody's triangle is 20 square inches. The length of its base is 8 inches. The area of Althea's triangle is 15 square inches. The length of its base is 5 inches. What is the height of each of these triangles?

EXPLORATION 3

A box has a square base of length x. The height of the box is 3 times the length of the base. Write an expression for each of the following:

1. Volume of the box.
2. Surface area of the box.

Try sketching the box.

Now let's explore the data you collected in Exercise 6 from Section 1.6.

EXPLORATION 4

Recall the formula for the *circumference* C of a circle in terms of its *radius* r, $C = 2\pi r$ and in terms of its *diameter* d, $C = \pi d$.

1. What is the relationship between d and r? Write an equation to represent this relationship.
2. Solve the equation $C = \pi d$ for π in terms of C and d.
3. For Exercise 6 from Section 1.6 you measured the circumference and diameter of the base of some cylinders you found in your house. Discuss with your neighbors why you chose the objects you did, how you measured the circumference and any difficulties that arose.
4. As a group record the measurements of the objects from around your house in a table. Using a calculator compute $\frac{C}{d}$ for each object. What should this equal?

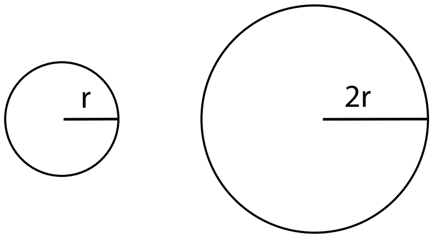
Name of Object	d (mm)	C (mm)	$\frac{C}{d}$

5. Use the results from your table to estimate the value of π.
6. Press the π key on your calculator. How does this number compare to your estimate above?

The next example explores the relationship between different circles.

EXAMPLE 2

If we examine a second circle with a radius twice the radius of the first, what is the relationship between the circumference of the second circle to the circumference of the first?



SOLUTION Let $r =$ the radius of the first circle. Then the radius of the second circle $= 2r$. The circumference of the first circle is $C_1 = 2\pi r$. The circumference of the second circle, $C_2 = 2\pi(2r) = 4\pi r = 2(2\pi r) = 2C_1$. Therefore, we conclude that if we double the radius of a circle, then the circumference doubles.

PROBLEM 3

What happens to the circumference of a circle if the radius of a circle is tripled? What happens to the circumference of a circle if the radius of a circle is quadrupled? Do you see a pattern?

PROBLEM 4

Consider a box with length L, width W and height h. The formula for the volume is $V = LWh$.

1. If the volume is 20 cubic centimeters, what is the formula for the width W?
2. If the surface area is 60 square centimeters what is the formula for

the height H?

In many cases, the most difficult part of using a literal formula is figuring out what each variable represents. This can especially be true in geometry where different names can be used to describe the same figure. For example, in this book we have referred to the length and width of a rectangle. So we write the formula for the area as $A = LW$. However, you could also compute the area as $A = bh$ where b is the base and h is the height. These formulas are really the same but the names and letters used to describe the rectangle are different. The table at the end of this section shows more formulas from geometry. Use these formulas for the next problem.

PROBLEM 5

For each of the following situations:

- determine the formula to use
- specify what each variable means
- compute the area or volume

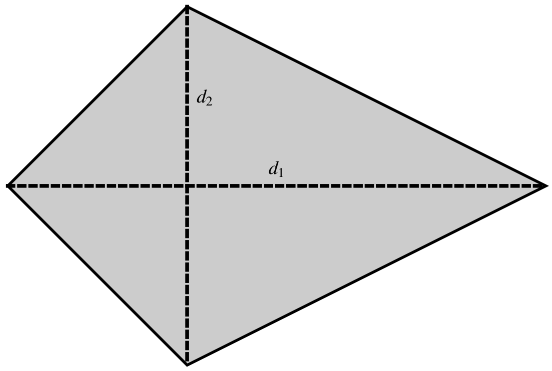
1. A model of a square pyramid has base edges of 10 inches and height of 12 inches. The slant height is 13 inches. What is its lateral area?
2. What is the volume of a sphere with a 6cm radius?
3. What is the volume of a cone with a height of 15 inches and a radius of 4 inches for its circular base?
4. What is the volume of a cylinder with a height of 15 inches and a radius of 4 inches for its circular base?

Formulas from Geometry

CIRCUMFERENCE		
Circle	$C = 2\pi r$	$C = \pi d$
AREA		
Triangle		$A = \frac{1}{2}bh$
Rectangle or Parallelogram		$A = bh$
Trapezoid		$A = \frac{1}{2}(b_1 + b_2)h$
Circle		$A = \pi r^2$
SURFACE AREA		
	Lateral	Total
Prism	$S = Ph$	$S = Ph + 2B$
Pyramid	$S = \frac{1}{2}Pl$	$S = \frac{1}{2}Pl + B$
Cylinder	$S = 2\pi rh$	$S = 2\pi rh + 2\pi r^2$
VOLUME		
Prism or cylinder		$V = Bh$
Pyramid or cone		$V = \frac{1}{3}Bh$
Sphere		$V = \frac{4}{3}\pi r^3$

EXERCISES

1. The area of a triangle is one half of its base times its height: $A = \frac{1}{2}bh$.
   a. Solve for the base b in terms of the area A and height h.
   b. Juan wants to make a triangle whose area is 1 cm^2 and height is 1 cm, how long should the base be?
2. The area of a rectangle is its length times its width: $A = lw$.
   a. Solve for the length l in terms of the area A and width w.
   b. Julie wants to make a rectangular garden. Since the garden is along the side of her house, she knows the width will be 5 feet. How long should she make the garden if she wants to have an area of 40 square feet?
3. The area of a kite is one half of the product of its diagonals: $A = \frac{1}{2}d_1d_2$. See the figure below.



   a. Solve for the diagonal d_2 in terms of the area A and the other diagonal d_1.
   b. Nate wants to build a kite whose area is 1 square meter. He already has one stick (diagonal) that is 80 cm long. How long should the other stick be?
4. The volume of a rectangular prism (in other words a box) is the area of its base B times its height h: $V = Bh$. Of course the base is a rectangle, so the area of the base is the length times the width $B = lw$
   a. Solve for the height h in terms of the volume, and the area of the base.
   b. Hiro wants to build a swimming pool. Its base will a rectangle

with length $l = 15$ meters, width $w = 4$ meters. She wants the volume equal to be 120 cubic meters. What should the height of the pool be?

c. Jian wants to have a pool that holds twice as much water Hiro's, but with the same base. What should the height of Jian's pool be?

d. Both Hiro and Jian need to paint the inside of the pool, that is, the 4 inside walls and the bottom. How many times more paint does Jian need than Hiro?

5. The average of two numbers is their sum divided by two. $A = \frac{(m+n)}{2}$. Copy and fill in the blanks in the following table.

m	n	A
1	4	
3		9
	6	-1
x		10
	y	A
x	y	

6. The temperature in Fahrenheit is 1.8 times the temperature in Celsius plus 32: $F = 1.8C + 32$.

a. Find F if $C = 0$ and when $C = 100$. (Note: 0°is the freezing temperature for water, and 100°is the boiling point for water).

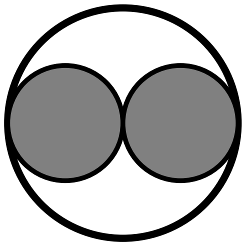
b. Solve for the temperature in Celsius C in terms of the temperature in Fahrenheit F.

c. Find the high temperature in your town today in Fahrenheit and in Celsius.

d. Ricardo lives in Chile where they use Celsius for temperature. Ricardo calls his friend Pete in Texas. When Pete asks Ricardo about the weather, Ricardo says, "It's crazy! Today's high temperature was twice as high as the low temperature." Ricardo is thinking in degrees Celsius; do you think his statement would still be true if he thought about the temperatures in Fahrenheit? Explain.

7. One idea in formulas or literal equations is to use letters for the variables that remind you what each variable represents. The following are formulas from mathematics or science you may not be familiar with. For each one, after looking at the word description and the formula, label what each variable means.
   a. The density of an object is its mass divided by its volume: $D = \frac{m}{V}$.
   b. The electric current through a wire is the amount of electric charge that goes through the wire divided by the amount of time over which this charge is measured. $I = \frac{Q}{t}$.
   c. The surface area of a sphere is 4 times π times its radius squared: $S = 4\pi r^2$.
   d. The area of a trapezoid is one half its height times the sum of its 2 bases. $A = \frac{1}{2}h(b_1 + b_2)$
8. Rewrite each formula as indicated.
   a. Solve $D = \frac{m}{V}$ for V.
   b. Solve $I = \frac{Q}{t}$ for Q.
   c. Solve $A = \frac{1}{2}h(b_1 + b_2)$ for h.
9. Come up with 3 literal formulas that describe quantities that you deal with in your life.
10. Ms. Foss asked her students to come up with the literal formula for the area of a triangle. Most of her students gave the answer of $\frac{1}{2}bh$, where b represents the base of the triangle and h represents the height of the triangle. However, a few students gave the answer of $\frac{1}{2}xy$, where x represents the base of the triangle and y represents the height, and one student gave the answer of $\frac{1}{2}bh$, where b represents the height of the triangle and h represents the base of the triangle. Which students are right?
11. Alfredo increased the sides of a square by 3 cm and the area increased by 189 square cm. What is the area of the bigger square?
12. A cone with an open circular base at the top is filled with water to $\frac{1}{2}$ of its full height. What fraction of its full volume is filled with water?

13. Two pizzas fit side by side on a large circular tray. See the figure below. If each pizza has a circumference of 36 inches, write an equation that could be used to find d, the diameter of the tray.



14. **Investigation:**
We have seen that doubling the radius of a circle also doubles the circumference.
    a. What happens to the perimeter of a rectangle when we double the length and width? When we triple the length and width?

    b. What happens to the perimeter of a triangle when we double the length of its sides? When we triple the length of its sides?

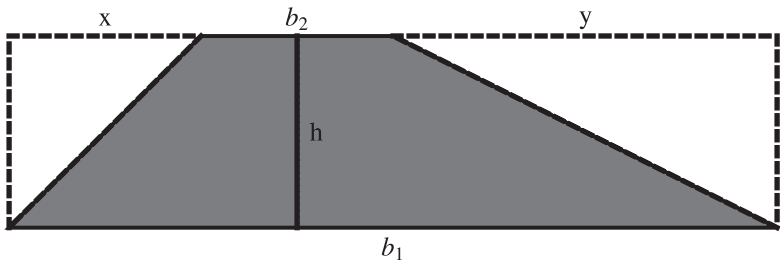
    c. What do you think the effect on the perimeter will be if we increase the lengths of the sides by a factor of r?

15. **Investigation:**
In this investigation we will investigate the relationship between increasing the radius or sides of figures and the change in its area.
    a. What happens to the area of a circle when we double its radius? When we triple its radius?
    b. What happens to the area of a rectangle when we double the length of its sides? When we triple the lengths of its sides?

    c. What happens to the area of a triangle when we double the length of its sides? When we triple the lengths of its sides?

    d. What do you think the effect on the area will be if we increase the lengths of the sides of a polygon by a factor of r?

16. **Investigation:**
The number π is very important in mathematics. Mathematicians have been studying it for four thousand years. Use the internet to research the following questions about the history of π.
    a. When was π first discovered? What was the estimate of its value?
    b. Archimedes is the first mathematician to write about a method to calculate π. What was his estimate?
    c. π is an irrational number. What does that mean?
    d. What is the best estimate of π known today? Why isn't it included in the book?
17. A cylindrical flower vase is filled with 200 cm^3 of water. If the height of the vase is 10 cm and the radius of the base is 5, what percentage of the total volume of the vase is filled with water?
18. **Ingenuity:**
The Rope Around the Earth Puzzle: Imagine a rope tied around the Earth's equator. If we increased the length of the rope by exactly one meter, a gap between the rope and the Earth's surface will form all the way around. How large is the gap that is formed?
19. **Ingenuity:**
Show how to derive the area of a trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$. The figure with the shaded trapezoid may be helpful.



20. **Investigation:**
A company produces hollow 3-D shapes to use in schools. One packet of shapes has three shapes all with the same circular base with radius of 5 cm. The three shapes are
   - a cylinder with a height of 5 cm
   - a cone with a height of 5 cm
   - hemisphere (half of a sphere)

Imagine filling each of the objects with water.
   a. Which of the objects has the largest volume? Which of the objects has the smallest volume?
   b. If we fill the cone with water and then use this water to try to fill the cylinder, what fraction of the cylinder will be full?
   c. If we fill the cone with water again, but now use this water to fill the hemisphere, what fraction of the hemisphere will be full?

SECTION 1.8 CHAPTER REVIEW

Key Terms

algebraic expression
algebraic method
circumference
constants
equivalent equations
equivalent expressions
integers
irrational numbers
multiplicative inverse
natural numbers
perimeter
radius
rational numbers
reciprocal
rectangle
set notation
sets
solution set
subset
upper and lower bounds
variable
whole numbers

Properties and Theorems

Additive Inverse Property:
$n + (-n) = 0$

Multiplicative Inverse :
$n \cdot \frac{1}{n} = 1$

Addition Property of Equality:
If $a = b$, $a + c = b + c$

Subtraction Property of Equality:
If $a = b$, $a - c = b - c$

Multiplication Property of Equality:
If $a = b$, $ac = bc$

Division Property of Equality:
If $a = b$ $c \neq 0$, $\frac{a}{c} = \frac{b}{c}$

Associative Property of Addition:
$(x + y) + z = x + (y + z)$

Commutative Property of Addition:
$x + y = y + x$

Associative Property of Multiplication:
$(xy)z = x(yz)$

Commutative Prop. of Multiplicatio
$xy = yx$

Distributive Property:
$x(y + z) = xy + yz$

Double Opposite Theorem
$-(-n) = n$

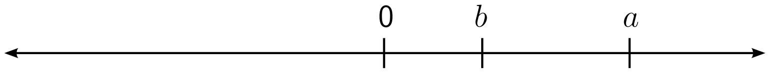
Formulas

Distance = Rate · Time   Circumference = $2\pi r$

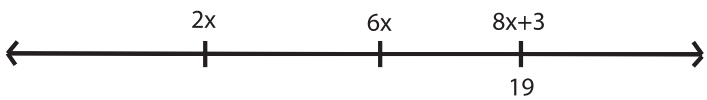
Practice Problems

1. Compute:
   a. $34 + (-6)$
   b. $-21 + (-19)$
   c. $\frac{2}{3} + \frac{1}{5}$
   d. $5\frac{1}{4} - 2\frac{2}{3}$
   e. $-11 \cdot (-6)$
   f. $3\frac{2}{3} \cdot 4\frac{1}{6}$
   g. $-7 \div \frac{3}{5}$
   h. $-2\frac{1}{2} \div 1\frac{1}{3}$
2. Evaluate:
   a. $-6 + 3(4)$
   b. $10 - (-3)(1)$
   c. $3 - (-4)(-6)$
   d. $-13 + (-6 - 3)$
3. Compute the distance between each of the following pairs of numbers:
   a. -3 and 5
   b. 2 and 12
   c. $-\frac{1}{2}$ and 7
   d. 5 and x
4. Plot a point that represents each expression and label it:

$$\frac{1}{2}(a+b), \quad \frac{1}{2}a + \frac{1}{2}b.$$



5. Consider the number line below.



   a. Given the location of the two expressions find the locations of x and 0.
   b. Use the number line above to solve $8x + 3 = 19$.
6. Write the following as algebraic expressions:
   a. x is 3 units more than y.
   b. x is 4 times greater than y.
   c. x is 6 units less than y.

d. x is three-fourths of y.
e. x is y units more than z.

7. Max is M years old now. How old will he be in 10 years? Write an expression that models this.
8. A triangle has a base of length x cm.
   a. When the height of the triangle is 20 cm, write an expression for the area of the triangle.
   b. When the height is 4 units longer than the base, write an expression for the area of the triangle.
9. The high temperature in San Antonio, Texas on July 9 was 107 °F and the low was 88 °F. Write the temperatures for July 9 in set notation and graph this set on the number line.
10. Solve on the number line.
    a. $3x = 24$
    b. $8x = 32$
    c. $2x = 12$
    d. $12x = 30$
11. Write an equation that models the problem and solve.
    a. A rectangular field has a length that is twice as long as its width. If the perimeter of the field is 326 m, what is the length of the field?
    b. The sum of three consecutive integers is 66. What are the integers?
    c. Eric has $30. He promised to give Xavier $10 and give one-fourth of the left over money to Anahi. How much money does Anahi get?
12. Solve each equation. Tell which property of equality you use in each step.
    a. $y + 4 = 6y - 8$
    b. $3x - 8 = x + 6$
    c. $3y - 2 = -y + 3$
    d. $4y + 1 = 5 - 3y$
13. Solve each equation. Tell which property of equality you use in each step.
    a. $18(x - \frac{2}{3}) = 5x$
    b. $13x - [3\,(4x - 2)] = 6\,(x - 1)$
    c. $33 - 10(c + \frac{2}{5}) = \frac{2}{3}(9c - 6)$

14. Suppose n is a number such that if you triple it and subtract 1, you get the same as if you double the sum of n and 2. What is n?

15. The volume of a rectangular pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid.
    a. Solve the equation for the height.
    b. If the volume of a rectangular pyramid is 24 cm^3 and the area of the base is 3 cm^2, what is the height?

16. The density of an object is its mass divided by its volume.
    a. Write an expression for the density of an object.
    b. If the object's density is 100 g per cubic cm, and its weight is 50 g, what is its volume?

17. A swimming pool has the shape of a rectangular prism. The volume of a rectangular prism is $V = Bh$ where B is the area of the base.
    a. Suppose the length of the pool is 25 m and the width is 10 m. If we want the volume of the water in the pool to be 750 m^3, how deep should the pool be?
    b. Suppose we want to double the volume of the pool but leave the base alone. How tall should the walls of the pool be now?

EXPLORING FUNCTIONS

2

SECTION 2.1 FUNCTIONS

The idea of a function is central to our study of algebra. We use functions to build mathematical models for real world problems, and these math models give us a way to talk carefully and precisely about many different kinds of problems. A *function* is a special kind of relationship between two variables. For example, $T =$ the number of raffle tickets purchased and $C =$ the total cost of the tickets. One of the variables is called the *independent variable*, or input, while the other variable that depends on the input is called the *dependent variable*, or output. Independent here means that you are free to choose the value of the input. Dependent means that once you have picked a value for the independent variable, the value for the output is determined, depending on the choice of input.

EXPLORATION 1

1. For each of the following examples, decide which variable is the dependent variable. Explain your choice.
   a. If you buy $4 raffle tickets, the number of tickets T purchased and the total cost of the tickets C.
   b. If your car gets 35 miles per gallon, the number gallons G used

and the number of miles M driven.

2. Brainstorm and discuss settings and other relationships between two quantities that could be labeled with variables. Which would be the dependent variable?

EXAMPLE 1

Mary drives a car for t hours at a constant rate of 50 miles per hour. She drives d miles during this time. The time t and the distance d are related. Find an expression for d in terms of t.

SOLUTION

One way to approach this problem is to make a chart with two columns. Use the first column for the independent variable or inputs. In our case, we let t be the independent variable representing the time traveled in hours. The second column is for the dependent variable or the outputs. In our case, we let d be the dependent variable representing the distance traveled, measured in miles. Complete the following chart.

Independent Time $= t$ hours	Dependent Distance $= d$ miles
1	50
2	
3	
4	
t	

The bottom line in the chart above represents an important relationship between your input and output: $d = 50t$. In words, you would say that: "for each value of t, if you multiply t by 50 you will find the distance d traveled."

It is important to emphasize that the distance d depends on t. We have a special functional notation that allows us to write functions using an equation. We do this by giving our function a name. You may use any letter such as f to name a function, or h or p. In our case, we will use d for distance. Using function notation, the distance traveled is $d(t) = 50t$. This is read as "d of t equals $50t$." We are now using the letter d to denote the distance function, and $d(t)$ is the distance traveled in t hours. The notation $d(t)$ is used to tell us the value of the function d with input value t:

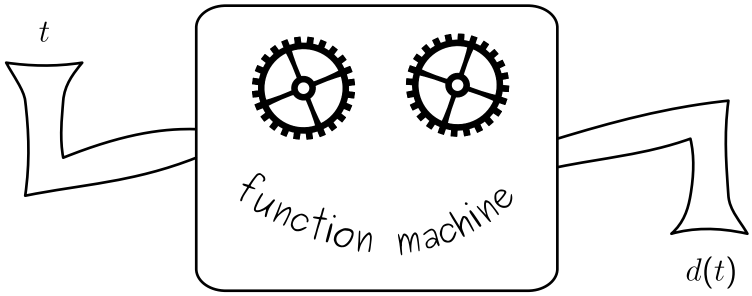
$$
\begin{aligned}
d(1) &= 50 \cdot 1 = \ \ 50 \\
d(2) &= 50 \cdot 2 = 100 \\
d(3) &= 50 \cdot 3 = 150 \\
d(5) &= 50 \cdot 4 = 200 \\
d(t) &= 50 \cdot t \ = \ 50t
\end{aligned}
$$

PROBLEM 1

What is the value of $d(8)$ and what does it represent in words?

This is a subtle point. On the one hand, d is the dependent variable in the equation $d = 50t$. On the other hand, we are using d as the name of the distance function, and since d depends on the input variable t, we write $d(t)$ to be the value of the dependent variable d with input value t. You might notice that we have used the letter d to represent two different things: first we used it to represent the number of miles driven and then we used it to name the function $d(t)$. Normally, you do not use the same letter to represent two things in one math problem. However, here the two different representations are actually the same, since the function d computes the number of miles driven.

One way to picture how a function works is to think of it as a machine. The function d takes inputs t and produces outputs $d(t)$.



Domain and Range of a Function

For a function it is important it important to think about what inputs should be allowed and what outputs are possible.

EXPLORATION 2

For each of the following situations, think about what the inputs and outputs represent. For each case, do negative inputs make sense? Do the inputs have to be whole numbers? How about the outputs? Describe all the possible inputs and outputs.

1. Mary rides in a car for t at a constant rate of 50 miles per hour. The distance traveled is $d(t) = 50t$.
2. Raffle tickets cost \$4 per ticket. The total cost for T tickets is $C(T) = 4T$.
3. The car get 35 miles per gallon. The number of miles you can drive using G gallons is $M(G) = 35G$.

The set of all permissible inputs of a function is called the *domain* of the function, and the set of possible outputs is called the *range* of the function. An important question that always arises when working with functions is "How do you determine the set of all possible inputs or domain of the function?" In Exploration 2, we saw that the domain depends on the context of the problem. Often, mathematicians find it useful to consider functions without referring to a real world situation. In this case, we will take the domain to be the set of all possible inputs for which the formula makes sense.

EXAMPLE 2

For each of the functions determine the its domain:

1. $P(x) = 4x$, the perimeter of a square whose sides have length x.
2. $f(x) = 4x$.
3. $h(x) = \frac{1}{x}$.

SOLUTION The domain is the set of all values x for which the function makes sense in the given context.

1. In the context we are told that x is the length of the side of the square. So the domain consists of all positive numbers $x > 0$.
2. No context is given. We can multiply any number x by 4, so the domain is all numbers.
3. Again no context is given, however, we can not divide 1 by 0. So the function does not make sense when $x = 0$. The domain consists of all numbers except 0.

Note that the functions P and f have the same formula. However, the context changes what we include in the domain.

Formally we can state the definition of a function as follows:

FUNCTION
A *function* is a rule that assigns to each element of a given set of inputs one and only one output value. The set of all permissible inputs is the *domain*. The set of all the corresponding outputs is the *range*.

Finding the Rule for a Function

The rule for a function is usually given as a formula. In Example 1, the rule was $d(t) = 50t$. Let's now practice with some more problems where we try to find the rule for different functions.

EXAMPLE 3

Suppose that tickets to the movies sell for \$3 each. How much does it cost to purchase 1 ticket? 2 tickets? 5 tickets? x tickets?

SOLUTION Let's define and use a function to solve this problem.

1. *What is the input or independent variable?*
   It is the number of tickets you want to buy. You may find that x is often used as the independent variable, so let's choose x to represent the number of tickets to buy. However, we could have chosen any other letter.
   What is the output or dependent variable?
   It is the cost of the tickets we buy, which depends on the number x of tickets bought. Let C denote the cost in dollars in this problem. We could have chosen any letter, as long as it's different from the one used for the number of tickets.
2. *Write the relationship between the independent and dependent variables in words and then write this relationship as an equation using functional notation.*
   In words, the total cost of the tickets in dollars is 3 times the number of tickets, as each costs \$3. We can represent the relationship as an equation, $C = 3x$, or using functional notation, $C(x) = 3x$.
3. *Make a table for these inputs and outputs and label each column.*
   As only whole tickets can be purchased, x is an integer. Since we are buying tickets, not selling them, x has to be non-negative. If we buy no tickets, $x = 0$ is allowed and $C(0) = 0$.

Now complete the table below with the values of $C(x)$.

Number of Tickets	Cost $=$
1	$C(1) =$
2	$C(2) =$
3	$C(3) =$
4	$C(4) =$
5	$C(5) =$
x	$C(x) =$

EXPLORATION 3

Discovering the rule for a function is an important step in understanding a function. Consider the function with inputs and outputs as described below. Do you see a pattern? Complete the table. Describe the pattern in words. Write down the formula of the function.

Input $= x$	Output $= f(x)$
1	11
2	12
3	13
4	
5	
x	

Diagrams of Functions

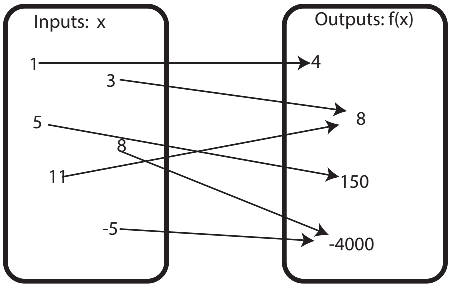
A function is not always written as a formula. You just need a rule to assign an output value to each permissible input value. This rule can be described in words, or shown in a diagram or table. However, the rule must assign one and only one output to each input. In terms of our function machine, if we put a particular input in, we want to know for sure what comes out.

EXPLORATION 4

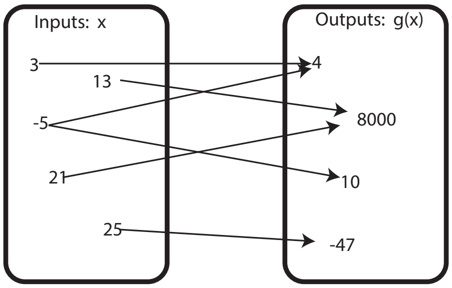
1. For each of the diagrams below, the arrows represent a rule for assigning output values to the given inputs x. For each case,

determine if the rule describes a function.

A.



B.



2. Make up your own rule that is a function.
3. Make up your own rule that is not a function.

Functions Described in Words

In Example 3 you determined the formula for the cost of buying x tickets. In some instances determining the word description of a function is the most important step when dealing with a function.

EXAMPLE 4

Isabel uses the function $C(x) = 8x$ to describe the cost (in dollars) of buying x tickets to a 3D movie. Describe in words what this function represents.

SOLUTION $C(x) =$ the cost (in dollars) of buying x tickets to a 3D movie $=$ \$8 times the number of tickets, x. So the tickets cost 8\$ per ticket.

In the next exploration, you get to make up your own story to match a formula!

EXPLORATION 5

Suppose $f(x) = 3x + 2$.

1. Imagine you are a teacher and you want to make word problem so that your students will come up with the formula for $f(x)$. Make up a story that matches the formula. Describe in words what $f(x)$ represents. What does the 3 represent in your story? What about the 2?
2. Add to your story. Now you want the students to solve the equation $17 = 3x + 2$.
3. Share your story with your neighbors. How are their stories similar? How are they different?

Ordered Pairs

We can also describe a function f using the set of *ordered pairs*, $(x, f(x))$, for all x in the domain. Note that the pair is "ordered", because we always write the input first and then the output.

EXAMPLE 5

Suppose that the rule for a function f is given by the equation $f(x) = 5x$. Find $f(2)$, $f(3)$, $f(5)$, $f(10)$, $f(t)$, and $f(3t)$. Write the input and output as ordered pairs.

SOLUTION

Input $= x$	Output $= f(x)$	Ordered Pair
2	10	$(2, 10)$
3	15	$(3, 15)$
5	25	$(5, 25)$
10	50	$(10, 50)$
t	$5t$	$(t, 5t)$
$3t$	$5(3t) = 15t$	$(3t, 15t)$

Finding all of these pairs would be the same process as finding the function, even if it took a long time to write. However, using variables, we can write all of these pairs easily. For Example 4, we could have written f as all of the ordered pairs of the form $(x, 5x)$. See if you can represent the functions in Examples 1–3 using ordered pair representation.

In Example 5, you began with the rule $f(x) = 5x$ and created the tables of ordered pairs. In Exploration 3, you went in the other direction. Starting with a list of 3 ordered pairs, you recognized a pattern and found the rule for that pattern. Then this rule can be used to find all the ordered pairs which follow the same pattern.

When the domain of the function is a small set of values, we can completely represent a function with a list of ordered pairs. In this case, it not always possible or necessary to find a nice formula for the rule of the function. The rule is simply, "assign the second coordinate to first coordinate." However, not any list of ordered pairs represents a function. Remember a function must assign one and only one output to each input. So a list of ordered pairs can represent a function as long as the first coordinates have no repeats. The list $\{(1, 1), (1, 2), (3, 3), (4, 4)\}$ is not a function, because the number 1 is repeated as the first coordinate and is associated with two different outputs. Different inputs can result in the same output, however. So the list $\{(1, 1), (2, 1), (3, 3), (4, 4)\}$ is a function.

PROBLEM 2

In Exploration 4, diagrams with arrows were used to represent the rule for a function. For each list below, represent the rule with a diagram. Label the domain and the range. Finally, decide if the list represents a function.

1. $\{(1,3),(2,5),(3,4),(7,8)\}$
2. $\{(-2,-4),(-3,5),(-5,10),(3,-2),(4,11)\}$
3. $\{(-2,4),(-1,5),(0,2),(-2,-4),(-2,5)\}$
4. $\left\{\left(\frac{1}{2},3\right),\left(\frac{3}{2},3\right),\left(\frac{5}{2},3\right),\left(\frac{7}{2},3\right)\right\}$

EXERCISES

1. Suppose f is the function defined by $f(x)=2x+5$ for all x. Compute each of the following:

   a. $f(0)$ d. $f(-3)$

   b. $f(1)$ e. $f(b)$

   c. $f(2)$ f. $f(3b)$

2. The garden center is selling native grass seeds for $8 per pound. Let the cost of x pounds of seed be represented by C.
   a. Write an equation relating x and C.
   b. Write the relationship in functional notation.
   c. Determine an appropriate domain for this function.
3. For the function defined by the equation $z(x)=4x+1$, make up 5 different inputs, and for each input find the corresponding output.

4. Suppose that $f(0)=10$ and when the input increases by 1 the output increases by 5. Make up 5 different inputs, and for each input find the corresponding output. Find a general formula that describes the function.
5. Lola wants to make a garden that is 2 feet longer than it is wide and enclose it with a fence. If x is the width of the garden in feet,

let $f(x)$ be the length of fence in feet that she needs to enclose the garden.

a. Find a formula for the function f.
b. What is the meaning of $f(4)$?
c. What is the value of $f(4)$?
d. What dimensions of her garden would use 52 feet of fencing?
e. What dimensions of her garden would use 42 feet of fencing?
f. For what values of x does $f(x)$ make sense? In other words, what is the domain of f?

6. A square field has sides of length x meters. A walkway is built around the square that is 1 meter wide, so that the field is enclosed by a larger square. Let $A(x)$ denote the area of the walkway. *Hint: This is similar to the pool problem in Exploration 1 from section 1.5.*
a. If $x = 4$, find the area of the walkway.
b. If $x = 10$, find the area of the walkway.
c. If $x = 15$, find the area of the walkway.
d. Write a formula for $A(x)$ in terms of x.

7. Susan goes on a bike ride traveling at a constant rate of 12 miles per hour. She rides for 2.5 hours.
   - Make a table for how far she has ridden at intervals of 15 minutes.
   - Define variables for the time she has been riding and the distance she has traveled. Write the relationship between distance and time using functional notation.
   - Find the domain and range of this function.

8. Paul needs to drive to a town that is 100 miles away.
   a. If he travels at 10 miles per hour, how long will it take? At 20 miles per hour? At 50 miles per hour? Note: in this case, the independent variable is the rate, not the time.
   b. If he travels at x miles per hour, the trip takes t hours. Express t in terms of x.
9. Sam is 5 years older than Paul.
   a. If Paul is 4 years old, how old is Sam?
   b. If Paul is 10 years old, how old is Sam?
   c. If Paul is x years old, how old is Sam?
   d. What is the independent variable?
   e. What is the dependent variable?
10. Suppose h is the function such that for each x, $h(x) = -3x + 15$.
    a. Find 5 different input-output pairs.
    b. If $x = -2$, what is $h(x)$?
    c. If $x = 0$, what is $h(x)$?
    d. Determine all values of x for which this function makes sense. This set is the domain of the function h.
11. A rectangular room is twice as long as it is wide. Denote the width measured in feet by w. What is the area as a function of the width?
12. Now consider a room where the length is two feet more than the width.
    a. Make a table of different data points, and compare this to the exercise above.
    b. Find an expression for the length $L(w)$ and the area $A(w)$ as a function of the width w.
    c. Find an expression for the perimeter, $P(w)$, as a function of the width w.
13. Let $f(x) = 3x - 5$. Imagine you are a teacher and you want to make word problem so that your students will come up with the formula for $f(x)$ and solve the equation $f(x) = 0$. Make up two stories that matches the formula. For each story:
    a. Describe in words what $f(x)$ represents.
    b. What does the 3 represent in your story?
    c. What does the -5 represent in your story?

14. Match the function with table and the diagram.
Function Rule:
a. $f(x) = \frac{1}{3}x - 2$ for $x = -9, -6, -3, 0, 3, 6, 9$
b. $f(x) = x^2 - 1$ for $x = -9, -6, -3, 0, 3, 6, 9$
c. $f(x) = -2x + 3$ for $x = -9, -6, -3, 0, 3, 6, 9$

Tables:

i)

x	f(x)
-9	21
-6	15
-3	9
0	3
3	-3
6	-9
9	-15

ii)

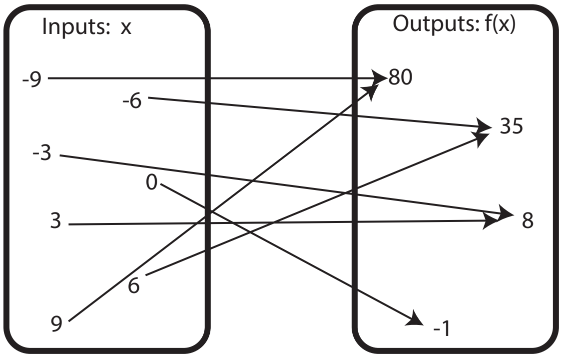
x	f(x)
-9	80
-6	35
-3	8
0	-1
3	8
6	35
9	80

iii)

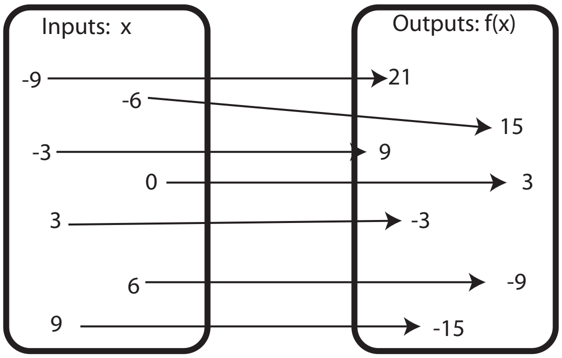
x	f(x)
-9	-5
-6	-4
-3	-3
0	-2
3	-1
6	0
9	1

Diagrams:

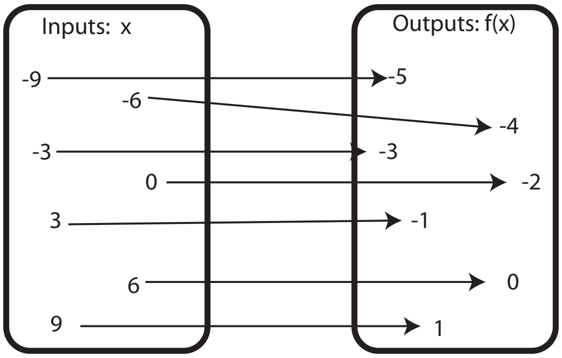
A.



B.



C.



15. Determine which of the following are functions.

a. $\{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$

b. $\{(4,2),(4,-2),(1,1),(1,-1),(0,0)\}$

c. All the pairs $(x, 2x-3)$ where x is a number.

Consider the following tables.

d.

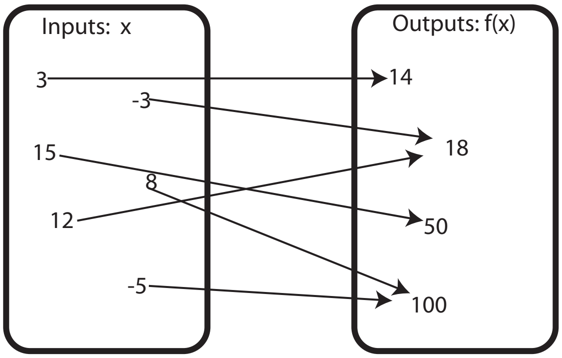
x	$h(x)$
-3	-2
-1	-1
-3	2
-1	2

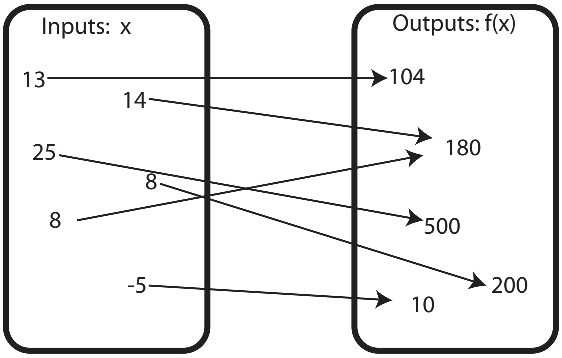
e.

x	$f(x)$
-3	1
-1	2
1	3
3	4

f. Carlos is running home. He is now 10 miles from home and starts running at a tenth of a mile per minute. Let $d(x)$ be his distance from home after running x minutes.

g. Sodas cost \$.50 each and candy bars cost \$1 each. Let $N(x)$ be the number of items (sodas, candy bars, or both) we can buy with x dollars.

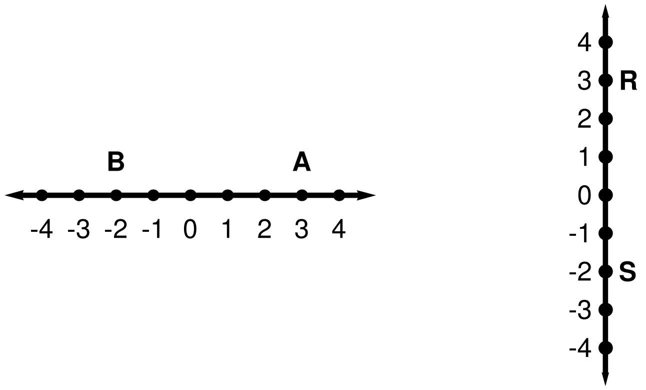
h. 

i. 

SECTION 2.2 FUNCTIONS AND THEIR GRAPHS

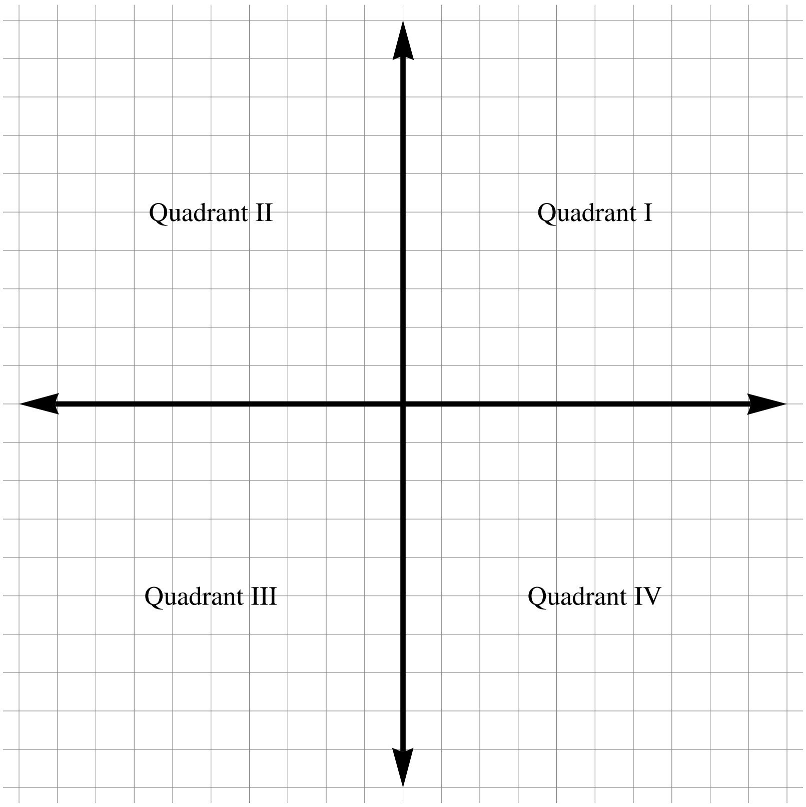
Coordinate Plane

We have used the number line to represent numbers. To each point on the number line we associated a number or *coordinate*. For example to *graph* or plot a point A with the number or coordinate 3 on the number line, we go 3 units to the right of 0. If the point B has coordinate -2 then we go 2 units to the left of 0. The graph below shows the number line with the points A and B plotted. Notice that we move left or right along the number line if the number line is horizontal. We move to the right of 0 if the number is positive and to the left of 0 if the number is negative.



We can also put the number line in a vertical position with the positive side up, like a thermometer. The point R has coordinate 3 (or 3 above 0) and the point S has coordinate -2 (or 2 below 0).

Now put the two number lines together to create a plane. The *Cartesian* or *coordinate plane* has two number lines, one horizontal and one vertical. The number lines intersect at the zero point of each number line and this point is called the *origin*. Values increase to the right and upward. The horizontal number line is typically called the x*-axis* and the vertical number line is called the y-*axis*. Thus each point in the plane is associated with an *ordered pair* (x, y) known as the *coordinates* of the point. Often the first coordinate is called the x coordinate and the second coordinate is called the y coordinate. The origin has coordinates $(0, 0)$. The axes divide the plane into four regions. We call these regions Quadrant I, Quadrant II, Quadrant III, and Quadrant IV.



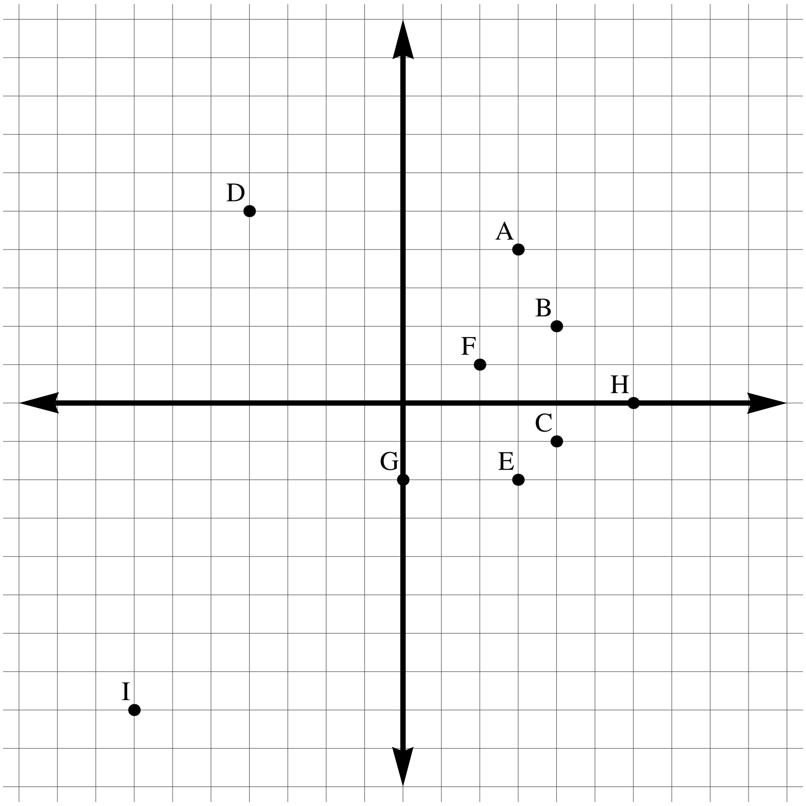
EXPLORATION 1

For each of the following points, determine which quadrant they are in. Then plot the ordered pairs on a coordinate plane.

a. $(2, 5)$	d. $(5, -2)$	h. $(0, -3)$
b. $(5, 2)$	e. $(-2, -5)$	i. $(3, 0)$
c. $(-2, 5)$	f. $(0, 3)$	j. $(-3, 0)$

PROBLEM 1

Find the coordinates of the points labeled in the graph below.



Graphing Functions on the Plane

In Section 2.1, the concept of function was introduced as a rule relating values of a dependent variable (input) to the values of an independent variable (output). We have seen that an effective way to discover or explore the pattern of the rule is to make a table of the inputs and outputs. Organizing the list of inputs and outputs into a table can help us see a pattern in the outputs.

Another way to visualize this relationship is to use the input and output as an ordered pair of numbers. We call this pair of numbers a point and plot it on a coordinate plane. The input is designated as the first coordinate x and its corresponding output as the second coordinate $y = f(x)$. In the ordered pair (x, y), x is from the domain, and the second coordinate $y = f(x)$ belongs to the range. We take all of the ordered pairs (x, y) and plot them on a Cartesian plane. This is called the *graph of the function*.

One important point to observe is that each element (input) x from the domain is paired with one and only one output, called $f(x)$, which acts as the second coordinate of the point $(x, f(x))$ of the graph of f. Note, however, that if the value of $f(x)$ is given, it does not necessarily determine x uniquely.

EXPLORATION 2

Make a table of 8 input-output pairs for each of the following functions. Include some negative input values. Plot these ordered pairs as points on a coordinate plane using graph paper. Guess how all the other points that satisfy the rule for f would be plotted. Lightly sketch in the rest of the graph of each function.

1. $f(x) = 2x - 3$
2. $f(x) = 4 - x$
3. $f(x) = x^2$

What do you notice about the graphs? How are the graphs helpful in understanding each of the functions?

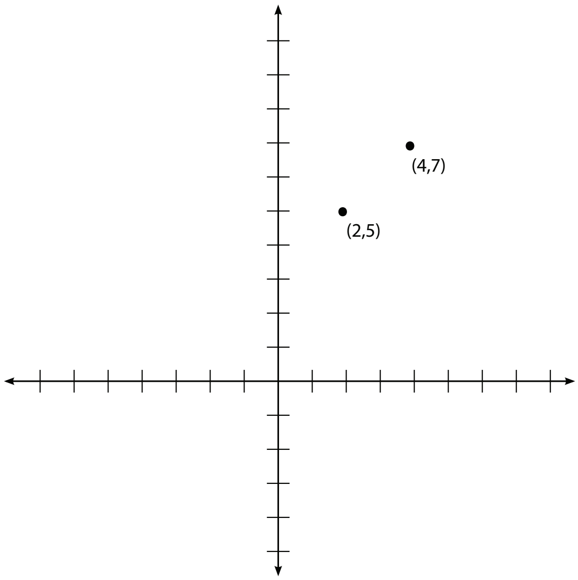
GRAPH OF A FUNCTION
The *graph* of a function is a plot of the set of all its input-output pairs.

As we saw in Exploration 2, not every function has a straight line as its graph. Functions whose graphs are straight lines are called *linear functions*. Let's explore how to find the rule for a linear function.

EXAMPLE 1

What is the rule for the function with $f(2) = 5$ and $f(4) = 7$ that has a straight line as its graph?

SOLUTION To answer this question, consider the graph of this function. As $(2, 5)$ and $(4, 7)$ are input-output pairs, points with these coordinates are on the graph of the function. The coordinate plane below has points $(2, 5)$ and $(4, 7)$. Draw a straight line passing through both these points. This is the graph of f.



You might already be able to tell the formula of the function by looking at the line. However, if you do not see the formula by looking at only the graph, don't worry. Let's look at the input-output pairs in a table format. Organizing the inputs in order will help see the pattern of change in the outputs:

Input $= x$	Output $= f(x)$
-4	
-3	
-2	
-1	2
0	3
1	
2	5
3	6
4	7
5	
6	9
7	
x	

Fill in the gaps by looking at the graph of the function.

How many points do you need in your table before you see the pattern of how the output depends on the input? Write a rule or formula for $f(x)$ in terms of each number x.

By comparing the numbers, we see that each output is 3 more than its corresponding input and so the formula for f is $f(x) = x + 3$.

EXAMPLE 2

What is the formula of the function f that has the following input-output pairs and has a graph of a straight line?

Input $= x$	Output $= f(x) = y$
1	2
3	8

SOLUTION First, investigate how to use these input-output pairs to find more of them, or find the function value at an arbitrary input. We can write the pairs above as $(1, 2)$ and $(3, 8)$. Now plot each of these points on a coordinate plane. Since the graph of f is a straight line, draw the straight line through these two points.

Use the graph to find the values of $f(0)$, $f(2)$, $f(4)$, and $f(5)$. Label the corresponding points on the graph.

The key to finding the rule is to see how the output changes as the input changes. Extend the table above so that we will have consecutive input values:

Input $= x$	Output $= f(x) = y$
0	
1	2
2	
3	8
4	
5	

You can either use the table or the graph to work out the formula for $f(x)$. Let's think about both and think about how could you could translate them to an algebraic expression.

The first thing that you might see is that each time the input increases by 1, the output increases by 3. This is much like the function given by the rule $g(x) = 3x$, but if you extend the table with a third column for the values of the expression $3x$, and graph this function on the same coordinate plane, it does not quite work. Compare the second and the third columns.

Input $= x$	Output $= f(x)$	$3x$
-1	-4	-3
0	-1	0
1	2	3
2	5	6
3	8	9
4	11	12
5	14	15
x	$f(x) =$	$3x$

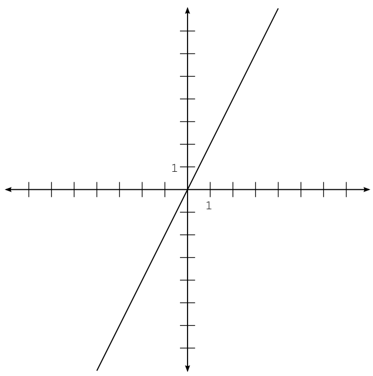
What is the rule is for the function f? Check to see if your rule gives the same point $(6, f(6))$ as the graph.

How do the ordered pairs (x, y) satisfying $y = 3x$ compare with the ordered pairs $(x, f(x))$ that form the function f?

We could describe the function by listing all ordered pairs. If we label the output with y, then the function is the set of all pairs (x, y) satisfying the relationship $y = f(x) = 3x - 1$.

PROBLEM 2

Consider the graph below of $y = f(x)$:

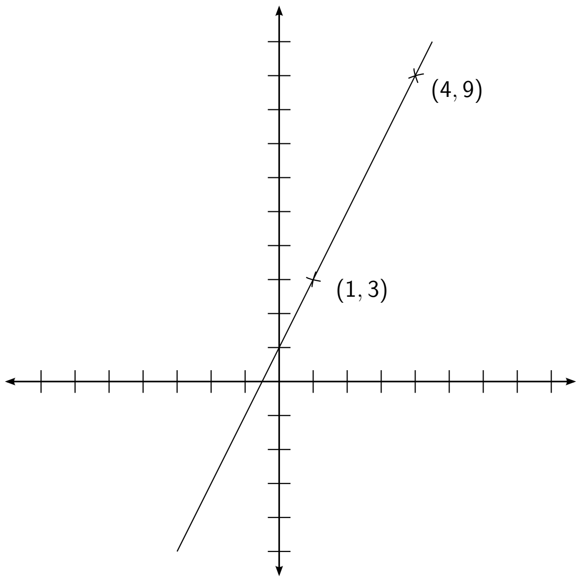


Identify 5 points on this graph. How much does the output y change when the input x increases by 1? Using this fact, what is the rule or formula for the function?

EXAMPLE 3

Plot the points $(1, 3)$ and $(4, 9)$ on a coordinate plane and draw a straight line through them. Identify other points on the graph. Find the rule for the linear function that yields this graph.

SOLUTION



Other points on the graph include $(2, 5)$, $(3, 7)$, $(0, 1)$, $(-1, -1)$, $(-2, -3)$, $(-3, -5)$.

Put some pairs with $x > 0$ in a table and observe that y increases by 2 each time x increases by 1. Use this to find a pattern.

x	y	Pattern	Connection to input
0	1	1	$1+2\cdot 0$
1	3	$1+2$	$1+2\cdot 1$
2	5	$1+2+2$	$1+2\cdot 2$
3	7	$1+2+2+2$	$1+2\cdot 3$
4	9	$1+2+2+2+2$	$1+2\cdot 4$

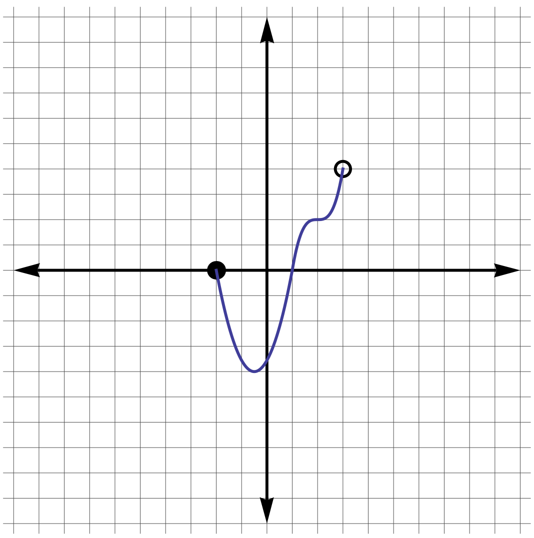
Notice that the number of 2's in the output is the same as the input. So the formula for the rule is $f(x) = 2x+1$. You can check to see that this formula correctly predicts that $f(-2) = -3$. We say the graph of the line consists of the pairs (x, y) satisfying the relationship $y = f(x) = 2x+1$.

Domain and Range on the Graph

Sometimes we want to determine the domain and range of a function just by inspecting its graph.

EXAMPLE 4

The graph of $f(x)$ is shown below. Use the graph to determine the domain and range of $f(x)$. Represent them using set notation.



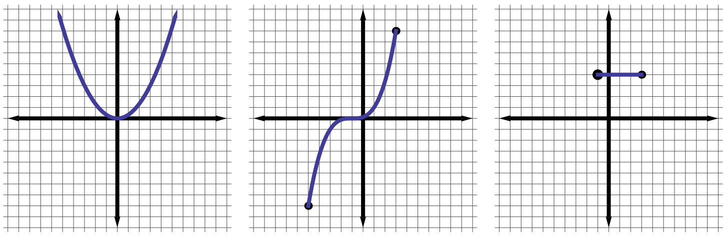
SOLUTION Notice the graph includes a shaded circle on one end and an unshaded circle on the other. These circles were also used in Chapter 1 to represent sets on the number line. Recall that the unshaded circle means that the point in the center of the circle is **not** included in the graph. So the point $(3, 4)$ is not part of the graph of the function.

To determine the domain, we need to find the set of inputs. This means we must include all the x-coordinates that appear on the graph of the function. We see that there are points on the graph with x-coordinates from -2 up to but not including 3. For the range, we include all the y-coordinates that appear on the graph of the function. The graph includes y-coordinates from -4 up to but not including 4. Now we are ready to use set notation discussed in Section 1.3. Be careful to use $\leq$ when the point is included and $<$ when it is not.

Domain: $\{x| - 2 \leq x < 3\}$
Range: $\{y| - 4 \leq y < 4\}$

PROBLEM 3

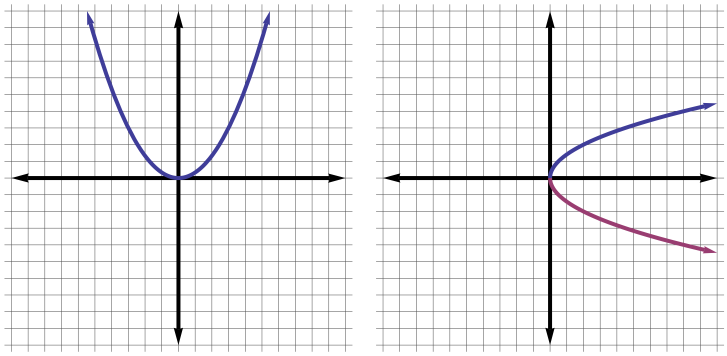
For each of the graphs use set notation to describe the domain and range.



Is it a Function?

Not all graphs represent functions. In a function each input is only assigned to one output, but different inputs can be assigned to the same output. Now we explore what this means for the graph of a function.

EXPLORATION 3



Look at the two graphs above. One is a function and one is not.

1. For the graph on the left, when $x = 1$ what is the value of y? Is there more than one value of y assigned to $x = 1$?
2. For the graph on the right, when $x = 1$ what is the value of y? Is there more than one value of y assigned to $x = 1$?
3. Make a table of 6 ordered pairs for each graph. Did you repeat any of the input values? Why is this important in determining if the graph represents a function?

EXERCISES

1. For each of the functions, make a table of input-output pairs and graph the function. Find the point where the graph crosses the y-axis. What do you notice about this point? How do you use the formula for f to compute this point?
   a. $f(x) = 2x + 3$
   b. $f(x) = 2x - 3$
   c. $f(x) = -2x$
   d. $f(x) = -2x + 6$
2. For each function
   - Make a table of at least 4 points of the graph.

- Graph the function, and find the point where the graph crosses the y-axis.
- Compute the value of the output $f(0)$.
- Explain the connection of the value of $f(0)$ to the point where f crosses the y-axis.

a. $f(x) = \frac{x}{2} = \frac{1}{2}x$
b. $f(x) = 5 - 2x$
c. $f(x) = \frac{3x+1}{2}$

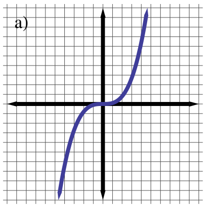
3. Consider the function s described by the table:

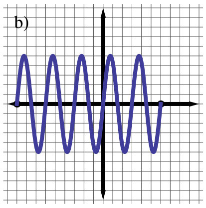
Input n	Output $s(n)$
1	9
2	19
3	29
4	39
5	49

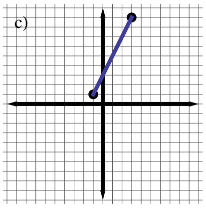
Graph the corresponding ordered pairs. Find a formula for the output $s(n)$ in terms of the input n.

4. Consider the function g whose graph is a straight line containing the two points $(3, 2)$ and $(5, 6)$.
   - Mark the points on a coordinate system and draw the graph of the function.
   - Make a table containing 3 other points on this line.
   - Find a rule of the function.

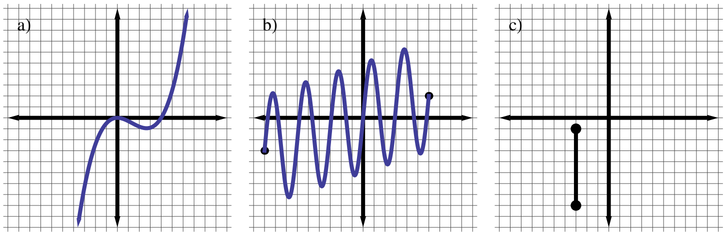
5. For each of the following use set notation to describe the domain and range.







6. Determine if the graphs below represent functions. Explain.



7. Consider the table of coordinate pairs below. Plot the points and then determine if the points could be on the graph of a function. Explain.

x	y
1	5
2	8
3	11
2	2
3	-1

8. Graph the following functions using a graphing calculator. Discuss how they are similar and different.

$$f(x) = x^3$$
$$g(x) = x^3 + 2$$
$$h(x) = x^3 - 2$$
$$j(x) = -x^3$$

9. Let's revisit the function f given by $f(x) = x^2$. Fill in the table for f below and draw a careful graph of these points with inputs between -2 and 2. You may use a calculator to compute the outputs.

x	-2	-1.5	-1	$-.7$	$-.4$	$-.2$	$-.1$	$-.06$
$f(x)$								

x	$-.02$	0	.02	.06	.1	.2	.4	.7	1	1.5	2
$f(x)$											

What do you notice about the graph? Compare it to the graph you made in Exploration 2.

10. Find a function containing the points in the table below.

x	$f(x)$
-5	26
-4	17
-3	10
-2	5
-1	2
0	1
1	2
2	5
3	10
4	17
5	26

Graph these points. How does this function compare with the one in the previous exercise?

11. Make a table of input-output pairs for the function $f(x) = x^2 - 4x$. Plot the pairs on a coordinate plane. How many points do you need to plot before you have a good idea of the shape of the graph of f?

12. **Ingenuity:**
    a. Make a table of at least 10 points of the graph of the function $f(x) = \frac{1}{x}$ for each number $x \neq 0$. Make sure to include the numbers $\frac{1}{2}, \frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}$ as inputs in your table.
    b. Graph the points on a coordinate plane.
    c. Fill in the remaining points of the graph.
    d. Compare your graph with one made with a graphing calculator. Do they agree? Discuss.

13. **Ingenuity:**
Define the function g by the equation $g(x) = 2x$ if x is positive or zero, and $g(x) = 3 + x$ if x is negative. Sometimes it is written as:

$$g(x) = \begin{cases} 2x & \text{if } x \geqslant 0 \\ 3 + x & \text{if } x < 0. \end{cases}$$

Note: This is called a *piecewise* defined function. Do you see why? Evaluate the following:

a. $g(3)$
b. $g(10)$
c. $g(-3)$
d. $g(0)$
e. $g(-2)$
f. $g(-1)$
g. $g(-\frac{1}{2})$
h. $g(-.1)$
i. Draw the graph of g. What interesting features does the graph have? Where do these features occur?

SECTION 2.3 PATTERNS AND SEQUENCES

Functions give us a way to describe relationships precisely. We now use functions to describe the pattern of a list of numbers. If the domain of a function is the natural numbers, the range will be a list of numbers. We call this list of outputs a *sequence*.

EXAMPLE 1

Let the function A be defined by $A(n) = 3n + 1$ for all natural numbers n.

1. Complete the table below:

n	1	2	3	4	5
$A(n)$	4				

2. Write the outputs of the function as a list.
3. When a sequence is written as a list, each member of the sequence is called a *term* of the sequence. Compare consecutive terms (terms that are next to one another) in the sequence. What relationship do you notice? Use this relationship to find the next three terms in the sequence.

SOLUTION

1. Substitute the different values of n into the rule for $A(n)$ to complete the table.

n	1	2	3	4	5
$A(n)$	4	7	10	13	16

2. The list of all outputs is the set $\{4, 7, 10, 13, 16, \ldots\}$. Notice we used $\ldots$ in the list to show that the list goes on forever.
3. Terms of a sequence are often denoted using subscripts instead of parentheses. For this sequence, $a_1 = A(1) = 4$, $a_2 = A(2) = 7$, and so on. If we compare consecutive terms: $a_1 = 4$ vs. $a_2 = 7$, $a_2 = 7$ vs. $a_3 = 10$, $\ldots$, we notice that each term is 3 more than the previous term. This is another way to find terms in the sequence:

$a_5 = 16$, so $a_6 = a_5 + 3 = 16 + 3 = 19$, $a_7 = a_6 + 3 = 19 + 3 = 22$, $a_8 = 25$.

In Example 1 we found different ways to compute the terms of a sequence. In part 1, we used the rule of the function $a_n = A(n) = 3n + 1$. This is also called the formula for the n-th term. In part 3, we used the relationship between consecutive terms: $a_{n+1} = a_n + 3$. This is called the *recursive formula*. Let's explore how to find the formula for the n-th term.

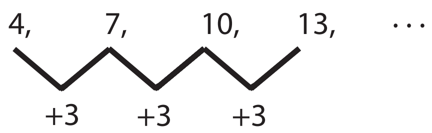
EXPLORATION 1

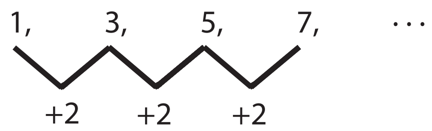
Find a formula for the n-th term of the sequence:

$$1, 3, 5, 7, 9, 11, \ldots$$

1. Compare consecutive terms in the sequence. Describe the relationship between them. Use this relationship to find the next two terms in the sequence.
2. Make a table like the one used in Example 1.
3. Find the formula for the n-th term of the sequence.

The two sequences considered so far had similar recursive formulas. The differences between consecutive terms were constant.





Sequences with this type of recursive formula have a special name.

ARITHMETIC SEQUENCE
A sequence $a_1, a_2, a_3, \ldots$ is an *arithmetic sequence* if there is a number c such that for each natural number n, $a_{n+1} = a_n + c$, that is $a_{n+1} - a_n = c$.

PROBLEM 1

Consider the sequences:

- $2, 4, 6, 8, \ldots$
- $-3, 1, 5, 9, 13, \ldots$
- $2, 5, 10, 17, \ldots$
- $15, 13, 11, 9, \ldots$

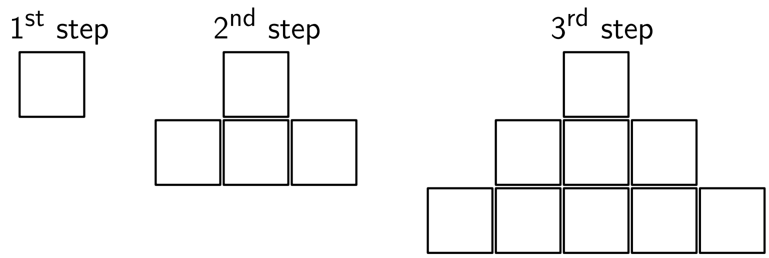
For each sequence,

1. Determine if it's arithmetic or not.
2. Make a table using the natural numbers as inputs and the sequence as outputs.
3. Graph the points of the table on a coordinate plane. Compare the graphs of the arithmetic sequences. How are they similar?

Not all sequences are arithmetic. In the following exploration, you will investigate different sequences that arise from a geometric pattern of blocks. Some are arithmetic, others are not.

EXPLORATION 2

Suppose that you have a large supply of blocks. Make a sequence of stacks of blocks in the following way. First you place one block down. In step 2, place three blocks in a row under the first block, centered horizontally. In step 3, place five blocks under the stack in step 2 to form a new stack.



1. Draw the next 2 stacks in steps 4 and 5 by adding a new row of blocks with 2 more blocks than the row above it.
2. If we continued this process, make a list of what changes step to step. (Try to find 5 different things that change.) In groups, share your lists and make a group list.
3. We are interested in changes that can be described by a numerical value. For each quantity that changes, we want to define a function that gives us this numerical value as its output. The input for each of these outputs will be the step number. For example, we let $h(n)$ be the height of the stack at step n. What is $h_1 = h(1)$ and $h_2 = h(2)$? For another example, we let $B(n)$ denote the number of blocks in the n^{th} step.
4. In each group, pick 3 things that change from the class list. Make a table for each of these functions. Try to find a rule for at least one of these functions.

Notice that in all of the functions that we defined in Exploration 2, the domain contains only natural numbers. For example, $B(\frac{1}{2})$ does not make sense because there is no step $\frac{1}{2}$. Even though the rules for these functions could apply to all numbers, the domains only contain numbers that make sense in the context of the geometric pattern in the figure above.

PROBLEM 2

For each of the functions below, restrict the domain to the natural numbers and:

- write out the first 4 members of the sequence
- determine if the sequence is arithmetic or not
- use the pattern to write the next two members of each sequence without computing with the rule

1. $f(x) = x^2 + 1$
2. $f(x) = 2x + 3$
3. $f(x) = 3x + 2$
4. $f(x) = \frac{1}{x}$
5. $f(x) = \frac{1}{2x-1}$
6. $f(x) = 2^x$

EXERCISES

1. For each function, restrict the domain to the natural numbers and list the first 6 numbers in the sequence.
   a. $s(n) = n - 10$
   b. $m(x) = -2x + 5$
   c. $f(z) = \frac{z}{5} + 2$
   d. $g(y) = 4y - 3$
   e. $f(n) = 6n - 4$
2. For each function, restrict the domain to the natural numbers, and list the first 6 numbers in the sequence. Determine if each sequence is arithmetic or not. Explain your answer.
   a. $r(x) = 3x + 5$
   b. $f(y) = 7y - 2$
   c. $g(n) = \frac{3n}{2} - \frac{1}{5}$
   d. $s(x) = 8x + 1$
   e. $q(y) = y^2$
3. For each sequence, find a rule for the n^{th} term:
   a. $1, 2, 3, 4, \ldots$
   b. $3, 6, 9, 12, \ldots$
   c. $2, 5, 8, 11, \ldots$

d. $-4, -1, 2, 5, \ldots$

e. $0, 0, 0, 0, \ldots$

f. $-5, -7, -9, -11, \ldots$

g. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

h. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, \ldots$

4. Matt started with only 3 marbles. He is a very good marbles player and wins 2 marbles every day. How many marbles does he have at the beginning of each day? What are the first 4 elements of the sequence?

5. Anne walks up the stairs of a tall building. Each flight of stairs has 10 steps and she must walk up two flights of stairs to get to the next floor. How many steps must she walk up to get to the n^{th} floor? Write out the rule for the sequence and the first 4 elements.

6. **Investigation:**
The recursive formula is not enough information to write down a sequence. You also need to know the value of one of the terms. Typically you are given the first term. Suppose an arithmetic sequence satisfies the recursive formula $a_{n+1} = a_n + 4$.
   a. What does the recursive formula mean in words?
   b. Let $a_1 = 0$. List the first 5 terms of the sequence. Find the formula for the n^{th} term of the sequence.
   c. Let $a_1 = 2$. List the first 5 terms of the sequence. Find the formula for the n^{th} term of the sequence.
   d. Let $a_1 = 5$. List the first 5 terms of the sequence. Find the formula for the n^{th} term of the sequence.
   e. Compare the three sequences. How does the formula depend on the value of the first term?
   f. Graph the sequences on the same coordinate plane. How are the graphs similar? How are they different?

7. **Investigation:**
Suppose an arithmetic sequence satisfies the recursive formula $a_n = a_{n-1} + c$ with $a_1 = 0$.
   a. What does the recursive formula mean in words?
   b. Let $c = 1$. List the first 5 terms of the sequence. Find the

formula for the n^{th} term of the sequence.

c. Let $c = 3$. List the first 5 terms of the sequence. Find the formula for the n^{th} term of the sequence.

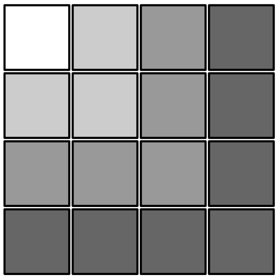
d. Let $c = -2$. List the first 5 terms of the sequence. Find the formula for the n^{th} term of the sequence.

e. Compare the three sequences. How does the formula depend on the value of c?

f. Graph the sequences on the same coordinate plane. How are the graphs similar? How are they different?

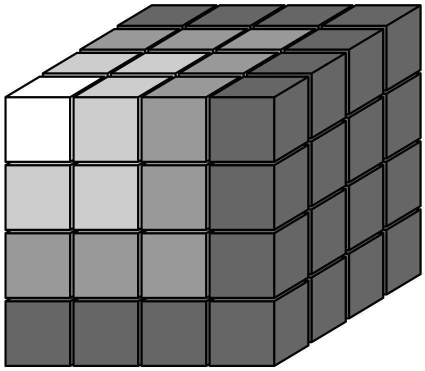
8. **Investigation:**

Suppose you have a pile of blocks. Make a sequence of block squares in the following way. In Step 1, place one block down. In Step 2, place 3 blocks down to form a 2 block by 2 block square. In Step 3, place 5 blocks down in order to form a 3 block by three block square. In each of the next steps, add the appropriate number of blocks in order to make the next larger square. This pictures shows each step with a different shade.



Find 3 different quantities that change. Then, like you did in Exploration 2, find a rule for how these quantities change.

9. **Investigation:**
Now suppose you have a pile of cubes. Make a sequence of block cubes in a similar way: in Step 1, place one cube down. In Step 2, place 7 cubes down to form a 2 block by 2 block by 2 block cube. In Step 3, place 19 additional blocks down in order to form a 3 block by 3 block by 3 block cube. In each of the next steps, add the appropriate number of blocks in order to make the next larger cube. This pictures shows each step with a different shade.



Find 4 different quantities that change and come up with a rule for how these quantities change.

SECTION 2.4 APPLICATIONS AND FUNCTIONS

In previous grades you may have learned about rates and unit rates. What is a *unit rate*? In the following exploration, we will connect these ideas to functions.

EXPLORATION 1

Individually, make a list of 5 rates that you know. In groups, share your list. Each group should choose 4 unit rates. For each unit rate:

1. Define an independent variable (the input).
2. Define the dependent variable (the output).
3. Name the function and find a rule for the function.
4. What is the set of inputs (domain) and the set of outputs (range)?
5. Sketch a graph of each function. You may use a graphing calculator to check these graphs. Should you put these graphs on the same coordinate system? Why?

You probably noticed that most of these functions have graphs that are straight lines. Some of them have domains that include all non-negative numbers, and some only make sense for the whole numbers. Nevertheless, we still sketch the graph with all of the points filled in. We do this in order to understand the pattern of the sequence better. If we can understand the function for all numbers, then it is easier to understand the function as a sequence of the natural numbers. When we first learn about a function, it is natural to ask questions such as "What is $f(1)$?" or "What is $f(5)$?" In general, given an input x, what is the output $f(x)$? This is the process that helped us build tables and plot the graphs of the functions.

However, when using functions in real applications, it is often the reverse question that is most useful. In many situations, we want to know for what input x will we obtain a certain output. This is a much trickier question for many functions.

EXAMPLE 1

Mary's brother gives her 14 DVDs the beginning of summer. From then on, at the end of each week, she buys 3 more DVDs. When will she have 50 DVDs in her collection?

SOLUTION We denote the function describing the number of DVDs Mary has after x weeks with $N(x)$. In the beginning, when zero weeks have passed, she starts with 14. Making a table of inputs and outputs, we get:

Week x	# of DVDs $N(x)$
0	14
1	$14 + 3 = 17$
2	$14 + 3(2) = 20$
3	$14 + 3(3) = 23$
4	$14 + 3(4) = 26$

What is the rule for $N(x)$, and how is it used to compute the number of weeks it takes for Mary to accumulate 50 DVDs? Again, rewriting 14 as $14 + 3(0)$ and $14 + 3$ as $14 + 3(1)$ gives us

x	$N(x)$
0	$14 + 3(0) = 14$
1	$14 + 3(1) = 17$
2	$14 + 3(2) = 20$
3	$14 + 3(3) = 23$
4	$14 + 3(4) = 26$

It is now easy to see that the formula is $N(x) = 14 + 3x$.

We need to find the number of weeks x it takes Mary to have 50 DVDs.

$$\begin{aligned} N(x) &= 50 \\ 14 + 3x &= 50 \\ 3x &= 50 - 14 \\ 3x &= 36 \\ x &= 12 \end{aligned}$$

So it takes Mary 12 weeks to get to 50 DVDs.

EXAMPLE 2

Jack sells hats for \$12 per hat and Bethany sells T-shirts for \$16 per shirt. If a soccer coach has \$576 to spend on her team, how many hats could she buy? How many T-shirts could she buy? How would your answers change if the coach had \$588 to spend?

SOLUTION Let $H(x)$ be the cost of x hats and $T(x)$ be the cost of x T-shirts. We see that the rule for H is $H(x) = 12x$ and the rule for T is $T(x) = 16x$.

Caps: Let x be the number of hats the coach can buy with \$576:

$$\begin{aligned} H(x) &= 576 \\ 12x &= 576 \\ x &= 48 \end{aligned}$$

T-shirts: Let y be the number of T-shirts she can buy with \$576:

$$\begin{aligned} T(y) &= 576 \\ 16y &= 576 \\ y &= 36 \end{aligned}$$

So the coach can get 48 hats or 36 T-shirts for her team.

If the coach had \$588 to spend, we would write similar equations and solve:

$$\begin{aligned} H(x) &= 588 \\ 12x &= 588 \\ x &= 49 \end{aligned}$$

$$\begin{aligned} T(y) &= 588 \\ 16y &= 588 \\ y &= 36.75. \end{aligned}$$

The answer is 49 hats or 36.75 T-shirts. Since you cannot buy $\frac{3}{4}$ of a T-shirt, the coach can only buy 36 T-shorts with \$12 left over.

EXAMPLE 3

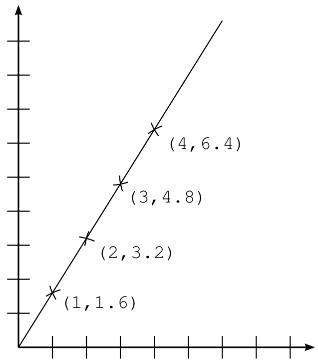
Jack is participating in a Walk for Diabetes to raise money for the American Diabetes Association (ADA). He raises \$1.60 for each mile he walks. Write a function that relates the number of miles he walks with the total amount of money he raises. Represent the function as ordered pairs. Graph the function and indicate the domain and range.

Jack has raised \$50 for ADA. How many miles did he walk for this?

SOLUTION Let $D(x)$ represent the donation in dollars Jack raises if he walks x miles.

Miles walked $= x$	Donation $= D(x)$
0	0
1	1.60
2	3.20
3	4.80
4	6.40
x	$1.6 \cdot x$

Let's make a graph marking all these ordered pairs: $(1, 1.6)$, $(2, 3.2)$, $(3, 4.8)$, $(4, 6.4)$. Again, when we connect the points, we get a straight line. So if Jack walks x miles, what will the donation $D(x)$ be? In this example, each time the input x increases by 1, the output (donation) $D(x)$ will increase by 1.6. So the equation is $D(x) = 1.6x$.



How many miles does Jack have to walk to raise \$50? Let's use the function $D(x)$ to solve this problem:

$$D(x) = 50$$
$$1.6x = 50$$
$$x = 31.25$$

So Jack had to walk 31.25 miles to raise \$50.

EXERCISES

1. Suppose f is the function such that for each x, $f(x) = 4x - 5$.
   a. Fill in the missing inputs in the table below:

Input $= x$	Output $= f(x)$
	11
	23
	5
	-2
	0

   b. Sketch the graph of f. Did you use the points from the table above? If not, check to see that they are on your graph.

2. Suppose h is the function such that for each x, $h(x) = 10 - 4x$.
   a. Fill in the missing inputs in the table below:

Input $= x$	Output $= h(x)$
	1
	3
	5
	7
	9

   b. What do you notice about the pattern of inputs from these outputs?
   c. Sketch the graph of h using these points.

3. Suppose g is the function such that for each x:

$$g(x) = \frac{2x + 5}{3}.$$

   a. Fill in the missing inputs in the table below:

Input $= x$	Output $= g(x)$
	0
	2
	4
	6
	8

b. What do you notice about the pattern of inputs from these outputs?
c. Sketch the graph of g using these points.

4. A store sells pencils for $.30 each. Let $R(x)$ be the revenue in dollars for selling x pencils.
   a. Write a formula for $R(x)$.
   b. Compute $R(15)$ and explain what this number means.
   c. If Diana spends $6.90 (before tax), how many pencils did she buy?
5. At the beginning of the year, a young pig weighs 8 pounds. A manual predicts that the pig will gain weight steadily at the rate of 12 pounds per month. Let $W(x)$ be the weight of the pig in pounds at the end of x months.
   a. What is the predicted weight at the end of 4 months?
   b. What is a formula or rule for $W(x)$ in terms of the number of months x?
   c. What is the domain of the function W?
   d. How much is the pig expected to weigh at the end of 2 and a half months?
   e. When will the pig weigh 92 pounds?
   f. When will the pig weigh 110 pounds?
6. Ramon has a print shop that uses 3 boxes of paper each day. At the beginning of day 1, he has 42 boxes of paper in his storage room.
   a. Use a function to describe the relationship between time and the number of boxes. Define your function in words.
   b. How many boxes will he have left at the end of third day?
   c. How many boxes will he have at the end of the x^{th} day?
   d. On what day will his paper run out?
7. Julia is performing an experiment in the lab. She hooks up a vat

containing 24 liters of chemical A and lets it flow into another container at the rate of 3 liters per hour.

a. What is the rule or formula for the amount of chemical A in the vat after x number of hours?
b. After how many hours will the vat be emptied?
c. Can we come up with a rule for the amount of chemical A in the second container after x number of hours?

8. At 6:00 PM in a rain forest in Central America, Alex starts collecting rain water at the steady rate of 4 liters per hour. At the same time, Alejandra begins collecting 4 specimens of insects at the end of each hour from the traps she has set. She starts with no insects in her collection at the beginning of the first hour.
   a. Define 2 functions, W for water and B for bugs, to describe these activities. Find the rule for each, specify the domain and discuss the relationship between $W(0)$ and $B(0)$.
   b. By what time will Alex have collected 22 liters of rain water and Alejandra 22 bugs? Compare your answers.

9. **Investigation:**
   For each of the following functions, describe how the output changes if you increase the input by 1 unit. Is there something in the rule for each function that would help to predict the pattern of how the output changes?
   a. $f(x) = 2x + 1$
   b. $g(x) = 3x - 2$
   c. $h(x) = -3x + 1$
   d. $j(x) = \frac{1}{2}x + 3$
   e. $k(x) = -4x + 8$

SECTION 2.5 CHAPTER REVIEW

Key Terms

arithmetic sequence
coordinate plane
dependent variable
domain
function
graph of a function
independent variable
n^{th} term
ordered pair
range
rate
recursive formula
revenue
sequence
unit rate

Formulas

Arithmetic Sequence: $a_n = a_{n-1} + c$

Practice Problems

1. Suppose $f(x) = 6x - 2$ for all x. Compute the following:

   a. $f(1)$
   b. $f(-3)$
   c. $f(0)$
   d. $f(\frac{1}{2})$
   e. $f(d)$
   f. $f(\frac{2}{3})$

2. Stephanie wants to build a dog run that is 3 ft longer than it is wide. This dog run will be enclosed by a fence. If w is the width in feet, let $f(w)$ be the length of fencing required to enclose the dog run.
   a. Create a formula for the function f.
   b. What is the meaning of $f(6)$? What is the value of $f(6)$?
   c. What are the dimensions of the dog run that would use 36 ft of fencing?

3. Valerie needs to empty a large liquid storage container. The container has 1450 liters of used oil. She pumps the used oil into a tanker truck at a rate of 75 liters per minute.
   a. Write a rule for the amount of used oil in the tanker after x

minutes.

b. Write a rule for the amount of used oil in left the large storage container after x minutes.

c. How long will it take to empty the storage container?

d. Graph the amount of used oil in the storage container and in the tanker over time. Use the same grid.

e. Using the graphs determine at what time each container has the same amount of used oil in it. How much do they have at this time?

4. For each of the following functions, generate the first 5 numbers in the sequence, starting with $n = 1$.

a. $f(n) = 3n - 7$

b. $g(n) = -5n + 4$

c. $k(n) = n^3$

d. $p(n) = \frac{3}{4} - \frac{n}{2}$

5. For the portions of each of the sequences create a rule shown below, create a rule $q(x)$ starting with $x = 1$.

a. $8, 13, 18, 23, 28, \ldots$

b. $7, 2, -3, -8, -13, \ldots$

c. $-15, -11, -7, -3, \ldots$

d. $12, 5, -2, -9, -16, \ldots$

6. Patty has 12 songs on her mp3 player. She plans to download 14 songs a month.

a. When will she have 110 songs on her mp3 player?

b. Write a function that models this situation.

c. Create a table that fits this situation and make a graph based on the table.

d. How could you use your graph from part 6c to answer the question from part 6a?

7. Given the function $k(x) = -3x + 2$:

a. Complete the table.

Input $= x$	Output $= k(x)$
	5
	-1
	-7
	-13

b. Using the points from the table, create a graph of the function. Describe the pattern you see.

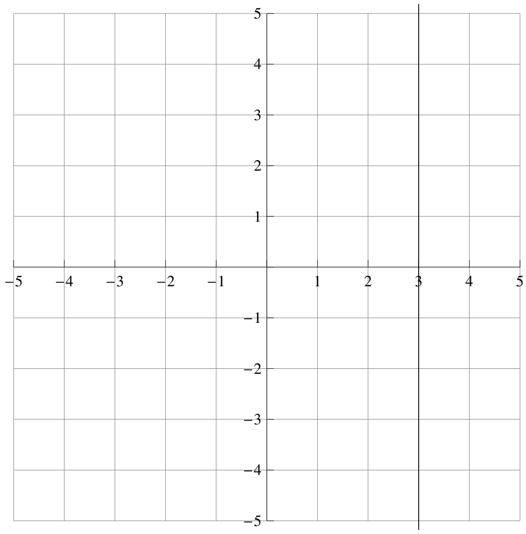
STRAIGHT LINES

3

SECTION 3.1 HORIZONTAL AND VERTICAL LINES

EXPLORATION 1

Consider the graph of the *vertical line* below. Label 5 different points on the line. What property do all the points share?



In this chapter we will learn how to express the property that points on a line share using algebra. We want to come up with a "point tester." A *point tester* is a condition that points on the line satisfy and points not on the line fail to satisfy. Let's use the graph above as an example. We can state the point tester in words, "a point is on the line if and only if the first coordinate is equal to 3". Using algebra, we can express the point tester by saying "a point (x, y) is on the line if $x = 3$". We call $x = 3$ the *equation of the vertical line*.

EXPLORATION 2

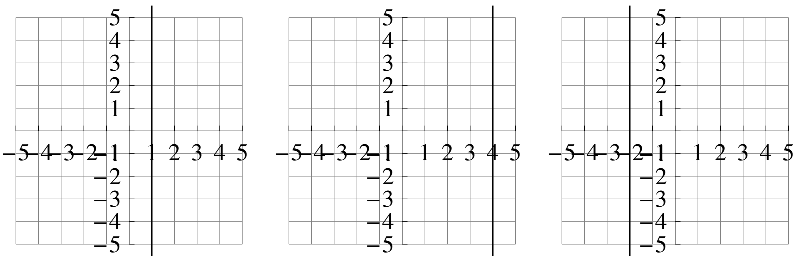
Use "a point (x, y) is on the line if $x = 3$" to test which of the following points are on the vertical line shown in Exploration 1.

1. $(3, 0)$
2. $(0, 3)$
3. $(3, 2)$
4. $(3, -6)$
5. $(2, 1)$
6. $(-1, 3)$

For those points that are not on the line, how can you tell if they are to the left or to the right of the line? Explain. Can you tell if the points are close to the line or far from the line? Explain.

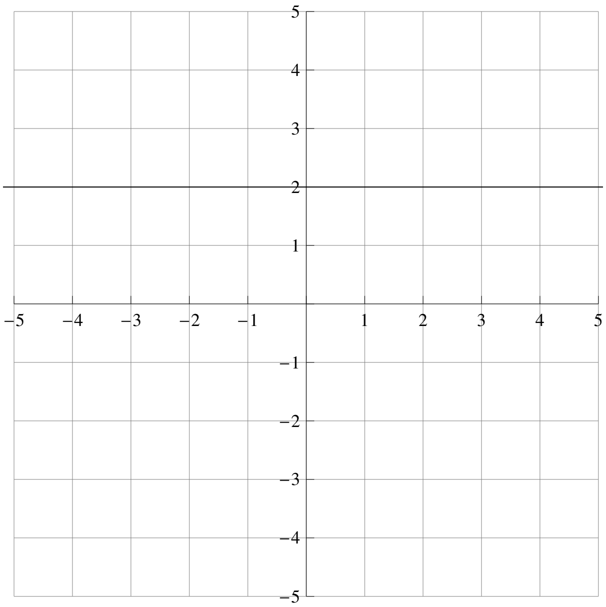
PROBLEM 1

Consider the vertical lines below. On each line, label 3 different points, and describe the property the points share. Write a statement using an equation that can be used to test what points are on the line.



EXPLORATION 3

Consider the graph of the *horizontal line* below. Label 3 different points on the line. What property do all the points share? Write a statement using an equation that can be used to test if a point is on the line.



In the next exploration we use the equation from above to check if points are on the horizontal line.

EXPLORATION 4

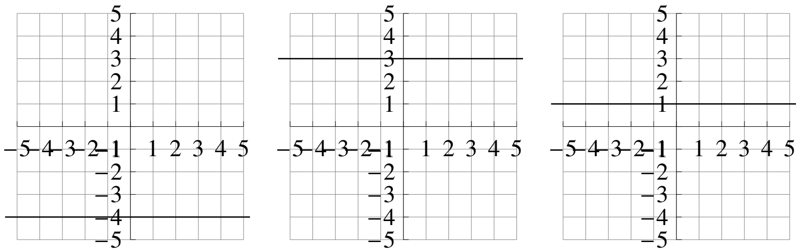
Use the point tester "a point (x, y) is on the line if $y = 2$" to determine which of the following points are on the horizontal line shown in Exploration 3.

1. $(0, 2)$
2. $(0, 3)$
3. $(3, 2)$
4. $(2, -6)$
5. $(2, 1)$
6. $(-1, 3)$

For those points that are not on the line, how can you tell if they are above or below the line? Explain. Can you tell if the points are close to the line or far from the line? Explain.

PROBLEM 2

Consider the horizontal lines below. For each line, label 3 different points, and describe the property the points share. Write a statement using an equation that can be used to test what points are on the line.



EXPLORATION 5

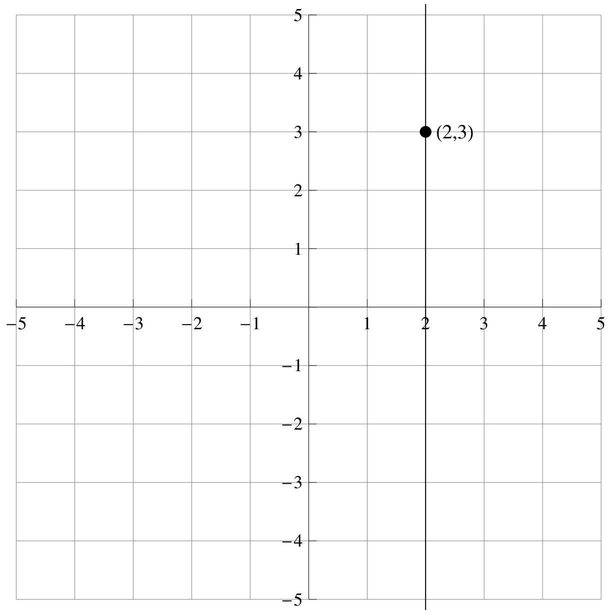
In Problem 1 we noticed that points on the vertical lines were described by the equation $x = a$ for some number a. Do you think that all vertical lines are described by equations like this? Explain. How about horizontal lines, what do their equations look like?

One of the common tasks in algebra is to find the equation for a line through a given point that satisfies a given property.

EXAMPLE 1

Find the equation of the vertical line that passes through the point $(2, 3)$. Graph the line.

SOLUTION All the points (x, y) on a vertical line share the same first coordinate and satisfy the equation $x = a$. The first coordinate of the point $(2, 3)$ is 2, so a point (x, y) is on the line if $x = 2$.



PROBLEM 3

Find the equation of the horizontal line which passes through the point $(-2, 4)$. Graph the line.

PROBLEM 4

1. We call the horizontal line through the origin the x-axis. Find its equation.
2. We call the vertical line through the origin the y-axis. Find its equation.

One of the most powerful tools in mathematics is the combination of algebra and geometry. Geometry often helps us understand why certain algebraic results are true, and algebra is very useful in solving problems in geometry. In this chapter we will look at the relationship between the geometric concepts of parallel and perpendicular lines and the algebraic forms of those lines. In the next exploration we start by focusing on the simplest case: vertical and horizontal lines.

EXPLORATION 6

1. What does it mean to say two lines are *parallel*?
2. What does it mean to say two lines are *perpendicular*?
3. On one coordinate grid, make a graph of the vertical and horizontal lines described by these equations:
   a) $x = 4$ b) $y = -1$ c) $x = -2$ d) $y = 1$ e) $y = \frac{3}{2}$
4. Which lines in part 3 are parallel? Which pairs of lines are perpendicular?

EXERCISES

1. Graph the vertical line through the point $(-3, 5)$. Label 3 points on the line. Write a statement using an equation that can be used to test if a point is on the line.
2. Graph the horizontal line described by the point tester "a point (x, y) is on the line if $y = 2$." For each of the following points decide if the point is on the line, above the line, or below the line. Explain how you can use the point tester to determine this.
   a. $(2, 3)$
   b. $(3, 2)$
   c. $(-4, 2)$
   d. $(4, -2)$
   e. $(2, 5)$
3. Suppose a line passes through the point $(4, 2)$ and is a vertical line. What is its equation?
4. Suppose a line passes through the point $(4, 2)$ and is a horizontal line. What is its equation?
5. Identify each as a horizontal or vertical line:

   a. $x = 3$ d. $y = 0$ g. $x = 7$

   b. $x = -2$ e. $y = 10$

   c. $y = 4$ f. $y = -3$

6. At what point does the line described by the equation $x = 3$ intersect the line described by the equation $y = 5$?

7. Explain why the line given by the equation $x = 2$ does not intersect the y-axis.
8. Explain why the line given by the equation $y = 4$ does not intersect the x-axis.
9. Find the equations of two lines which are parallel to the vertical line described by the equation $x = -1$.
10. Find the equations of two lines which are perpendicular to the vertical line described by the equation $x = -1$.
11. Find the equations of two lines which are parallel to the horizontal line described by the equation $y = 6$.
12. Consider the two points $(1, 4)$ and $(2, 6)$.
    a. Find the equation of the horizontal line that passes through $(1, 4)$. Graph the line and label the point $(1, 4)$.
    b. Find the equation of the vertical line that passes through $(2, 6)$. On the same grid as 12a, graph the line and label the point $(2, 6)$.
    c. Label the point of intersection of the horizontal line from part 12a and the vertical line from 12b.
    d. Draw the line segment from $(1, 4)$ to $(2, 6)$. Consider the right triangle that is formed. What is the length of the horizontal side? What is the length of the vertical side?
13. Consider the two points $(2, 6)$ and $(1, 4)$.
    a. Find the equation of the horizontal line that passes through $(2, 6)$. Graph the line and label the point $(2, 6)$.
    b. Find the equation of the vertical line that passes through $(1, 4)$. On the same grid as 13a, graph the line and label the point $(1, 4)$.
    c. Identify and label the point of intersection of the horizontal line from 13a and the vertical line from 13b.
    d. Draw the line segment from $(2, 6)$ to $(1, 4)$. Consider the right triangle that is formed. What is the length of the horizontal side? What is the length of the vertical side? Compare your answers to Exercise 12.
14. Consider the two points $(2, 5)$ and $(3, 0)$.
    a. Find the equation of the horizontal line that passes through

(2, 5). Graph the line and label the point (2, 5).

b. Find the equation of the vertical line that passes through (3, 0). On the same grid as part 14a, graph the line and label the point (3, 0).

c. Label the point of intersection of the horizontal line from 14a and the vertical line from 14b.

d. Draw the line segment from (2, 5) to (3, 0). Consider the right triangle that is formed. What is the length of the horizontal side? What is the length of the vertical side?

15. Consider the two points (2, 3) and (a, b) for numbers $a \neq 2$ and $b \neq 3$.

a. Find the equation of the horizontal line that passes through (2, 3).

b. Find the equation of the vertical line that passes through (a, b).

c. Find the point of intersection of the horizontal line from part 15a and the vertical line from part 15b.

16. **Investigation:**

Functions. In section 2.2 we considered the graphs of functions. Many of those graphs were straight lines. In this investigation we begin to look at the connections between the functions studied there and the equations studied in this section.

a. Consider the function $f(x) = 5$ for any number x. Graph the function. What kind of line is it?

b. Graph the vertical line described by the equation $x = 4$. Look at the definition of a function. Can the vertical line be the graph of a function? Explain why or why not.

17. **Ingenuity:**
Let's consider an unusual graph.
   a. Round each of the following numbers to nearest integer: $\{-1.9, -1.5, -1.4, -1, -.7, 3, 3.4, 3.5, 3.6, 3.9, 4, 4.2, 4.4, 4.5\}$. *Note: for the sake of this problem, round* $-.5, .5, 1.5, 2.5,$ *etc. to the next greatest integer.*
   b. Describe in words what it means to round a number.
   c. Graph the following points:

      i. $(3, 3)$
      ii. $(0, 3)$
      iii. $(3.4, 3)$
      iv. $(3.6, 3)$
      v. $(3.6, 4)$
      vi. $(-1.2, -1)$
      vii. $(-1.6, -2)$
      viii. $(-1.5, -2)$
      ix. $(-1.4, -1)$

   d. Think about the points whose second coordinate equals first coordinate rounded to the nearest integer. We might write this as "all points (x, y) where $y = \text{round}(x)$." Circle all the points you have graphed that pass this test (in other words those that belong to the set). What pattern do you see?
   e. If we graphed all the points in the set, what would the graph look like? *Hint: This graph is not a line.*

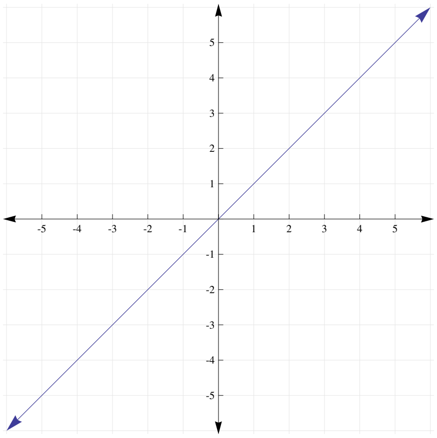
SECTION 3.2 SLOPE

Lines Through the Origin

In this section we will start our study of non-vertical and non-horizontal lines. We begin by focusing only on lines which pass through the origin. In the first exploration we will find the point tester for such a line.

EXPLORATION 1

Consider the graph of the line below. The line bisects the first and third quadrants. This means that the angle between the line and the x-axis is equal to the angle between the line and the y-axis. Identify and label 5 points on the line. What property do all the points share? Write an equation that tests if a point is on the line.



In Exploration 1, for every point on the line, the first coordinate is always equal to the second coordinate. Using algebra we can write, "A point (x, y) is on the line if $y = x$". Notice that the equation in the point tester for this line, unlike those in the previous section, involves both variables. The "slanted" lines we focus on in this section will have equations involving both variables.

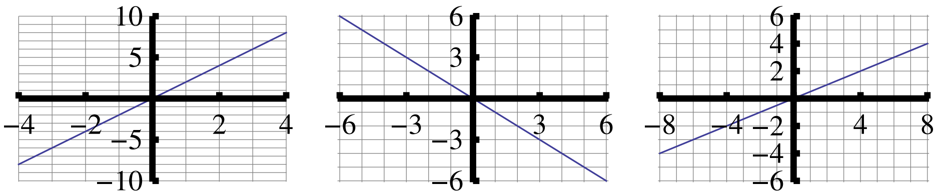
EXPLORATION 2

Use "a point (x, y) is on the line if $y = x$" to test which of the following points are on the line from Exploration 1.

1. $(-1, -1)$
2. $(0, 2)$
3. $(\frac{1}{2}, \frac{1}{2})$
4. $(-3, -6)$
5. $(4, 1)$
6. $(-1, -1.2)$

PROBLEM 1

Consider the lines shown below. For each line,label 3 points, and describe the property the points share. Write a statement using an equation that can be used to test if a point is on the line.



In Problem 1 you may have noticed that there is more than one way to describe the property the points share and more than one way to write the equation. For example, for the third graph, you could have said that the first coordinate is twice the second coordinate, $x = 2y$, or equivalently, the second coordinate is one half of the first coordinate, $y = \frac{1}{2}x$. Either of these descriptions is correct. To compare different lines, however, it is often convenient to write all the equations with y isolated on one side of the equal sign.

PROBLEM 2

For each of the following equations, solve for y in terms of x.

$$2y = x$$
$$\frac{1}{2}y = x$$
$$\frac{1}{3}y = x$$
$$2y = 8x$$

EXPLORATION 3

1. Compare the graphs of the lines represented by the following 5 equations. Do you need to make tables? (Optional: Use a graphing calculator.)

$$y = \frac{1}{2}x$$
$$y = x$$
$$y = 2x$$
$$y = 3x$$
$$y = 4x$$

2. What do you notice about these graphs? How are the graphs similar? How are they different?
3. What is the equation for a line that is steeper than all of them? What is the equation for a line that is flatter than all of them?
4. What predictions about a line can you make by looking at the equation before you plot points? Can you tell by looking at the equations what similarities and differences the lines will have? Explain.

EXPLORATION 4

Compare the graphs of the line described by the following 5 equations. Do you need to make tables? (Optional: Use a graphing calculator.) What do you notice about these graphs? How are the graphs similar? How are they different? How are the graphs similar to those in Exploration 3? How are they different?

$$y = -\frac{1}{2}x$$
$$y = -x$$
$$y = -2x$$
$$y = -3x$$
$$y = -4x$$

As you have seen, equations of lines can be written in several different ways. The form we used the most in this section is to isolate y. When slanted lines pass through the origin they will have the form $y = mx$, where m is number. Let's take a closer look at m in order to come up with its definition and understand its meaning.

PROBLEM 3

Determine the values of m for each of the equations in Explorations 3 and 4. What does the number m tell you about the graph of the line?

When we look at a graph, mathematicians typically read it from left to right. In Exploration 3 we see that the y coordinates of the lines increase as the x coordinates increase. In other words, the line goes up as we read it from left to right. You can also see in Exploration 3 that $m > 0$. Meanwhile, in exploration 4 the y coordinates of the lines decrease as the x coordinates increase. In other words, the line goes down as we read it from left to right.

The Geometric Meaning of Slope

EXAMPLE 1

Graph the straight line whose equation is $y = 2x$. Compute 6 ordered pairs to plot as points on this line with x-values of $-2, -1, 0, 1, 2, 3$.

SOLUTION

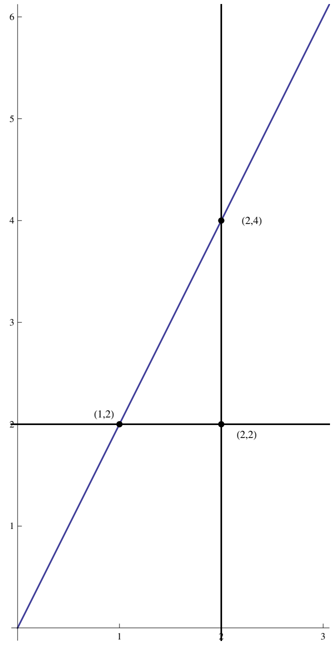
We choose the x-values $-2, -1, 0, 1, 2,$ and 3 because we are looking for a pattern of how the second coordinates are changing as the first coordinates increase by 1 unit. Organizing these points in a table is helpful when looking for patterns.

x	y
-2	$2(-2) = -4$
-1	$2(-1) = -2$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$
3	$2(3) = 6$

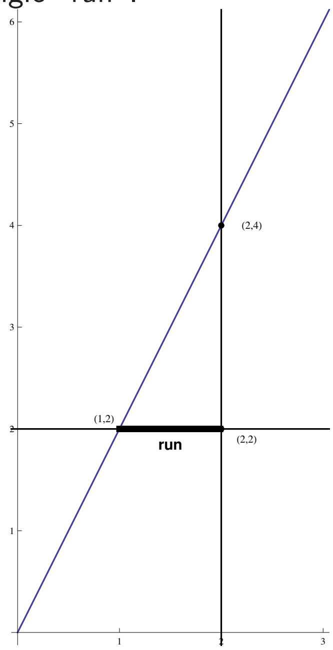
Notice that each time the x-value increases by 1, the y-value increases by 2. Now let's interpret what this means geometrically.

We can start by forming a right triangle using two points from the table:

1. Pick any two points on the line from the table, say $(1, 2)$ and $(2, 4)$. Draw a horizontal line through the first point $(1, 2)$. Draw a vertical line through the second point $(2, 4)$. Label the point where the vertical and horizontal lines intersect. Notice you have formed a right triangle with two vertices at the points $(1, 2)$ and $(2, 4)$ and one vertex at the point of intersection of the horizontal and vertical lines.

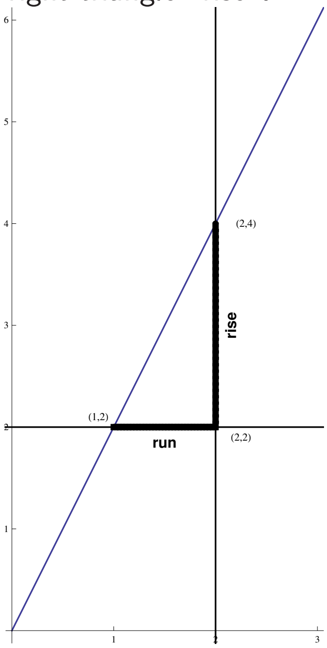


2. Shade the segment of the horizontal line between the first point $(1, 2)$ and $(2, 2)$, the point of intersection you found in step 1. Label this side of the right triangle "run".



Now shade the segment of the vertical line between $(2, 2)$ and $(2, 4)$.

Label this side of the right triangle "rise".



How far did you run? We say that you have run a distance of 1 unit. The "run" refers to horizontal movement to the right.

How far did you rise? We say that you have risen a distance of 2 units. The "rise" refers to vertical movement.

Do this again, moving from the point $(2, 4)$ to the point $(3, 6)$.

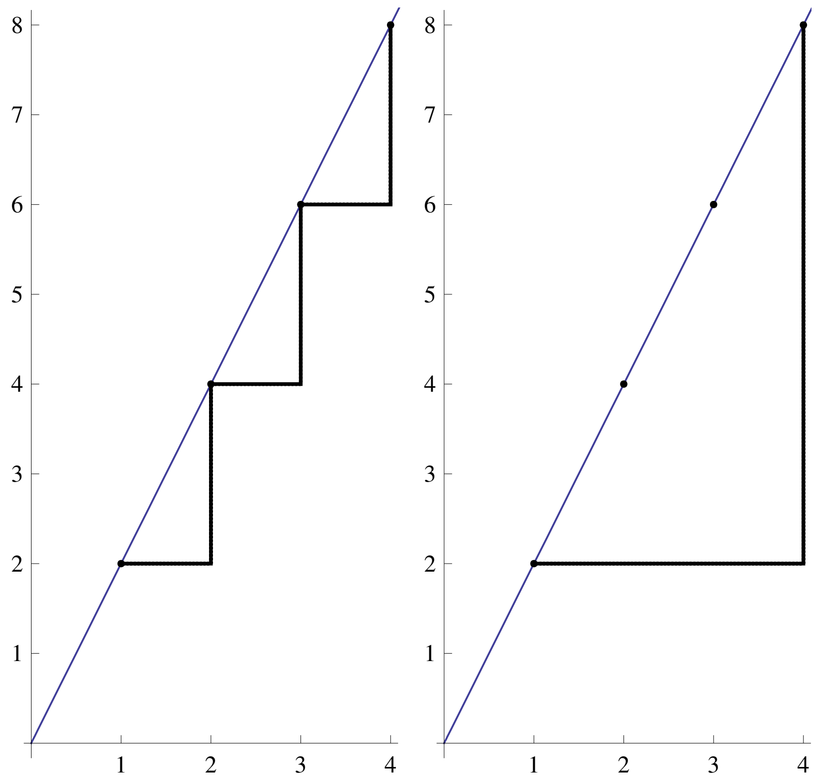
If you started at any point on the graph, each time you run 1 unit to the right, how far do you need to rise?

We define the *slope* of the line as the ratio of the rise to the run. We will use the letter m to represent the slope. In this example, the slope equals $m = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$.

Make a right triangle with the line segment between $(3, 6)$ and $(4, 8)$ as the hypotenuse. Do you get the same slope? Why?

Let's look at this process in a different way. Begin with $(1, 2)$ as the first point and $(4, 8)$ as the second point. What is the rise and what is the run?

You can think of going from $(1, 2)$ to $(4, 8)$ in three steps. First form the triangle with the points $(1, 2)$, $(2, 2)$ and $(2, 4)$. Second, form the triangle with the points $(2, 4)$, $(3, 4)$ and $(3, 6)$. Finally, form the triangle with the points $(3, 6)$, $(4, 6)$ and $(4, 8)$.



Notice that the total run is the sum of the smaller runs, $1 + 1 + 1 = 3$. Similarly, the total rise is the sum of the smaller rises, $2 + 2 + 2 = 6$. However, the ratio is the same since $\frac{6}{3} = \frac{3 \cdot 2}{3 \cdot 1} = \frac{2}{1}$.

We could also calculate the slope by drawing one large triangle. Form the triangle with the points $(1, 2)$, $(4, 2)$, and $(4, 8)$. The run here is 3 and the rise is 6. What do you notice about the triangles? They are all right triangles with the same shape.

Here, the ratio is still the same: $\frac{6}{3} = \frac{2}{1} = 2$.

Now pick two points that are further apart. Calculate the rise and the run. Then compute the ratio of the rise over the run.

In the next example, we look at the geometric meaning of the slope m when it is negative.

EXAMPLE 2

Graph the straight line whose equation is $y = -2x$. Compute 6 ordered pairs to plot as points on this line with x-values of $-2, -1, 0, 1, 2, 3$. Draw a triangle that shows the slope.

SOLUTION

We choose the x-values $-2, -1, 0, 1, 2$, and 3 because we are looking for a pattern of how the second coordinates change as the first coordinate increases by 1 unit.

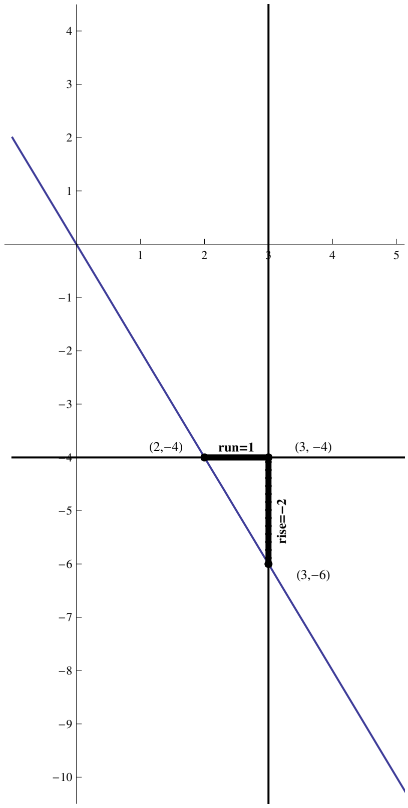
x	y
-2	$-2(-2) = 4$
-1	$-2(-1) = 2$
0	$-2(0) = 0$
1	$-2(1) = -2$
2	$-2(2) = -4$
3	$-2(3) = -6$

Observe that each time the x-value increases by 1, the y-value decreases by 2.

As we did for Example 1, form a right triangle using two points from the table.

1. Pick any two points on the line from the table, say $(2, -4)$ and $(3, -6)$. Draw a horizontal line through the first point $(2, -4)$. Draw a vertical line through the second point $(3, -6)$. Label the point where the vertical and horizontal lines intersect. Notice you have formed a right triangle with two vertices at the points $(2, -4)$ and $(3, -6)$ and one vertex at the point of intersection of the horizontal and vertical lines.
2. Shade the segment of the horizontal line between the $(2, -4)$ and $(3, -4)$, the point of intersection you found in step 1. Label this side of the right triangle "run."
3. Shade the segment of the vertical line between $(3, -4)$ and $(3, -6)$. Label this side of the right triangle "rise."

Notice that the second coordinate of the second point is less than the second coordinate of the first point. So you might think we should call this side of the triangle the "drop", but we don't. We still use the word rise. However, since the line is going **down** as we read from left to right, our rise will be a **negative** number. Why do we do this? Because we want to be able to distinguish between the slopes of lines that go up like the ones in Exploration 3 and lines that go down like the ones in Exploration 4. The sign of the rise will do that for us. So we end up with the following triangle:



How far did you run? Remember the “run” refers to horizontal movement to the right, so the run = 1. How far did you rise? The “rise” refers to vertical movement. In Example 1 our rise was positive, but here our rise is −2.

So the slope of the line equals $m = \frac{\text{rise}}{\text{run}} = \frac{-2}{1} = -2$.

The Definition of Slope

So far in this section we have focused only on lines that pass through the origin. However, slope can be defined for other lines as well. See the figure below. We can use the same ideas from before to find a general formula for the slope of line that goes through two points, (x_1, y_1) and (x_2, y_2).

First, how do you compute the run? In this case we need to run from x_1

to x_2, so the run is $x_2 - x_1$.

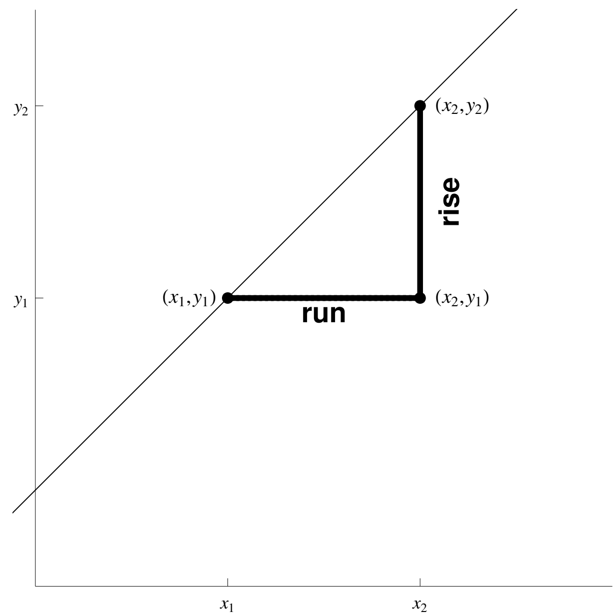
Similarly, how do you find the rise? The rise is $y_2 - y_1$. This leads to the definition of slope:

SLOPE OF A LINE

If (x_1, y_1) and (x_2, y_2) are two points on a line, then the *slope* of the line is the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1},$$

provided $x_2 - x_1 \neq 0$.

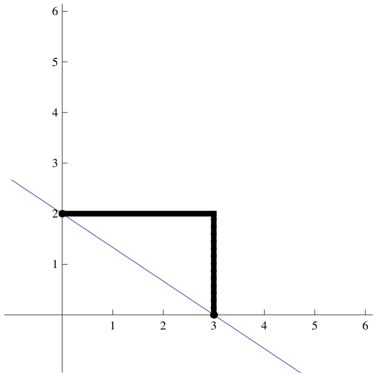


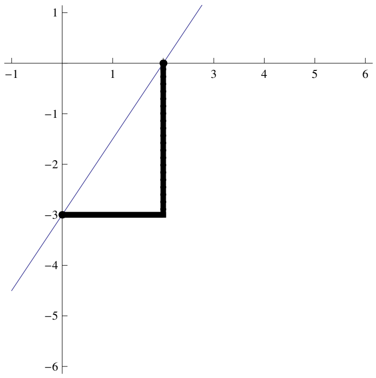
EXAMPLE 3

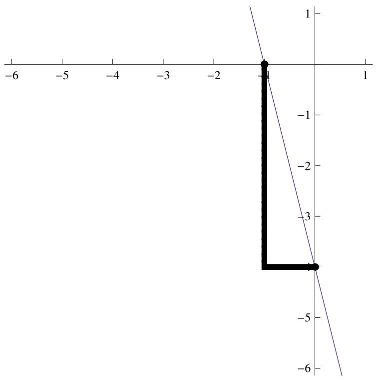
For each pair of points below, find the slope of the straight line using the formula in the definition. Sketch a graph of each line and verify by drawing a triangle that the rise over run gives the same result.

1. $(0, 2)$ and $(3, 0)$
2. $(0, -3)$ and $(2, 0)$
3. $(0, -4)$ and $(-1, 0)$

SOLUTION For each pair of points, we substitute the values of the coordinates into the formula in the definition. Pay close attention to the sign and be careful with your arithmetic. Drawing a graph (or even just picturing one in your head) can help you make sure the sign of your slope is correct. If the line is increasing as we read left to right, the slope must be positive. If the line is decreasing as we read left to right, the slope must be negative.







1. $(0, 2)$ and $(3, 0)$: $m = \frac{2-0}{0-3} = -\frac{2}{3}$
2. $(0, -3)$ and $(2, 0)$: $m = \frac{-3-0}{0-2} = \frac{3}{2}$
3. $(0, -4)$ and $(-1, 0)$: $m = \frac{-4-0}{0-^{-}1} = -4$

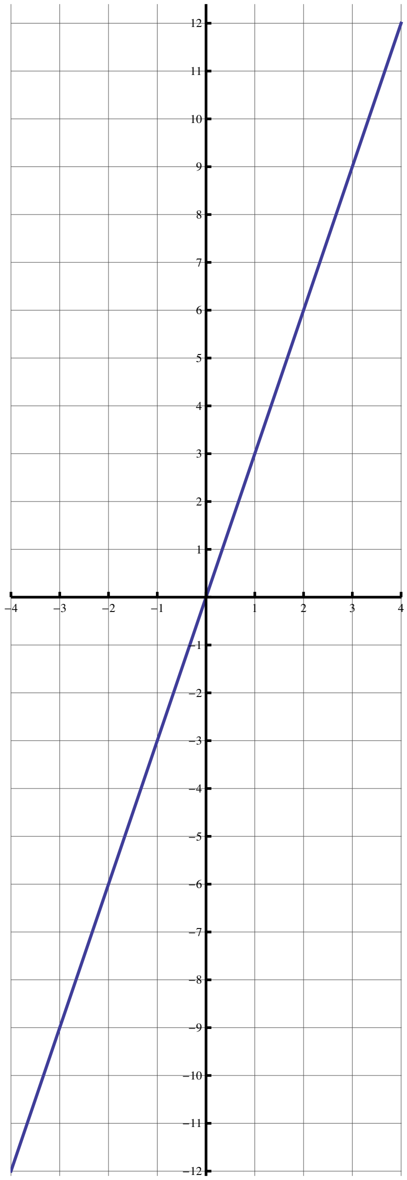
PROBLEM 4

For each pair of points below, find the slope of the straight line with these two points using the formula in the definition. Sketch a graph of each line and verify by drawing a triangle that the rise over run gives the same result.

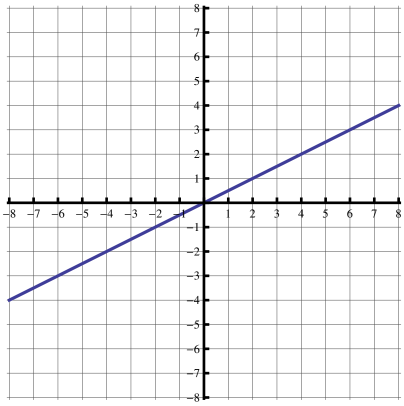
1. $(0, 2)$ and $(1, 5)$
2. $(2, 5)$ and $(4, 9)$
3. $(2, 3)$ and $(4, 1)$
4. $(-4, 3)$ and $(2, 9)$

EXERCISES

1. Find 5 points on the graph of the line shown below. Try to figure out what pattern exists between the coordinates. You may want to use a table to help you see a pattern. What equation can be used to test if a point is on the line?



2. Find 5 points on the graph of the line below. Use a table to help you see a pattern what pattern exists between the coordinates. What equation can be used to test if a point is on the line?



3. We are told that a point (x, y) is on the line if $y = 5x$. For each of the following points decide if the point is on the line or not.
   a. $(1, 5)$
   b. $(-1, -5)$
   c. $(-2, -1)$
   d. $(5, 1)$
   e. $(1, 1)$

4. For each of the following points, consider the line that passes through the origin and the given point. Find the equation of the line. What is the slope of the line?
   a. $(1, 3)$
   b. $(1, -2)$
   c. $(2, 8)$
   d. $(4, 12)$
   e. $(-1, 3)$

5. Graph the line described by each of the equations below.
   a. $y = 3x$
   b. $y = -4x$
   c. $y = \frac{1}{3}x$
   d. $y = 1.5x$
   e. $y = 0$
   f. $x = 3$

6. For each pair of points below, find the slope of the line that passes through the points. Sketch a graph for each pair and verify by drawing a triangle that the rise over run gives the same result.
   a. $(0,3)$ and $(2,5)$
   b. $(-2,3)$ and $(2,7)$
   c. $(-1,-3)$ and $(4,2)$
   d. $(1,7)$ and $(4,-2)$
   e. $(2,6)$ and $(4,0)$
7. Two students Kate and Ally are asked to find the slope of the line passing through two points $(2,6)$ and $(4,12)$. Kate labels the points $(x_1, y_1) = (2,6)$ and $(x_2, y_2) = (4,12)$ and computes the slope as

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12 - 6}{4 - 2} \\ &= 3 \end{aligned}$$

   Ally decided to label the points in the reverse order $(x_1, y_1) = (4,12)$ and $(x_2, y_2) = (2,6)$. If Ally uses the formula for slope, will she get the same answer as Kate? Explain.
8. Suppose a line passes through the origin and the point $(1, m)$ for some positive number m. Sketch a graph of the line. What is the slope? What is its equation? Explain.
9. Suppose a line passes through the origin and the point $(1, -m)$ for some positive number m. Sketch a graph of the line. What is the slope of the line? What is its equation? Explain.
10. **Investigation:**
    a. Place the points $(0,3)$ and $(2,7)$ on a coordinate plane. Draw a line through the points.
    b. Use the formula in the definition to compute the slope of the line.
    c. Carefully draw a line through the origin and parallel to the one you drew in part 10a. Label 3 points on this new line. Find the slope and the equation for this new line. What do you notice?

11. a. Graph the horizontal line described by the equation $y = 0$. Label two points on the line.
    b. Use the formula in the definition to find the slope of the line.
    c. Does the equation fit the form $y = mx$ where m is the slope? Explain.
    d. Think about another horizontal line, this time not through the origin. Look at the formula for slope. Explain why the slope for any horizontal line will be equal to zero.
12. a. Graph the vertical line described by the equation $x = 0$. Label two points on the line.
    b. Use the formula in the definition to try to find the slope of the line. What goes wrong?
    c. Does the equation fit the form $y = mx$ where m is the slope? Explain.
    d. Think about another vertical line, this time not through the origin. Look at the formula for slope. Explain why the slope for any vertical line will be undefined.
13. Draw the line that passes through the points $(1, 4)$, $(3, 3)$.
    a. Use the formula in the definition to compute the slope.
    b. Draw a right triangle whose hypotenuse lies on the line with a run of 2. Draw a another triangle, also with the hypotenuse on the line, but with a run of 4.
    c. Verify that that rise over run equals the slope.
    d. What do you notice about the triangles?
14. **Ingenuity:**
    All lines that pass through the origin are described by equations of the form $y = mx$ except for one. Which one? Explain.

SECTION 3.3 Slope and Proportions

EXPLORATION 1

A restaurant makes and sells a famous dish that contains rice and beans. The ratio of rice to beans in its secret recipe is 1 : 2, and the ratio of beans to rice is 2 : 1.

Beans (cups) x	Rice (cups) y

1. Fill in the table of possible amounts of rice and beans that the chef uses to make the dish.
2. Use the numbers in the table as coordinates of points. Make a graph using the data points. For each point, the number of cups of beans is the x-coordinate and the number of cups of rice is the y-coordinate. Describe the graph.
3. For each (x, y) pair compute the ratio of $\frac{y}{x}$. What do you notice about this ratio?
4. Write the equation of the line through the points.
5. What is relationship between slope of the line and the ratio from part 3?
6. Use the equation of the line to find:
   a. If we want to use 13 cups of beans, how many cups of rice do we need?
   b. If we want to use 10 cups of rice, how many cups of beans do we need?
   c. If we want to use 15 cups of rice and beans altogether, how many cups of beans do we need?
7. Explain how you could use the graph to answer 6a and 6b.

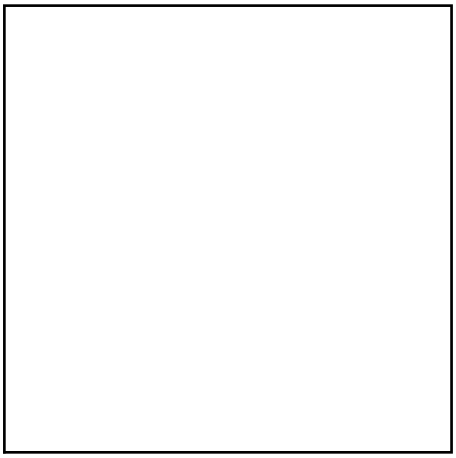
PROBLEM 1

Consider a basic recipe for vinaigrette which is 3 parts oil to 2 parts vinegar.

1. Make a table of amounts of oil (x) and vinegar (y) in tablespoons.
2. Compute the ratio of $\frac{y}{x}$ for each pair in your table.
3. Graph the ordered pairs in your table.
4. Write the equation of the line through the points.
5. What is the relationship between the slope and ratio from part 2?

PROPORTIONAL
Two variables have a proportional relationship if their ratio is always the same. We represent this relationship algebraically with the equation $\frac{y}{x} = m$ or $y = mx$, where x and y are the variables and m is the constant ratio. Sometimes this kind of relationship is also called direct variation or we say y varies directly as x.

EXAMPLE 1



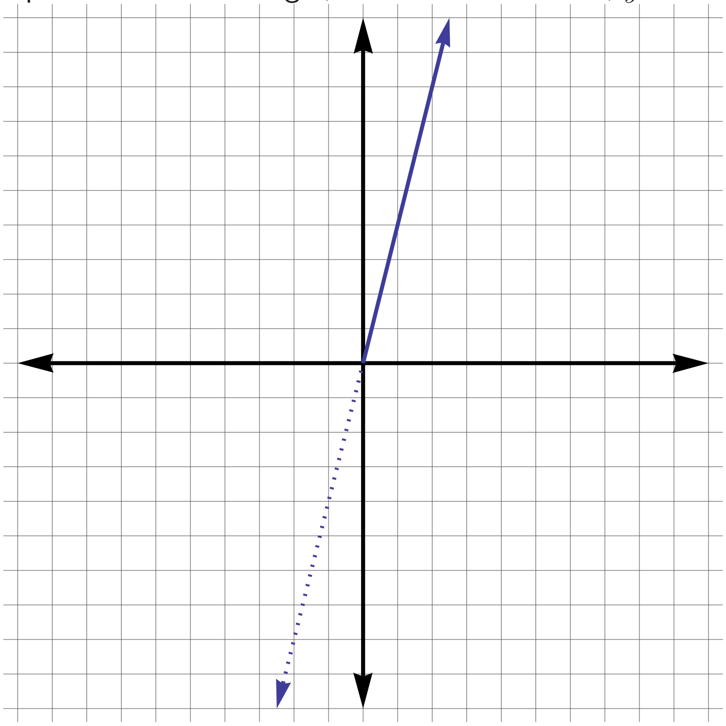
x

Let $y =$ perimeter of the square shown above.

1. Write an equation for the perimeter of the square in terms of x.
2. For a square, is the relationship between the perimeter and the length of a side proportional?
3. Make a graph of the perimeter versus the length of a side. Does the graph cross at the origin? Explain why this makes sense.
4. If we double the length of the sides, how does this affect the perimeter? What if we triple the length of sides?

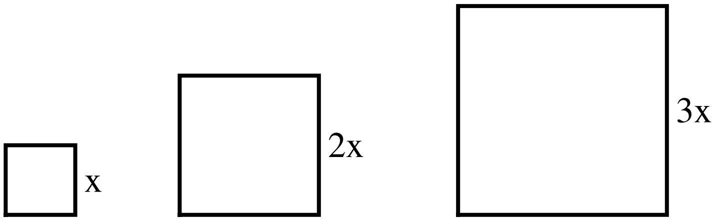
SOLUTION

1. The square has 4 sides each of length x, so the perimeter is $y = 4x$.
2. Yes, the perimeter is 4 times the length of the side. If we divide both sides of the equation $y = 4x$ by x, we see that $\frac{y}{x} = 4$.
3. The graph crosses at the origin, because when $x = 0$, $y = 4 \cdot 0 = 0$.



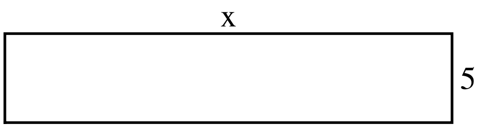
4. Consider the three squares shown below. Since we have three different

squares, we need to use three different symbols for the perimeter. Let $y_1 = 4x$ be the perimeter of the small square, $y_2 =$ the perimeter of the middle square and $y_3 =$ the perimeter of the large square. Notice the sides of the middle square are twice as long as the small square. The biggest square has sides three times as long as the small square. The perimeter of the middle square is four times its length or $y_2 = 4 * (2x) = 8x = 2(4x) = 2y_1$. The perimeter of the largest square is $y_3 = 4 * (3x) = 12x = 3(4x) = 3y_1$. Hence when we double the length of the side, the perimeter doubles. When we triple the length, we triple the perimeter.



Notice, if the side of the square is multiplied by any positive scale factor k, the perimeter is also scaled by that factor. $y_{scaled} = 4(kx) = 4kx = k(4x) = ky_1$

EXPLORATION 2

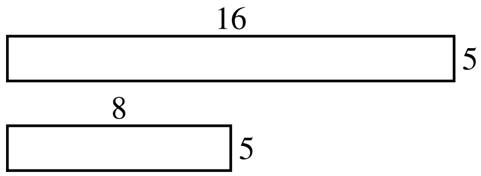


Let y be the perimeter of the rectangle shown above.

1. Make a table of possible values of x and the corresponding perimeter y.

length x	perimeter y

2. Is the perimeter proportional to x? For each pair (x, y) compute the ratio $\frac{y}{x}$. How does this compare to what happened in Example 1?
3. Plot the points on a grid. Notice that the points fall along a line. Draw the line. Does the line pass through the origin? Explain why this makes sense.
4. What happens to the perimeter if we double the length but leave the width the same? The rectangles shown below have width 5. One has length 8 and the other length 16. Determine the perimeter for each rectangle. Did the perimeter double?



Let's summarize some key results about proportional relationships:

1. The graph of the relationship is a line that passes through the origin.
2. The slope of the line is equal to the constant ratio between the two variables. Sometimes this is called the *constant of proportionality*.
3. If one variable is multiplied by a scale factor, the other variable is multiplied by the same scale factor.

EXPLORATION 3

Think of different contexts from your previous math classes or from your experience outside of school. Identify 5 relationships that are proportional.

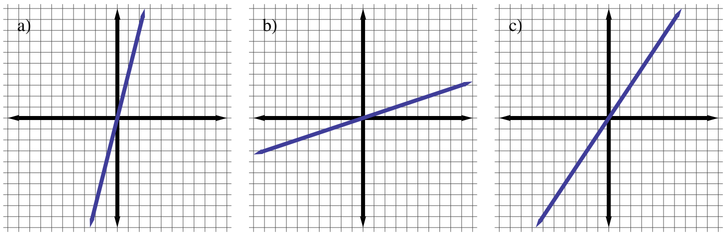
PROBLEM 2

Graph each of the following equations. Then determine if the relationship is proportional or not. Explain.

1. $y = 2x$
2. $y = \frac{2}{3}x$
3. $y = x^2$
4. $y = 2x + 3$

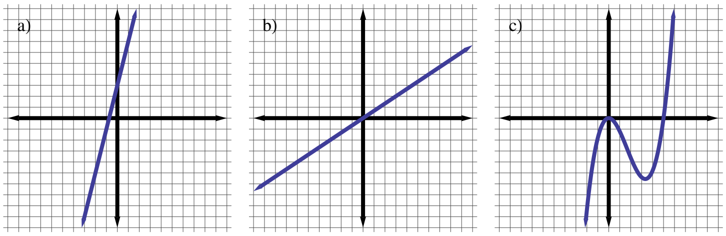
PROBLEM 3

Explain why all the graphs below represent proportional relationships. For each graph, determine the slope and write the equation of the line.



PROBLEM 4

For each of the graphs below determine if the graph represents a proportional relationship or not.



EXERCISES

1. Each of the following tables represent a proportional relationship between x and y. Complete each table, determine the slope and graph the line.

a)

x	y	$\frac{y}{x}$
0	0	undef.
1	3	
2	6	
3	9	
4	12	

b)

x	y	$\frac{y}{x}$
0	0	undef.
2	5	
4	10	
6	15	
8	20	

c)

x	y	$\frac{y}{x}$
0	0	undef.
1	3	3
2		
3		
4		

2. For each of the following complete the table, determine if the relationship is proportional.

a)

x	y	$\frac{y}{x}$
0	-8	undef.
1	-3	
2	2	
3	7	
4	12	

b)

x	y	$\frac{y}{x}$
0	0	undef.
2	1	
4	3	
6	6	
8	10	

c)

x	y	$\frac{y}{x}$
0	0	undef.
1	8	
2	16	
3	24	
4	32	

3. Magdalena has a skateboard business. The following table shows the cost y in dollars of manufacturing x of the most popular brand of skateboard.

Number x	Cost (\$) y
5	95
10	190
15	285
20	380
25	475

a. Is the cost proportional to the number of skateboards produced? Explain.
b. Plot the (x, y) ordered pairs.
c. Find the equation of the line that passes through the points.

d. Use your equation to predict how much it will cost to produce 50 skateboards.

4. Sales tax is computed as a percentage of pre-tax total. In Alamo Heights, Texas, the sales tax rate is 8%. Let $x =$ the pre-tax total and $y =$ sales tax.

a. Make a table of 5 different x and y values.
b. Graph the values in your table.
c. Explain why sales tax is proportional to the pre-tax total.

5. Gael is exploring the volumes of 3-dimensional objects in his math class. He and his partners have 4 pairs of objects. All the objects have the same height of 5 cm. Each pair has a cylinder and a cone with the same radius. There are four different radii. Gael's group fills each object with water and records the volume in ml in table. Use the table to answer the questions below.

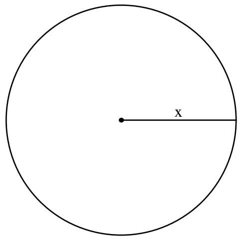
Radius (cm) r	Volume of Cylinder (ml) y	Volume of Cone (ml) z
2	63	21
4	251	84
8	1005	335
10	1570	523

a. Is the volume of a cylinder with height 5 cm proportional to its radius? Explain.
b. Is the volume of a cone with height 5 cm proportional to its radius? Explain.
c. The volume of a cone is proportional to the volume of a cylinder with same height and radius. Use the data in the table to determine the constant of proportionality?
d. Find the formulas for the volume of a cone and a cylinder. Explain how the formulas relate to your answer in part 5c.

6. Amanda owns an import/export business along the Texas-Mexico border. Let x represent the amount of her profit in US$. Let $y =$ her profit in Mexican pesos. The exchange rate is 13 pesos to 1 dollar.

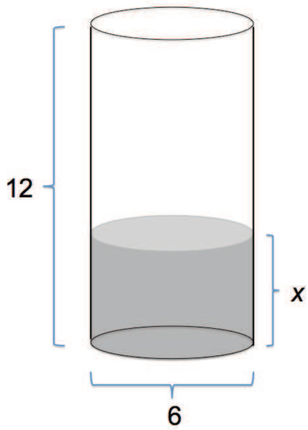
a. If x is negative, what does this mean?
b. Is the relationship between x and y proportional? Explain.
c. Write an equation for the relationship between x and y.

7. Let $y =$ circumference of the circle shown below.



a. Write an equation for the circumference of the circle in terms of x.
b. For a circle, is the relationship between the circumference and the length of the radius proportional? Explain.
c. Write an equation for the area of a circle in terms of x.
d. For a circle, is the relationship between the area and the length of the radius proportional? Explain.

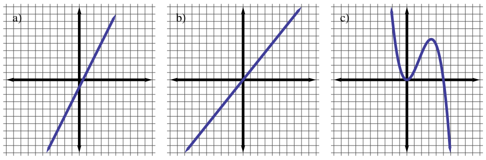
8. A cylindrical glass vase is 6 inches in diameter and 12 inches high. There are x inches of sand in the vase, as shown below.



a. What is the volume of the sand if the sand is 2 inches high? That is if $x = 2$.
b. Write an equation for the volume of sand in the vase in terms of x.
c. For this situation is the volume of sand proportional to the

height of the sand? Explain.

9. Which of the following graphs represent proportional relationships? Explain.



SECTION 3.4 SLOPES AND INTERCEPTS

In this section we will continue our study of non-vertical lines. Now we will find some general forms for lines even if they don't pass through the origin. In the first exploration we find the equation for such a line.

EXPLORATION 1

1. Plot the points $(2, 3)$ and $(5, 9)$. Draw a straight line passing through both of these points. Consider the points on this line and complete the table below.

First coordinate	Second coordinate
-2	
-1	
0	
1	
2	3
3	
4	
5	9
6	
7	
8	

2. Looking at the points on this line, what pattern do you see? For each ordered pair (x, y) in this table, describe in words the relationship between x and y. Find an equation that can be used to test if a point is on the line.

In Exploration 1 we notice a pattern or relationship between the coordinates. For example, if $x = 2$, then $y = 2 \cdot 2 - 1 = 3$, if $x = 3$ then $y = 2 \cdot 3 - 1$ and if $x = 5$ then $y = 2 \cdot 5 - 1 = 9$. We can describe this relationship as "the second coordinate is always equal to one less than twice the first coordinate", or express it algebraically as "a point (x, y)

is on the line if $y = 2x - 1$."

EXPLORATION 2

Use the fact that "a point (x, y) is on the line if $y = 2x - 1$" to determine which of the following points are on the line from Exploration 1.

1. $(-1, -1)$
2. $(0, 2)$
3. $(\frac{1}{2}, 0)$
4. $(-3, -6)$
5. $(4, 7)$
6. $(-1, -1.2)$

PROBLEM 1

Use the equation for the line from Exploration 1 to answer the following:

1. What point on the line has 10 as its x-coordinate?
2. What point on the line has 25 as its y-coordinate?

EXPLORATION 3

1. On the same coordinate plane, graph the lines described by each of the 3 equations below. [Optional: Use graphing calculator].

$$y = 2x + 3$$
$$y = 2x - 5$$
$$y = 2x$$

2. Comparing the graphs of each of these lines, what do you notice? What is similar? What is different?
3. Can you come up with another parallel line to put on the graph?
4. What do you notice about the equation of the new line you created and the equations from part 1? What is the same? What is different?
5. Now consider the following equations. What can you say about their graphs? What is similar? What is different? How are these lines

different from the ones above?

$$y = -3x + 2$$
$$y = -3x - 1$$
$$y = -3x$$

We will see that all non-vertical lines can be written in the form $y = mx + b$ where m is the slope and b is a number. Let's explore the meaning of b.

PROBLEM 2

Determine the values of m and b for each of the equations in Exploration 3. What does the number m tell you about the graph of the line? What does b tell you about the graph of the line?

The ideas from this problem generalize to an important theorem about parallel lines.

THEOREM 3.2: SLOPES OF PARALLEL LINES
The line described by the equation $y = mx + b$ will be parallel to any line with the equation $y = mx + c$ with $b \neq c$.

EXPLORATION 4

Graph each of the lines determined by the equations below. For each line, find the slope m and label the point where the line crosses the y-axis.

1. $y = 2x - 2$
2. $y = -3x + 4$
3. $y = \frac{1}{2}x + 2$

SLOPE-INTERCEPT FORM

We say $y = mx + b$ is the *slope-intercept form* of the line, where the number m is the slope of the line.

The number b is the y-*intercept* of the line, that is the value of the y-coordinate of the point where the line intersects the y-axis.

For example, the line described by $y = 3x + 2$ has slope $m = 3$ and the y-intercept is $b = 2$.

Let's explore this. Where do the y-axis and the line $y = mx+b$ intersect? The equation for the y-axis is $x = 0$. If $x = 0$, then $y = m \cdot 0 + b = b$. Thus the point $(0, b)$ is the point where the line intersects the y-axis.

EXAMPLE 1

In Exploration 1, the straight line goes through the points $(2, 3)$ and $(5, 9)$. Find the y-intercept of the line and write the equation in slope-intercept form.

SOLUTION

Recall that the slope is $m = 2$. This means that the slope-intercept form of the equation of the line is:

$$y = 2x + b.$$

The point $(2, 3)$ lies on the line and thus satisfies this equation. We can now determine b by replacing x by 2 and y by 3:

$$\begin{aligned} (3) &= 2 \cdot 2 + b \\ 3 &= 4 + b \\ -1 &= b \end{aligned}$$

If we compare with the results from Exploration 1, we see that this confirms that the point where the line crosses the vertical axis is $(0, -1)$. The graph and the calculations using the equation must always agree. So we can approach each problem using the graph, the equation or both. Which is best? It depends on the nature of the problem and which approach will help us understand the problem better. In many circumstances, we will understand the problem best if we use both approaches and go back and forth as needed.

You might wonder if using a different point would make any difference. Let's see if it does. Using the point $(5, 9)$, we get

$$\begin{aligned} 9 &= 2 \cdot 5 + b \\ 9 &= 10 + b \\ -1 &= b \end{aligned}$$

We get the same result.

EXPLORATION 5

Now let's explore a graphical method for finding the y-intercept of the line which has slope $m = 2$ and goes through point $(-3, 1)$.

1. Plot the point $(-3, 1)$ on a coordinate grid. Since the slope is positive, explain why the y-intercept of the line must be greater than 1.
2. Write the slope as a ratio of the rise and the run. What will the rise be for a run $= 1$?
3. We will use "slope" triangles to step from the point towards the y-axis. Make a slope triangle starting at the point $(-3, 1)$ with a run $= 1$. This triangle leads to another point on the line. What are its coordinates?

4. Continue drawing triangles, each with run $= 1$ until you reach the y-axis. What are the coordinates of the point where the line crosses the y-axis? What is the equation of the line?

EXAMPLE 2

Consider the two points $(1, 5)$ and $(2, 9)$. Find the slope, y-intercept and the equation of the straight line containing these two points.

SOLUTION

We will use the formula for slope to find the equation of the line. First we compute the slope of the line:

$$m = \frac{9-5}{2-1} = 4$$

Now let's find the y-intercept. We want to write the equation of the line in the slope-intercept form: $y = mx + b$. Since we know that the value of m is 4, then the equation must be

$$y = 4x + b$$

So let's compute b. How do we do this? Since the point $(1, 5)$ is on the line, it must satisfy the equation $y = 4x + b$. So when we substitute $y = 5$ and $x = 1$ into this equation, we get

$$5 = 4(1) + b$$
$$5 = 4 + b$$
$$b = 1$$

Hence the equation of the line is $y = 4x + 1$.

PROBLEM 3

Suppose a line has slope $\frac{1}{2}$ and contains the point $(3, 4)$. What is the y-intercept of this line? What is an equation for this line?

Now we will look at another way to approach the problem in Example 2.

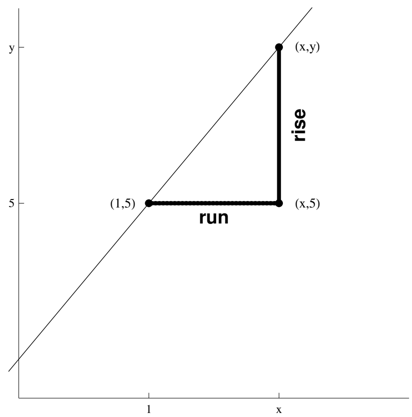
EXAMPLE 3

Consider the line which passes through the two points $(1, 5)$ and $(2, 9)$. Use the formula for the slope and an arbitrary point (x, y) on the line to find the equation of the line.

SOLUTION We start the same way we did in Example 2, by computing the slope using the formula.

$$m = \frac{9 - 5}{2 - 1} = 4$$

Now use one of the points, say $(1, 5)$, the slope $m = 4$ and let (x, y) be any other point on the line.



Form a triangle and compute the ratio of the rise over the run using the points $(1, 5)$ and (x, y). From this, we get the equation

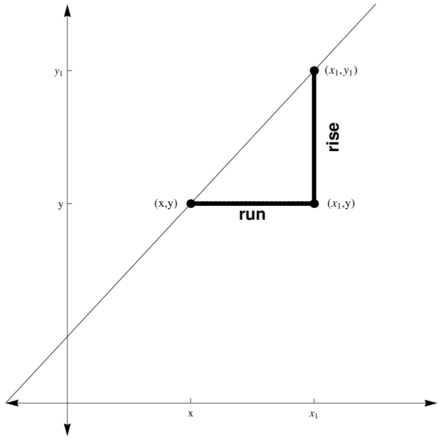
$$\begin{aligned} m &= \frac{y-5}{x-1} \\ 4 &= \frac{y-5}{x-1} \\ 4(x-1) &= y-5 \\ y-5 &= 4(x-1) \end{aligned}$$

We say this equation of the line is in *point-slope form*, because the coordinates of a given point $(1, 5)$ and the slope, 4, appear in the equation.

Now solve for y to get the *slope-intercept form*:

$$\begin{aligned} y-5 &= 4(x-1) \\ y-5 &= 4x-4 \\ y &= 4x-4+5 \\ y &= 4x+1 \end{aligned}$$

Now we explore a more general situation. Suppose you are given one point on a line (x_1, y_1) and the slope is m. How can you use this point and the slope to find the equation of the line?



Locate the point on the line and then pick any other point on the line and label it (x, y). Then you can use (x_1, y_1) and (x, y) to compute the slope. What is the run? What is the rise?

Computing the slope, we get the ratio

$$m = \frac{y-y_1}{x-x_1}.$$

Since the slope is always the same for any two points on the line, this equation is satisfied by any point (x, y) on the line. Multiplying both sides of this equation by $x - x_1$, you obtain:

$$m(x - x_1) = (y - y_1).$$

This gives the general definition of the point-slope form:

POINT-SLOPE FORM

$(y - y_1) = m(x - x_1)$ is the *point-slope form* of the line, where the number m is the slope of the line and the line passes through the point (x_1, y_1).

For example, if the line with slope $m = -3$ passes through $(2, 5)$, its equation in point-slope form is $(y - 5) = -3(x - 2)$.

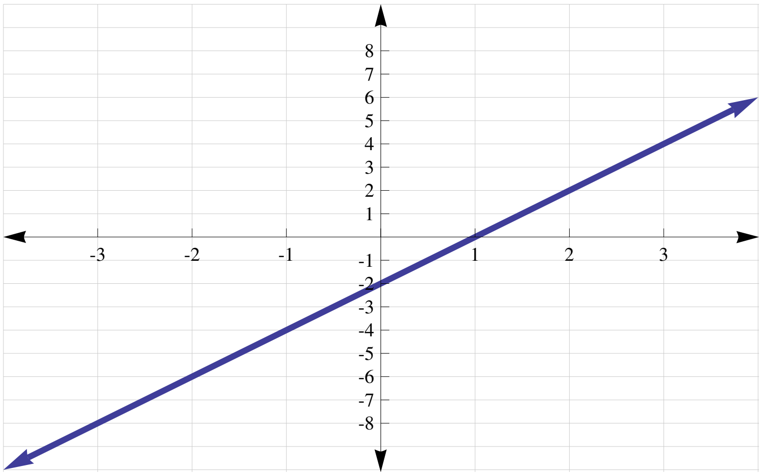
PROBLEM 4

A line has slope $\frac{1}{3}$ and passes through $(2, 1)$.

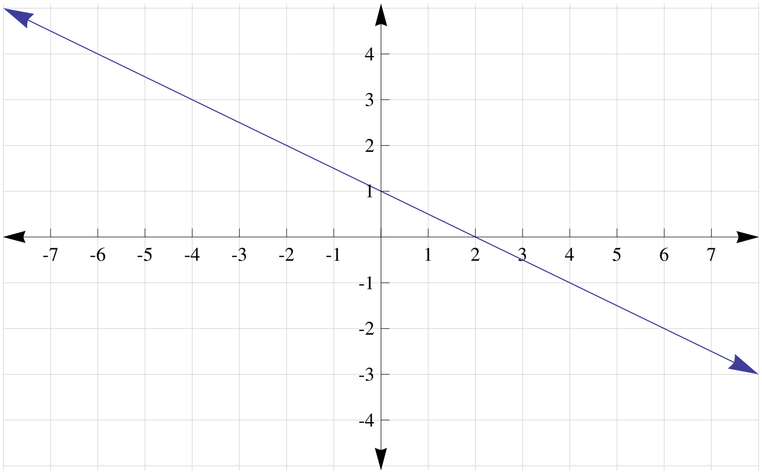
1. Write the equation of the line in point-slope form.
2. Write the equation of the line in slope-intercept form.

EXERCISES

1. Graph the line described by the equation $y = 2x + 3$. For each of the following points decide if the point is on the line or not.
   a. $(1, 5)$
   b. $(-1, -5)$
   c. $(1, -5)$
   d. $(5, -1)$
   e. $(1, 1)$
2. Find 5 points on the graph of the line on the next page. Try to discover a pattern between the coordinates. Write the equation of the line.



3. Find 5 points on the graph of the line below. Try to discover a pattern between the coordinates. Write the equation of the line.



4. For each table of coordinate pairs, determine if the points are on a line. If they are on a line, write the equation of the line in slope-intercept form.

a)	
x	y
-2	5
-1	3
0	1
1	-1
2	-3

b)	
x	y
3	4
4	7
5	13
6	22
7	34

c)	
x	y
-2	1
0	4
2	7
4	10
6	13

5. For each of the equations below, determine the slope and y-intercept of the line described by the equation.

a. $y = 4x + 8$

b. $y = -2x + 5$

c. $y = \frac{1}{2}x + \frac{1}{3}$

d. $y = -2x - \frac{5}{2}$

e. $y = x$

f. $y = -x$

g. $x + y = 3$

h. $2x + y = -4$

6. For each line described by the equations listed in Exercise 5, decide if the line goes up or goes down as you read it left to right.

7. For each line described by the equations listed in Exercise 5, which quadrants does the line cross? How can you tell?
In Exercises 8 and 9 use the graph to get a general sense of the slope by plotting points.

8. For each pair of points below, graph the points, draw a straight line through the two points and then answer the following questions: Is the slope positive or negative? If the slope is positive, is the slope greater than 1, equal to 1, or less than 1? Is the y-intercept positive or negative?
   a. $(0, 3)$ and $(2, 5)$
   b. $(-2, -4)$ and $(2, 0)$
   c. $(-1, 3)$ and $(2, -9)$
   d. $(1, -7)$ and $(4, 8)$
   e. $(2, 6)$ and $(4, 0)$

9. For each pair of points below, graph the points, draw a straight line through the two points and then answer the following questions: Is the slope positive or negative? If the slope is positive, is the slope greater than 1, equal to 1, or less than 1? Is the y-intercept positive or negative?
   a. $(2, 1)$ and $(4, 2)$
   b. $(2, 1)$ and $(4, 6)$
   c. $(4, 3)$ and $(2, -1)$
   d. $(1, 1)$ and $(-1, -5)$

e. $(3, -2)$ and $(6, -3)$
f. $(4, 5)$ and $(8, 8)$
g. $(3, -3)$ and $(6, 2)$
h. $(1, 2)$ and $(3, 3)$

10. For each pair of points in Exercise 8 use the formula for slope to compute the slope m. Then use one of the two methods discussed in Examples 2 and 3 to find the y-intercept b. Finally, write the equation of the line in the slope-intercept form: $y = mx + b$.

11. For each pair of points in Exercise 9 use the formula for slope to compute the slope m. Then use one of the two methods discussed in Examples 2 and 3 to find the y-intercept b. Finally, write the equation of the line in the slope-intercept form: $y = mx + b$.

12. Find the equations of three different lines with slope 5. Graph them on the same coordinate system. What do you notice?

13. Find the equations of three different lines with slope -2. Graph them on the same coordinate system. What do you notice?

14. Theorem 3.2 says that the line $y = mx + b$ will be parallel to any line with the same slope m and different y-intercept b. Explain why the y-intercepts have to be different. Why is the line $y = mx + b$ not parallel to a line with the same slope m and same y-intercept b?

15. a. Choose 4 points, one in each quadrant. Plot the 4 points on a coordinate grid.
    b. For each point sketch the graph of the line that passes through the point and has slope $m = 2$. Find the equation of each line. For two points use use the algebraic method described in Example 2 and for the other two points use the graphical method described in Exploration 5. Which method is easier?
    c. Make another graph of the points you chose in part 15a. For each point sketch the graph of the line that passes through the point and has slope $m = -\frac{1}{2}$. Find the equation of each line. For two points use use the algebraic method described in Example 2 and for the other two points the graphical method described in Exploration 5. Which method is easier?

16. Find the equations of the lines that contain the following pairs of points. Find the y-intercepts by using the point-slope form.

a. $(1, 2)$ and $(5, 6)$
b. $(-2, 3)$ and $(1, -3)$
c. $(1, -3)$ and $(-2, 6)$
d. $(2, 1)$ and $(4, 2)$

17. Find the equations of the lines that contain the following pairs of points. Find the y-intercepts by using the slope-intercept form.
a. $(3, 2)$ and $(6, 3)$
b. $(3, -3)$ and $(9, 1)$

18. Find the equation for the line which contains the points $(1, 3)$ and $(3, -1)$. Calculate the value of the slope and the y-intercept. Sketch a graph to verify your values of m and b are correct.

19. Find the equation for the line which contains the points $(2, 1)$ and $(6, -3)$. Calculate the value of the slope and the y-intercept. Sketch a graph to verify that your values of m and b are correct.

20. Find the value of the y-intercept for each line described below. Write the equation of the line in slope-intercept form.
a. Slope $m = 2$ and contains $(3, 10)$.
b. Slope $m = 2$ and contains $(2, 7)$.
c. Slope $m = -2$ and contains $(-1, 3)$.
d. Slope $m = \frac{2}{5}$ and contains $(5, 6)$.
e. Slope $m = \frac{3}{2}$ and contains $(2, 1)$.
f. Slope $m = -2$ and contains $(0, 3)$.
g. Slope $m = -\frac{1}{2}$ and contains $(0, -2)$.

21. Find the equation of the line that satisfies the given information. Compute the value of b by using the point-slope form or the slope-intercept form.
a. contains $(-2, 1)$ and $(3, 6)$
b. contains $(3, 4)$ and has slope -1
c. contains $(3, 6)$ and $(6, 8)$
d. has slope $-\frac{3}{2}$ and contains $(-2, 4)$
e. contains $(4, 6)$ and has y-intercept 3

f. contains $(1, \frac{5}{2})$ and $(3, \frac{13}{2})$

g. contains $(1, 1)$ and $(6, 3)$

22. Graph the line described by the equation $y = \frac{x+4}{2}$. Determine the slope of this line. What is its y-intercept?

23. Determine the slope and y-intercept for the lines described by the following equations.
    a. $y = \frac{2x-6}{3}$
    b. $y = \frac{3x+5}{4}$

24. Find the equation of the line that satisfies the given information.
    a. contains $(3, 7)$ and has y-intercept $b = 1$
    b. contains $(-2, 4)$ and has y-intercept $b = 2$
    c. contains $(6, 8)$ and has y-intercept $b = 4$
    d. contains $(-5, -4)$ and has y-intercept $b = -1$
    e. contains $(4, -6)$ and has y-intercept $b = 4$

25. Review the definition of proportional relationship from Section 3.3. Which of the following equations represent a proportional relationship between x and y? Explain.
    a. $y = 5x$
    b. $2y = 7x$
    c. $y = 3x + 2$
    d. $y = -2x$
    e. $y = \frac{1}{3}x + 2$
    f. $3y + x = 6$

26. Suppose x and y satisfy the equation $y = mx + b$ with $m \neq 0$ and $b \neq 0$.
    a. What will the graph of the line look like?
    b. Explain why the x and y are not proportional.

27. **Investigation:**
    Let's find out if the two lines described by the equations $y = 3x - 2$ and $y = -2x + 4$ intersect. If so, where do they intersect?
    a. Graph the two lines on the same coordinate grid. Notice that the lines intersect. Estimate the coordinates of the point of intersection.
    b. If we call the point of intersection (a, b), then this point must

be on both of the lines and must satisfy both equations. This means that $b = 3a - 2$ and $b = -2a + 4$. Notice the left hand side of both equations is the same number b. So we can write $3a - 2 = -2a + 4$.

i. Solve this equation for a.
ii. Compare the answer to your estimate of the x coordinate from 27a.
iii. Once we know the x-coordinate of the point of intersection we can use either equation to find the y-coordinate. Substitute for x in each equation and see if you get the same answer for the y-coordinate.
iv. Compare this answer to your estimate of the y coordinate from 27a.

28. **Ingenuity:**
Point A with coordinates $(8, 5)$ lies on a line with a slope of $\frac{3}{4}$. Find the following coordinates (*hint: use Pythagorean Theorem.*):
a. two points on the same line that are exactly 5 units from A.
b. a point on the line in the third quadrant that is 15 units from A.
c. a point on the line in the first quadrant that is 10 units from A.
d. a point on the line in the first quadrant that is 12 units from A.
e. a point on the line that is d units from A.

SECTION 3.5 FUNCTIONS VS. EQUATIONS

There are important connections between lines and their equations and the linear functions that we explored in Chapter 2. For example, the graph of the function f described by the formula $f(x) = 4x$, is also the graph of the line that satisfies the equation $y = 4x$. Which is the best way to think of this line? Although it might be a little confusing at times, we will use both of these ways of thinking of lines. In this section we will explore the connections between linear functions and equations and develop some sense when one might be easier to use one method over the other.

EXAMPLE 1

Susan has a bag of nickels with x nickels. Let $V(x)$ be the value in cents of the bag.

1. Determine the formula of the linear function $V(x)$. Specify its domain.
2. Make a graph of the function.
3. Find the equation of the line that passes through the points of the graph of the function.
4. Graph the line from part 3. How is the graph of this line different from the graph of the function? How is it the same?

SOLUTION

1. The function describing the value is $V(x) = 5x$. Recall that the domain is set of possible inputs. The number of nickels must be a whole number. So the domain is the set of whole numbers.
2. The graph of a function is the plot of all the possible input-output pairs: $(0, 0), (1, 5), (2, 10), \ldots$.
3. Now find the equation of the line that passes through points in the graph of the function. Since the graph must pass through $(0, 0)$ we

can see that y-intercept will be 0. We can choose any two points to find the slope. For example we could choose $(0,0)$ and $(1,5)$ and compute the slope to be

$$m = \frac{5-0}{1-0} = 5$$

Hence the equation of the line is $y = 5x$. This suggests a much easier way to find the equation of the line: simply set $y = V(x)$. In fact, this always works!

4. The graph of the line passes through the points in the graph of the function, but includes many more points. For example, the points $(-1,-5)$ and $(\frac{1}{2},\frac{5}{2})$ belong to the line but are not included in the graph of the function.

For a linear function given by $f(x)$, it is easy to find the corresponding equation of the line. The line given by the equation $y = f(x)$ will pass through all the points of the graph of the function, but may include many more points depending on the domain of the function. However, the graph of the line sometimes makes it easier to see the pattern between the inputs and the outputs. In Example 1, the domain of the function V was the whole numbers $\{0,1,2,\ldots\}$ while for the line $y = 5x$ any number x was possible (including negative and decimal numbers). If the domain of the function is restricted to a list of numbers, we say that the domain is *discrete*. If the domain includes all numbers, or all numbers in an interval, we say the domain is *continuous*. When the domain is continuous, you can draw the graph of the function without lifting your pencil. When the domain is discrete, the graph looks like a set of disconnected points.

PROBLEM 1

For each of the following, determine the formula for the linear function and specify the domain. From the context, determine if the domain is continuous or discrete. Sketch a graph of the function. Also, find the equation of the corresponding line and sketch the line.

1. a. Each baseball costs \$6. What is the formula for the cost $C(x)$ of x baseballs?
   b. In the bulk section of the grocery store, fancy granola costs \$6 per pound. What is the formula for the cost $G(x)$ of x pounds of granola?
2. a. A movie theater charges \$6.50 for the latest movie. How much does it cost a group of x friends to see the movie? (What is the function?)
   b. Ramon's pet snail can crawl 6.5 centimeters per minute. What is the formula for the distance $d(x)$ that the snail can crawl in x minutes?

As you can see in Problem 1, we can use either the function approach or equation approach. They are similar. So why do we have two approaches? Each approach has its advantages. Consider Susan's bag of nickels described in Example 1.

Advantages of the function approach:

- The label of the function V reminds us that the function represents the value of the nickels.
- The function notation $V(x)$ makes it clear that we are thinking of the number of nickels x as the independent variable (input) and the value $V(x)$ as the dependent variable (output) .
- In the function approach, the domain (set of inputs) must be specified. In many applications, including this one, restrictions on the domain are very important. For example, Susan can only have a whole number of nickels. On the other hand, in Problem 2b Ramon's snail can crawl for $x = \frac{1}{2}$ or $x = \frac{2}{3}$ minutes, but x cannot be negative. Notice how this makes the graph of the function different from the graph of the line. However, the graph of the line can help us see the pattern of the function.

Advantages of the equation approach:

- There are different forms of the equation of a line, for example, *slope-intercept* and *point-slope*. Though they are equivalent, these different forms give you more flexibility to manipulate the equations of the line.

- The geometrical properties of the line, slope and intercepts, are easier to see in the equation of the line.

In the rest of the section, you will relate slope and intercepts of a line to different situations where linear functions are used.

Arithmetic Sequences as Linear Functions

One important application of functions with discrete domains is sequences. A sequence is the list of outputs of a function whose domain is restricted to the natural numbers. In Section 2.3, we studied special sequences that have a constant difference c between consecutive terms. So the recursive formula is $a_{n+1} = a_n + c$. In the next exploration, you will investigate the relationship between lines and arithmetic sequences.

EXPLORATION 1

- $\{6, 9, 12, 15, \ldots\}$
- $\{10, 8, 6, 4, \ldots\}$
- $\{-3, -1, 1, 3, \ldots\}$

For each arithmetic sequence above:

1. Find the constant difference c.
2. Graph the sequence.
3. Notice that the points lie on a line. Draw the line through the graph.
4. Find the slope and y-intercept of the line. Write the equation of the line.
5. What role does the constant difference c play in the equation?
6. It sometimes useful to think about a $0th$ term. If the function form of the sequence is $a_n = A(n)$, then $a_0 = A(0)$. We can find a_0 by using the pattern to work backwards. What term should come before the first term? How does the 0^{th} term compare to the y-intercept? How does the first term a_1 compare to the y-intercept?

7. How can you use the equation of the line to find the formula for the n-th term?

Once you recognize the relationship between arithmetic sequences and lines, you can use the tools from this chapter to solve trickier problems about sequences.

PROBLEM 2

Two terms of an arithmetic sequence are given: $a_3 = 7$ and $a_5 = 11$. Write the first five terms of the sequence. Find the formula for the n^{th} term of the sequence.

Hint: try the following steps:

1. Plot the points $(3, 7)$ and $(5, 11)$.
2. Since the sequence is arithmetic, the graph of the sequence should follow a line. Draw the line through the two points. Find the equation of the line.

Interpreting Slope as Rate of Change

Recall from Section 3.2 that the slope is the ratio $\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$ for "slope" triangles associated with the graph of the line. In some cases, it is helpful to think of rise as the change in the y variable, and run as the change in the x variable. If we do this, then the slope can be thought of as the *rate of change* of y with respect to x. In the next exploration, you investigate what happens when the y variable represents distance traveled and the x variable time traveled.

EXPLORATION 2

A bus full of students is traveling down the freeway on a field trip. At 9:00 A.M., Carter looks out the window and sees the bus is passing mile marker 185. At 9:30 A.M., the bus passes mile marker 215. At 10:00

A.M., it passes marker 245. Carter wants to know how fast the bus is going.

1. What was the bus' average speed in miles per hour on the first leg of the trip from 9:00 to 9:30? From 9:30 to 10:00? Is the speed the same?
2. Graph time versus distance.
3. Draw a line through the graph. Find the equation of the line.
4. How does the speed found above compare with the slope?
5. Use the concept of slope to explain why, if the speed is constant, the graph of time versus distance will be a line.
6. At 10:00 A.M., Carter asks the bus driver how much longer until they reach their destination. The bus driver says, "We will exit the freeway in 1 hour." If the speed is constant, how many more miles until the bus exits the freeway?
7. Speed is often referred to as the rate of change. Explain why this term makes sense.

Interpreting Slope as Unit Rate

Example 1 is called a "unit rate" or "unit value" problem. In this type of problem each unit has a certain "value." If you know the number of units x, and the unit rate m, then the total value is mx = the unit rate times the number of units. In Example 1 the unit rate was 5 cents per nickel, and x represented the number of units, in this case nickels. This led to the formula for the function $V(x) = 5x$ and the equation of the line $y = 5x$. So in Example 1, the slope represents the change in value for each extra nickel added. Thus for linear functions, the unit rate corresponds to the slope of the line.

PROBLEM 3

Find the unit value for each function in Problem 1.

EXPLORATION 3

Suppose that $y = C(x) =$ cost in \$ to produce x widgets, and that all points (x, y) that satisfy the cost equation fall on a straight line. Also, it costs \$50 to produce 3 widgets, and \$60 to produce 4 widgets.

1. Two points on the line are given in the problem. Identify and plot them.
2. Draw a line through the points.
3. What is the slope and y-intercept of the line?
4. What is the unit cost of each widget?
5. What does the y-intercept represent?

The initial value and the y-intercept

In many applications the value of the function when $x = 0$ has special meaning. This is especially true when the domain of the function does **not** include negative values.

In Example 1, we can see from the statement of the problem that the graph of the function should go through the origin. No nickels means the bag is empty and has no value. In many cases, however, the y intercept is not 0. The y intercept represents some initial value or fixed cost. In Exploration 3 we found that the y-intercept was 20. This meant that, even if no widgets were produced, it would still cost \$20. Where does this cost come from? It may cost a certain amount of money to set up the machine or store the material, for instance.

The terminal value and the x-intercept

In other applications the value of x which the function equals 0 has special meaning. This is especially true when the range of the function (set of output values) does **not** include negative values. In the next exploration we interpret the graphical meaning of this special value of x.

EXPLORATION 4

Consider the line given by the function $y = 15 - 5x$.

1. What are the slope and y-intercept of the line?
2. Graph the line.
3. Find the x-coordinate of the point where the line crosses the x-axis. That is, find x so that $(x, 0)$ is on the line.

THE X-INTERCEPT
The x-coordinate of the point where a line crosses the x-axis is called its x-*intercept*.

EXAMPLE 2

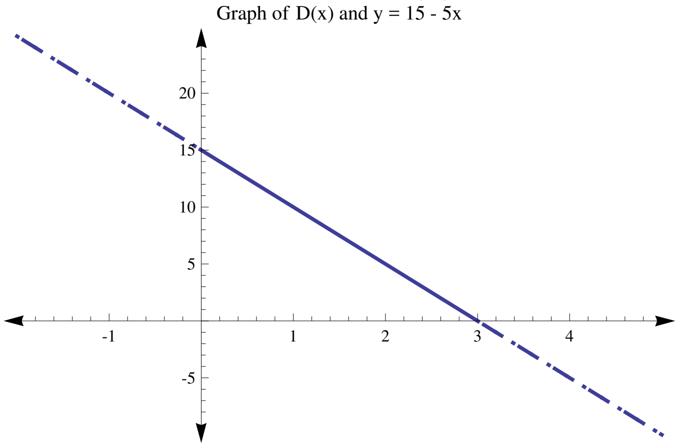
Juanita is 15 miles away from home. She walks at a constant rate of 5 miles per hour towards home for x hours.

1. Determine the formula for her distance in miles from home after walking for x hours, $D(x)$. What is the domain?
2. Sketch a graph of the function. Be careful to think about what are possible inputs and outputs.
3. Find and sketch the equation of the line $y = D(x)$.
4. Find and interpret the y and x intercepts of the line.

SOLUTION

1. Juanita starts 15 miles from home and every hour of walking gets her 5 miles closer, so $D(x) = 15 - 5x$. x is time walking in hours, so $x \geq 0$ is the domain.
2. Neither the inputs nor the outputs can be negative. So in our sketch,

the solid line represents the graph of $D(x)$.



3. $y = D(x)$, so $y = 15 - 5x$ is the equation of the graph of the function. This includes the entire line, both the dashed parts and the solid part.
4. From the equation we see that the y-intercept is 15. This represents the starting distance at $x = 0$ hours. To find the x-intercept we substitute $y = 0$ into the equation of the line which produces the linear equation $0 = 15 - 5x$. Now we solve for x, by subtracting 15 from both sides of the equation and dividing by -5, which gives:

$$\begin{aligned} 0 &= 15 - 5x \\ -15 &= -5x \\ 3 &= x \end{aligned}$$

So after $x = 3$ hours, Juanita's distance from home is 0. In other words, she is home!

In Example 2, the y intercept was the starting or initial distance and the x intercept was the ending or final time. This is often the case, but not always. In the following example, the domain and range are not restricted, so there is no initial or final value.

EXAMPLE 3

The Fahrenheit temperature scale was invented in 1724. In Fahrenheit, the boiling point of water is 212°and the freezing point is 32°. In 1744 scientists created a more logical set of units. Using water as the standard they created, the Celsius temperature scale lets 0 °C represent the freezing point of water and 100 °C the boiling point. Let x represent the temperature in Celsius, $F(x)$ represent the temperature in Fahrenheit and $y = F(x)$ be the corresponding equation of the line.

1. Two points on the line are given. Identify them.
2. Find the slope. What does it mean in this setting?
3. Find the equation of the line.
4. Find the x and y intercepts. Interpret them.
5. Is the temperature ever the same in Fahrenheit and Celsius? If so, when? If not, explain.

SOLUTION

1. The freezing point of water $(0, 32)$, the boiling point of water $(100, 212)$
2. $m = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$, a change of 1°C is the same as change of $\frac{9}{5}$°F.
3. The line has slope $m = \frac{9}{5}$. So the equation of the line is $y = \frac{9}{5}x + b$. The line passes through $(0, 32)$, so by substitution we see that $b = 32$. Hence the equation is $y = \frac{9}{5}x + 32$.
4. The y intercept is $b = 32$ which is the freezing point of water in °F. To find the x intercept, substitute $y = 0$ into the equation and solve for x.

$$y = \frac{9}{5}x + 32$$

$$0 = \frac{9}{5}x + 32$$

$$
\begin{aligned}
-32 &= \frac{9}{5}x \\
-32 \cdot \frac{5}{9} &= x \\
-\frac{160}{9} &= x
\end{aligned}
$$

This means that 0 °F is the same as $-\frac{160}{9} = -17.\overline{7}$ °C.

5. The temperature is the same in Fahrenheit and Celsius when $x = y$. So we can substitute into the equation of the line and try to solve.

$$
\begin{aligned}
y &= \frac{9}{5}x + 32 \\
x &= \frac{9}{5}x + 32 \\
x - \frac{9}{5}x &= 32 \\
\frac{5}{5}x - \frac{9}{5}x &= 32 \\
-\frac{4}{5}x &= 32 \\
x &= -\frac{5}{4}(32) = -40
\end{aligned}
$$

So -40 °F = -40 °C.

Proportional vs. Linear

In this chapter, we have studied lines and linear functions. In Section 3.3 you explored proportional relationships. Now, we carefully investigate the difference between the concepts of linear and proportional. We have seen that proportional relationships can be represented as straight lines that pass through the origin. So all proportional relationships are linear. But it is **not** true that all linear relationships are proportional. The following table summarizes the comparison between proportional and linear non-proportional relationships.

	Proportional	Linear Non-proportional
The ratio $\frac{y}{x}$	constant	not constant
The graph	line through origin	line with y-intercept $\neq 0$
The equation	$y = mx$	$y = mx + b$, $b \neq 0$

EXPLORATION 5

Think of different contexts from your previous math classes or from your experience outside of school. Identify 4 relationships, two that are proportional and two that are linear but not proportional. Explain how you can tell.

Let's examine the widget example from Exploration 3 more carefully.

EXAMPLE 4

In Exploration 3, the total cost in dollars of producing x widgets was $y = 10x + 20$.

1. Identify the fixed and variable cost.
2. Explain why the variable cost is proportional to the number of widgets produced.
3. Explain why the total cost is linear but not proportional to the number of widgets produced.

SOLUTION The *fixed cost* is the portion that must be paid no matter how many widgets are produced. The *variable cost* is the portion that increases as more widgets are made.

1. The fixed cost is equal to the y-intercept $b = 20$. The variable cost is $10x$.
2. If we divide the variable cost by the number of widgets produced we get $\frac{10x}{x} = 10$ the unit cost of making each extra widget.
3. The total cost includes the $20 fixed cost. When we divide the total cost by the number of widgets produced we get $\frac{10x+20}{x} = 10 + \frac{20}{x}$.

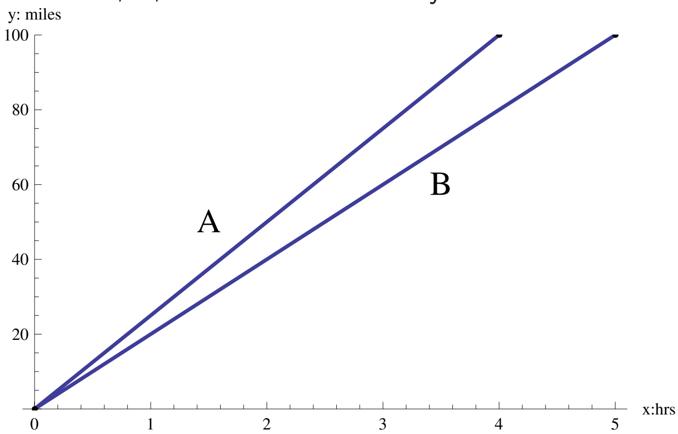
EXERCISES

1. For each of the following linear functions, determine the slope and sketch the graph for $-5 \leq x \leq 5$.
   a. $f(x) = 4x$
   b. $f(x) = -3x + 2$
   c. $f(x) = -x$
   d. $f(x) = \frac{1}{2}x - 1$
   e. $f(x) = 4$
   f. $f(x) = \frac{x}{3} - 4$
2. For each of the following functions, find the point(s) on the graph where the function crosses the y-axis and the x-axis.
   a. $f(x) = 20 - 4x$
   b. $G(x) = 2x + 8$
   c. $h(x) = 4x - 2$
   d. $g(x) = \frac{3x+5}{4}$
   e. $j(x) = \frac{2}{5}x + 1$
   f. $k(x) = 4$
3. For each of the following arithmetic sequences find the formula for the n^{th} term and graph the line that goes with this formula:
   a. $-4, -1, 2, 5, \ldots$
   b. $15, 10, 5, 0, \ldots$
   c. $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$
   d. $a_{n+1} = a_n + 6$, $a_1 = 5$
   e. $a_3 = 10$, $a_4 = 7$
4. The weight of a type of container is a function of its height. The weight of the container is 10 pounds more than twice its height in inches. Determine the formula for the weight of the container, $W(h)$, in terms of its height, h inches. What is the domain? Is it discrete or continuous? Write the corresponding equation of the line.
5. Joe drinks 8 glasses of water each day. If he works outdoors he drinks an additional 2 glasses for each hour he works. Determine the formula for the number of glasses of water he drinks, $C(x)$, for x hours of outdoor work. What is the domain? Is it discrete or continuous? Write the corresponding equation of the line.
6. Temperatures on the Warshauer scale are 3 degrees less than 5 times

the temperature measured on the McCabe scale. Write a formula for the temperature readings on the Warshauer scale, $B(x)$, in terms of a McCabe scale reading of x degrees. Write the corresponding equation of the line.

7. Amanda always talks at least 300 minutes per month on her cell phone. Amanda's cell phone company charges her \$22 per month plus 10 cents for each minute of use over 300 minutes. Determine the formula for Amanda's monthly bill, $B(x)$, in terms of using her cell phone for x minutes. What is the domain?
8. Billy's age is 1 more than 3 times Aaron's age. In 7 years, Billy will be twice as old as Aaron. How old is Aaron now?
9. Jane's weekly allowance is a function of how many hours of work she does at home. Two weeks ago she received \$22 and worked 3 hours. Last week she worked 5 hours and received \$34. Write an equation for her allowance as a function of the number of hours she works. Is her allowance proportional to number of hours she works?
10. The cost of a DVD at the local store is \$11. Let $C(x)$ be the cost of x DVD's. What is a formula for the function C? What is the equation of the straight line graph of C? If Ricardo spent \$187 (not including tax), how many DVD's did he buy? What is the slope of the line that is the graph of C?
11. Suppose that pencils sell for 20 cents each.
    a. What is the cost of 2 pencils? Of 3 pencils? Of 4 pencils?
    b. Let $C(x)$ equal the cost of x pencils. Consider the line given by the equation $y = C(x)$. What is the slope of the line? Interpret the meaning of the slope in this context.
    c. The line goes through the origin $(0, 0)$. Interpret the meaning of the origin in this context and explain why it makes sense.
12. Suppose that a car is traveling at a constant rate of 50 miles per hour. How far will the car travel in 1 hour? In two hours? In 3 hours? Let $d(x)$ be distance the car travels in x hours. Determine the formula for the function $d(x)$ and the corresponding equation of the line that is the graph of that function. What is the slope of the line? What is the meaning of the slope in this context? What is the meaning of the origin, that is the point $(0, 0)$, in this context?

13. The graph below shows the distance traveled in miles, y versus time traveled in hours, x, for two different bicyclists.



a. Which bicyclist is going faster? How can you tell?
b. Find the equation of each line.
c. What does the slope represent? What are its units?

14. Lemonade costs 15 cents per cup. Find the cost of 3 cups and of 5 cups. Let $x =$ number of cups of lemonade to be purchased. Find the total cost y of these x cups. What is the unit cost of a cup of lemonade?

15. In 1960, the cost of making x glasses of lemonade was $C(x) = 200 + 10x$ measured in cents. Notice that there was a fixed cost of 200 cents even if you didn't make any lemonade at all. This fixed cost might have come from purchasing supplies such as a pitcher. The variable cost of 10 cents per glass was used to pay for lemons, cups, etc.
a. Make a table for the cost of making different numbers of glasses of lemonade that includes the fixed cost.
b. Graph each of the points above on your graph paper.

16. Now suppose that you sell the lemonade for 50 cents per glass. The total amount of money the customers give you is called the revenue. Let x be the number of glasses you sell and R the revenue measured in cents. Because the revenue depends on the amount sold, we think about R as a function of x and write $R(x)$.
a. Write the rule for $R(x)$.
b. Make a table of input-output pairs for the function R. Plot these

points on the same graph paper from the previous exercise.

c. Compare the graph to that from Exercises 15. Is Revenue proportional to the number of glasses sold? How about Cost?

d. These graphs represent the amount of money you need to spend, $C(x)$, and the amount of money you receive, $R(x)$, in your business. The profit, also called net income, is the money you make running the business. This is calculated by subtracting the cost from the revenue. Now look at the graph of cost and revenue. Explain how you would be able to tell if the profit is positive (you are making money), or it is negative, (you are losing money), just by comparing the graph of C and R. Also, tell how this depends on x.

e. Write an equation for the profit P in terms of C and R. Note that P is a function of x, just like C or R. Write the formula for $P(x)$ in terms of x, using what you already know about $C(x)$ and $R(x)$.

f. For what number of glasses x will the profit be zero? Write this as an equation and solve for x. Compare this answer to where the graphs of C and R intersect. Explain.

17. Suppose that $C(x) =$ cost to produce x books, and that the graph of all points (x, y) that satisfy the cost equation is a straight line. Also, it costs $25 to produce 3 books, and $32 to produce 4 books.

a. Find the slope of the line.

b. Find the equation of the straight line.

c. How much does it cost to produce 20 books?

18. **Ingenuity:**

Vanessa and Danette are house painters. Vanessa can paint $\frac{1}{5}$ of a house in one day, and Danette can paint $\frac{1}{6}$ of a house in one day. Estimate how long it would take Vanessa and Danette to paint a house working together. How long will it take them to paint exactly one house?

19. Think about the graph of a line.

a. Why can the slope be interpreted as unit rate?

b. When can the y-intercept be interpreted as the initial value?

c. When can the x-intercept be interpreted as the terminal value?

d. Invent a word problem with a unit rate of 2, initial value of 10, and terminal value 7. What is the equation of the line that goes with your word problem?

20. **Ingenuity:**
A tank contains 400 liters of a 16% acid solution. Each hour 100 liters of the solution is drained from the tank and replaced with pure water. After draining and refilling 3 times, what percent of the solution is acid?

21. **Ingenuity:**
When 10 is subtracted from the numerator of a fraction and 10 is added to the denominator, the fraction that results is the reciprocal of the original fraction. Is the numerator a linear function of the denominator? Explain. Find three such fractions.

SECTION 3.6 STANDARD FORM OF A LINE AND ITS APPLICATIONS

In this chapter, we have explored several forms of the equation of a line including the point-slope form and the slope-intercept form. We will now explore a third form of a line called the *standard form*. The following equations are examples of this form:

1. $3x + 4y = 12$
2. $-2x + y = 5$
3. $2x - 5y = 6$
4. $x + y = 4$

What do you notice about these equations? They are each written in the standard form $Ax + By = C$ where A, B, and C are numbers. It is customary for each of A, B and C to be integers if possible with $A > 0$. Do you see how these equations are equivalent to equations that are of the slope-intercept form? Let's rewrite the first equation in the slope-intercept form.

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= -3x + 12 \\ y &= -\frac{3}{4}x + 3 \end{aligned}$$

Thus, we have transformed an equation in standard form to one that is in slope-intercept form. What is the slope and y-intercept of the line that these two equations describe?

PROBLEM 1

1. Rewrite each of the equations above in the slope-intercept form and identify the slope and y-intercept.
2. Rewrite the equation $Ax + By = C$ in slope-intercept form.

EXAMPLE 1

Convert the equation $y = \frac{2}{3}x + \frac{4}{3}$ into standard form $(Ax + By = C)$ and identify what A, B, and C are.

SOLUTION Start with the equation and multiply both sides by 3 in order to clear the fractions. Then add $-2x$ to both sides of the equation to get the term with x on the left side of the equation. This leaves only the number 4 on the right side. Finally one more step is done: multiply both sides of the equation by -1, which changes the first term on the left side from $-2x$ to $2x$. This makes the coefficient A positive.

$$\begin{aligned}
3y &= 3\left(\frac{2}{3}x + \frac{4}{3}\right) \\
3y &= 3 \cdot \frac{2}{3}x + 3 \cdot \frac{4}{3} \\
3y &= 2x + 4 \\
3y - 2x &= 4 \\
-(3y - 2x) &= -4 \\
2x - 3y &= -4 \\
2x + (-3)y &= -4
\end{aligned}$$

The equation is now in standard form: $Ax + By = C$ with $A = 2$, $B = -3$, and $C = -4$.

PROBLEM 2

Convert each of the following equations into standard form:

1. $y = -4x + 6$
2. $y = x - 3$
3. $y = \frac{4}{3}x - 1$

4. $\frac{y}{2} = \frac{2}{3}x + \frac{1}{6}$

EXPLORATION 1

Find the x-intercepts.

1. $y = 3x + 6$
2. $2x + 5y = 8$
3. $y = 4$

One advantage of standard form is that it is easy to compute the intercepts.

EXAMPLE 2

Compute the x-intercept and y-intercept for the line given by the equation: $6x + 5y = 30$.

SOLUTION To see how this standard form is useful for computing the intercepts, we first must remember that the x-intercept is the first coordinate of point where the line intersects the x-axis. This means that if we set $y = 0$, we can solve for x. So,

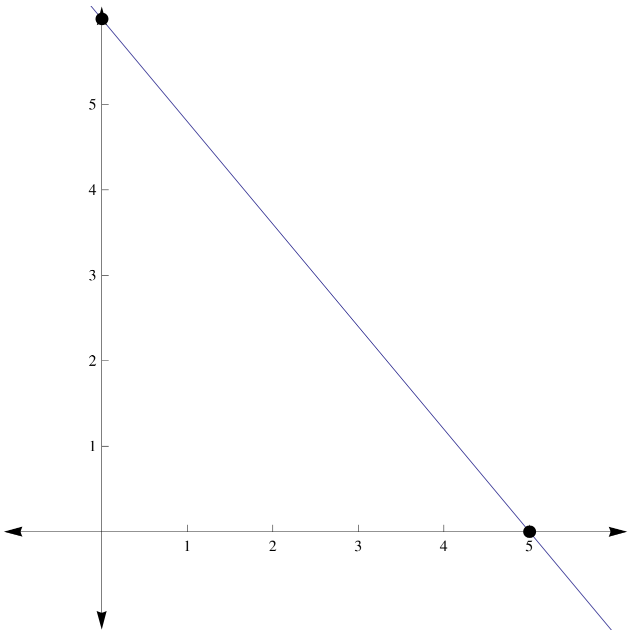
$$\begin{aligned} 6x + 5(0) &= 30 \\ 6x &= 30 \\ x &= 5 \end{aligned}$$

Thus, the x-intercept is 5 and the point $(5, 0)$ is where the line crosses the x-axis. We use the same method to compute the y-intercept by setting $x = 0$ and solving for y in the equation:

$$\begin{aligned} 6(0) + 5y &= 30 \\ 5y &= 30 \end{aligned}$$

$$y = 6$$

The point where the line crosses the vertical axis is $(0, 6)$. The intercepts are often good way to make a sketch of the line.



PROBLEM 3

Compute the x-intercept and y-intercept for each of the lines given by the equations:

1. $4x + 3y = 12$
2. $2x - 5y = 10$
3. $3x + 7y = 12$

Lines are useful in many real world applications and the standard form is often the most natural way to describe these lines.

The Standard Form in Applications: Graphical Method

EXPLORATION 2

A math class has 20 students. What are the different possibilities for the number of girls and the number of boys? Let x represent the number of girls and y represent the number of boys.

1. Make a table of the possibilities for x and y. Graph these pairs (x, y) as points on a coordinate system. Note that these points lie on a straight line.
2. What is the equation for this line?
3. What is the slope of this line? Explain why this slope makes sense for this problem.
4. What are the intercepts for this line? Explain why these two points make sense.
5. Are there points on the line that do not represent real possibilities for the number of girls and boys in this case? Explain why.

PROBLEM 4

Terry has total of $24 to spend at the grocery store on flour and sugar. Flour costs $2 per kilogram and sugar costs $3 per kilogram. Let x represent the amount (in kilograms) of flour and y represent the amount of sugar (in kilograms).

1. Write an equation that each possible pair (x, y) must satisfy. Is this the equation for a line? Explain.
2. What is the slope of the line? Why does it make sense?
3. What are the intercepts for this line? Explain why these two points make sense.
4. For each of the following points, determine if the point is on the line and if the x and y values represent real possibilities for the amount of flour and sugar.
   a. $(9, 2)$

b. $(\frac{5}{2}, \frac{19}{3})$
c. $(15, -2)$

The Standard Form in Applications: Algebraic Method

EXAMPLE 3

In the past, ships carried gold and silver bars. A treasure chest is discovered whose contents weighed 48 pounds. Each gold bar weighed 3 pounds and each silver bar weighed 2 pounds.

1. Write an equation in standard form describing the possible numbers of gold and silver bars in the chest.
2. Write the equation in slope-intercept form.
3. What is the y-intercept and why does it make sense?
4. Graph the line. Are there points on the line that do not represent real possibilities?
5. Make a list of the solutions for this equation that represent real possibilities. What patterns do you notice?
6. What is the slope? Why does the slope make sense when we look at the patterns in the solutions?

SOLUTION

1. Let's use the algebraic method to find the equation. In this method we first transcribe all the information in the problem by writing careful descriptions for the numbers given in the problem and assign variables where appropriate.

$$\begin{aligned} 3 &= \text{lbs. per gold bar} \\ 2 &= \text{lbs. per silver bar} \\ 48 &= \text{lbs. of total weight of bars} \\ x &= \text{\# of gold bars} \\ y &= \text{\# of silver bars} \end{aligned}$$

Next we combine numbers in ways that make sense for the problem. In this case, first we will multiply the number of bars times the weight per bar. Then we will add the weight for the two kinds of bars. The key is to pay attention to our descriptions.

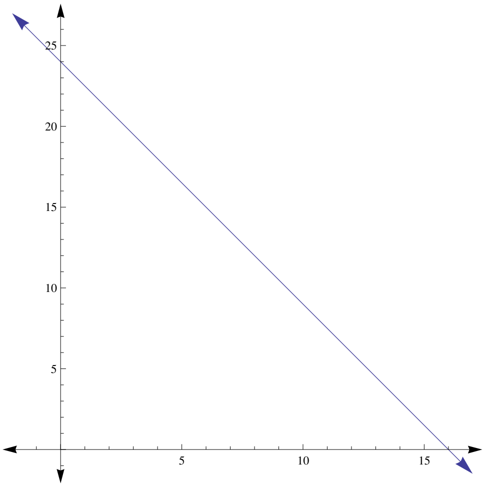
$$\begin{aligned} 3x &= \text{weight of } x \text{ gold bars in chest} \\ 2y &= \text{weight of } y \text{ silver bars in chest} \\ 3x + 2y &= \text{combined weight of gold and} \\ &\qquad \text{silver bars in treasure chest} \end{aligned}$$

From above we see that the total weight of bars is 48, so the equation must be $3x + 2y = 48$ which is in standard form.

2. To find the slope-intercept form, we solve for y.

$$\begin{aligned} 3x + 2y &= 48 \\ 2y &= 48 - 3x \\ y &= \frac{48}{2} - \frac{3x}{2} \\ y &= 24 - \frac{3}{2}x \\ y &= -\frac{3}{2}x + 24 \end{aligned}$$

3. The y-intercept is $b = 24$. If there are no gold bars, we need 24 silver bars to reach the total weight of 48 lbs.



4. Both the number of gold bars and the number of silver bars must be whole numbers. So there are many points on the line that do not represent real possibilities. The points $(-2, 27)$ and $(1, \frac{45}{2})$ are two examples.

5. To find a solution we let x equal a whole number, and then check if the y is a whole number. If we do this we find out that x must be an even number between 0 and 16. And y is always a multiple of 3 between 0 and 24. Below is a table of all the possible solutions.

x	0	2	4	6	8	10	12	14	16
y	24	21	18	15	12	9	6	3	0

6. The slope is $m = -\frac{3}{2}$. It is more difficult to interpret the slope because it is not an integer. First of all, we notice that the slope is negative and this makes sense. Because if we add gold bars we must take away silver bars in order to keep the weight the same. Second, as we saw above, the number of gold bars must be even. Since the weight of a gold bar is 3lbs and the weight of a silver bar is 2lbs, for each two gold bars we add, we add 6 lbs, so we must take away 6 lbs of silver which is 3 bars of silver. That is, take away 3 bars of silver for every 2 bars of gold we add, hence $m = -\frac{3}{2}$.

EXERCISES

1. Find the slope of the line $y = 2x + 8$. What is the y-intercept? What is the x-intercept?
2. Find the slope of the line $y = -2x + 10$. What is the y-intercept? The x-intercept?
3. For each of the following equations of lines, find the slope, the y-intercepts and the x-intercepts.
   a. $y = 2x + 8$
   b. $4x + 3y = 12$
   c. $2x + 4y = -5$
   d. $y = -2x + 7$
4. For each of the following equations in slope-intercept form, rewrite them in standard form: $Ax + By = C$. Identify A, B, and C.
   a. $y = -3x + 6$
   b. $y = \frac{3}{2}x + 1$
   c. $y = -\frac{1}{3}x - \frac{2}{3}$
   d. $y = \frac{3}{5}x + \frac{2}{5}$
   e. $y = x + 2$
5. Write each of the following equations in standard form using only integer coefficients.
   a. $y = -\frac{1}{2}x + 3$
   b. $y = -\frac{1}{2}(x + 3)$
   c. $2y = 3x + \frac{1}{3}$
   d. $2(y + 1) = 5(x - 2)$
   e. $3x - 2 = -4(y - 1)$
   f. $x = \frac{1}{3}y - 7$
6. Write each of the following equations in slope-intercept form.
   a. $2x - 3y = 15$
   b. $2y - 3x = -14$
   c. $\frac{1}{3}x + \frac{1}{6}y = 1$
   d. $4x - 5y = 1$
   e. $6x = 4y - 12$
7. For each of the following equations, use the intercepts to sketch a graph. Then rewrite the equations in slope-intercept form. Does the graph help confirm the slope? How?
   a. $10x + 5y = 15$
   b. $4x + 2y = 5$
   c. $2x + 3y = 12$
   d. $2x - 3y = 5$

e. $x - 4y = -8$

8. Julie goes into a candy store to buy bubble gum and jaw breakers. Bubble gum costs 3 cents per piece and jaw breakers cost 4 cents each. She has 24 cents and wants to spend all of it. Let x be the number of pieces of bubble gum that she buys and let y be the number of jaw breakers that she buys.
   a. Write an equation that each possible pair (x, y) must satisfy using the algebraic method. Represent the equation in words as well as using variables.
   b. Make a list of real possible values for x and y.
   c. Graph the line.
   d. What is the slope of the line? Why does it make sense?
9. From the list of equations in standard form, determine which ones describe lines that are parallel to another line in the list. *Hint: Recall that lines are parallel if they have the same slope.*
   - $2x + 5y = 9$
   - $3x - 2y = -4$
   - $4x + 10y = 10$
   - $2x - 5y = 3$
   - $-2x + 5y = 8$
   - $3x - 2y = 10$
10. Let $a \neq 0$ and $b \neq 0$ be numbers.
   a. Write an equation of the line that has x-intercept $(a, 0)$ and y-intercept $(0, b)$ and put the equation in standard form.
   b. Rewrite the equation $\frac{x}{a} + \frac{y}{b} = 1$ in standard form.
   c. How do your answers compare?
11. Suppose $B \neq 0$. What is the slope of the line $Ax + By = C$?
12. **Ingenuity:**
   Consider the line described by the equation $ax + by = c$. How will its graph compare to lines given by:
   a. $kax + kby = kc$ where $k \neq 0$.
   b. $ax + by = d$ where $d \neq c$.

SECTION 3.7 PERPENDICULAR LINES

In this section we will explore the equations of lines that are *perpendicular* to one another. To begin with, we focus only on lines that pass through the origin.

Lines Through the Origin

EXPLORATION 1

1. Find the equation of the line passing through the origin and the point $(1, 3)$. Also find the equation of the line passing through the origin and the point $(-3, 1)$. Plot the points and the two lines on the same coordinate grid. What are slopes of the two lines?
2. Find the equation of the line passing through the origin and the point $(1, \frac{1}{2})$. Also find the equation of the line passing through the origin and the point $(-\frac{1}{2}, 1)$. Plot the points and the two lines on the same coordinate grid. What are slopes of the two lines?
3. Find the equation of the line passing through the origin and the point $(1, -2)$. Also find the equation of the line passing through the origin and the point $(2, 1)$. Plot the points and the two lines on the same coordinate grid. What are slopes of the two lines?

For each pair of lines above, what do you notice about the slopes of the lines? What do you notice about the angle between the lines?

EXPLORATION 2

1. Pick a point $(1, m)$ for some positive m and plot it. Yes, you can choose any positive value that you want. Find the equation of the line passing through the origin and the point $(1, m)$. What is the slope of the line?
2. Locate the point $(-m, 1)$ on the coordinate grid from part 1. Find the equation of the line passing through the origin and the point

$(-m, 1)$. What is the slope of the line? Plot the line on the same coordinate grid.

3. What do you notice about the slopes of the lines? What do you notice about the angle between the lines?

PROBLEM 1

1. Graph the lines given by the equations $y = 2x$ and $y = -\frac{1}{2}x$ on the same coordinate grid.
2. Graph the lines given by the equations $y = \frac{1}{3}x$ and $y = -3x$ on the same coordinate grid.
3. Graph the lines given by the equations $y = -x$ and $y = x$ on the same coordinate grid.
4. What do you notice about the slopes of the lines? Use the corner of a piece of paper to test if the angle between each pair of lines is a right angle.

By this point, you probably have noticed that when you graph the lines through the origin given by the equations $y = mx$ and $y = -\frac{1}{m}x$, the two lines will be perpendicular.

PROBLEM 2

Find the equations of the lines through the origin which are perpendicular to lines given by each of the following equations:

1. $y = 5x$
2. $y = -\frac{1}{3}x$
3. $y = -4x$

Geometric Justification

In the explorations so far we have seen that if one line has slope m and another has slope $-\frac{1}{m}$, the graphs of the two lines will be perpendicular. But you may ask, how do we know this is always the case? How are we sure that the angles in the graphs from those explorations that looked like right angles really are 90°? Using the slope triangles discussed in section 3.2 and some geometry we will explore why this must be true for lines which pass through the origin.

EXPLORATION 3

Pick any point $(1, m)$. Draw the triangle with vertices $(0, 0)$, $(1, m)$, and $(1, 0)$. On the same coordinate grid locate the point $(-m, 1)$. Draw the triangle with vertices $(0, 0)$, $(-m, 1)$, and $(-m, 0)$. Write down at least 4 things you notice about these triangles.

Before we move on, let's recall some important facts from geometry.

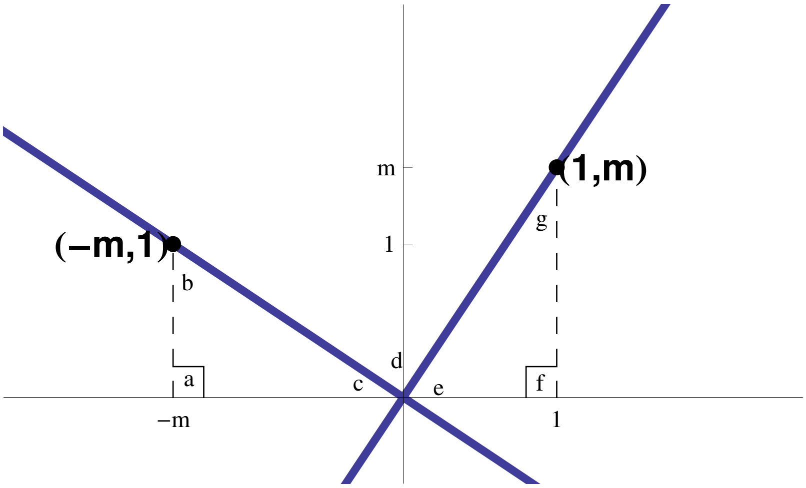
- The sum of the measures of the angles of a triangle is 180°
- The measure of the straight angle is 180°
- Corresponding angles of congruent triangles are equal

EXPLORATION 4

In this exploration we combine what we have learned in this section to justify that the lines $y = mx$ and $y = -\frac{1}{m}x$ are perpendicular.

1. Consider a line that passes through the origin and the point $(1, m)$ for some number $m > 0$. What is its slope? What is its equation? Label the line in the graph below.
2. Consider a line that passes through the origin and the point $(-m, 1)$. What is its slope? What is its equation? Label the line in the graph below.
3. Consider the angles on the graph below. What is the measure of angle a? What is the measure of angle f?
4. What is the sum of the measures of the angles b and c? Explain.
5. Explain why angle b has the same measure as angle e.

6. What is the sum of the measures of angles c and e? Explain.
7. What is the sum of the measures of the angles c, d, and e? Explain.
8. Using what you noticed in 7, find the measure of angle d. What does this say about the lines $y = mx$ and $y = -\frac{1}{m}x$?



Lines Perpendicular to a Given Line and Through a Given Point

EXPLORATION 5

Use a single coordinate grid for the following:

1. Graph the line given by the equation $y = 2x + 3$. What is its slope?

2. Find the equation of the line parallel to the line given by $y = 2x + 3$ which goes through the origin. What is its slope?
3. Find the equation of the line which goes through the origin and is perpendicular to line you found in part 2. Graph the line. What is its slope?
4. What do you notice about the graph of the line given by $y = 2x + 3$ and the line you found in 3? What do you notice about the slopes?

PROBLEM 3

Find the equations of the lines through the origin which are perpendicular to lines given by each of the following equations. Make a graph of each pair of perpendicular lines.

1. $y = 5x - 2$
2. $y = -\frac{1}{3}x + 1$
3. $y = -4x + 3$

THEOREM 3.3: SLOPES OF PERPENDICULAR LINES
The line given by the equation $y = mx + b$ with $m \neq 0$ will be perpendicular to any line with slope equal to $-\frac{1}{m}$.

In the next example we combine this result with the methods we learned in section 3.4 to find the equation of the line perpendicular to a given line and through a given point.

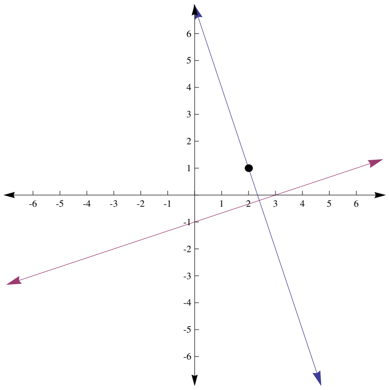
EXAMPLE 1

Find the equation of the line that is perpendicular to the line given by $y = \frac{1}{3}x - 1$ and passes through the point $(2, 1)$. Make a graph of the pair of perpendicular lines.

SOLUTION First we use Theorem 3.3 to find the slope of the new line. The equation of the line given is $y = \frac{1}{3}x - 1$, hence its slope is $m = \frac{1}{3}$. So the slope of the new line will be $-\frac{1}{m} = -\frac{1}{\frac{1}{3}} = -3$. Now find the equation of the line which has slope -3 and passes through the point $(2, 1)$. Using the point-slope form we learned in Section 3.4, we get

$$y - 1 = -3(x - 2) = -3x + 6$$

If we isolate y by adding 1 to both sides of the equation, we can rewrite this in the slope-intercept form: $y = -3x + 7$. Now let's graph the lines. Notice that the lines are perpendicular in the graph.



PROBLEM 4

For each of the following points, find the equation of the line that passes through the point and are perpendicular to the line given by $y = -\frac{1}{2}x + 3$:

1. $(0, 0)$
2. $(0, 3)$
3. $(-2, 5)$
4. $(3, 6)$

EXERCISES

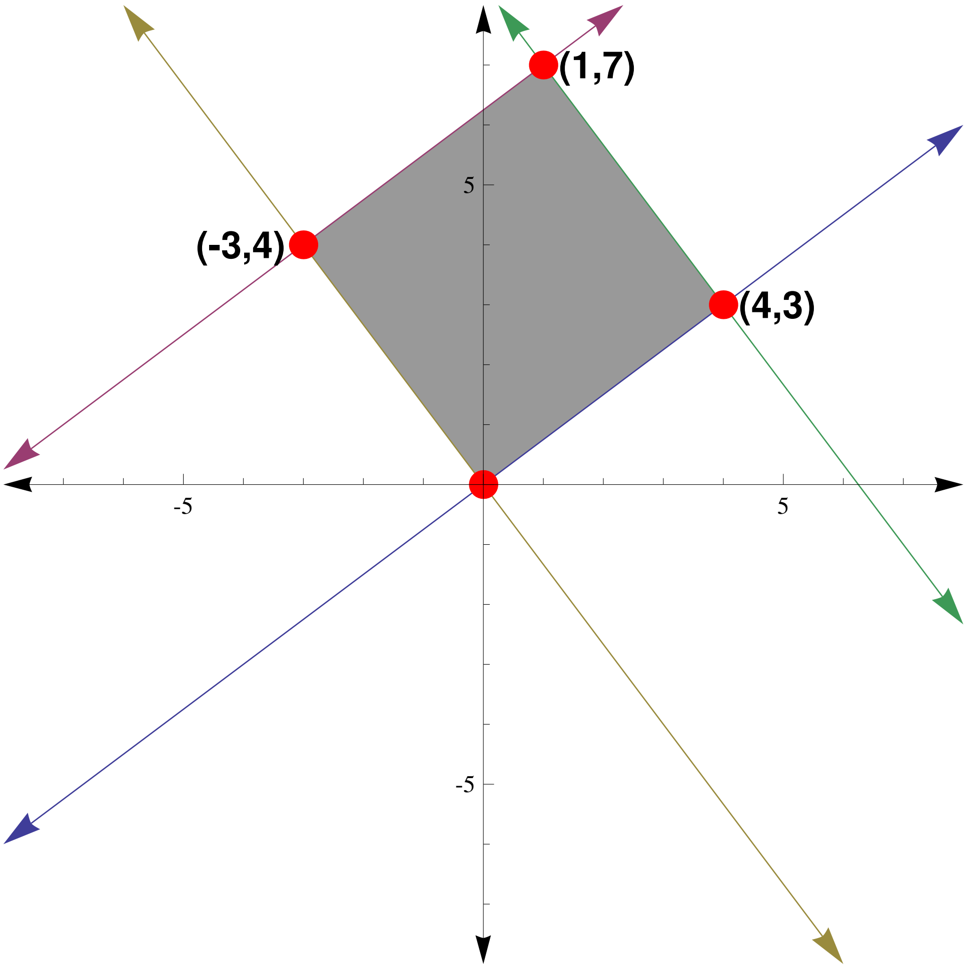
1. Find the equation of the line through the origin and the point $(1, 4)$. Also find the equation of the line through the origin and the point $(-4, 1)$. Plot the points and the two lines on the same coordinate grid. What are slopes of the two lines?
2. Find the equation of the line through the origin and the point $(1, 4)$. Also find the equation of the line through the origin and the point $(1, -\frac{1}{4})$. Plot the points and the two lines on the same coordinate grid. What are slopes of the two lines?

3. Find the equation of the line that is parallel to $y = 2x - 3$ and passes through:
   a. $(0, 0)$
   b. $(0, 3)$
   c. $(0, -2)$
   d. $(0, b)$ for some number b.
4. Find the equation of the line which is parallel to $y = 5x + 2$ and passes through:
   a. $(1, 3)$
   b. $(2, 3)$
   c. $(-1, 5)$
   d. $(-3, 4)$
   e. (a, b) for two numbers a and b.
5. Find equation of the line that passes through the origin and is perpendicular to:
   a. $y = 2x + 3$
   b. $y = -3x + 6$
   c. $y = 2x + 3$
   d. $y = \frac{2}{3}x + 2$
   e. $y = -(1/4)x - 2$
   f. $y = -2x$
   g. $y = mx$ for some number m
6. Find the equation of the line that contains the indicated point and is perpendicular to the specified line.
   a. $y = 2x + 3$ and contains $(1, 5)$
   b. $y = -3x + 6$ and contains $(1, 3)$
   c. $y = 2x + 3$ and contains $(-4, 1)$
   d. $y = \frac{2}{3}x + 2$ and contains $(4, -1)$
   e. $y = -(1/4)x - 2$ and contains $(1, 4)$
   f. $y = -2x$ and contains $(0, -5)$
   g. $y = 2x$ and contains (a, b) for some numbers a and b
   h. $y = mx$ and contains $(0, b)$ for some numbers m and b

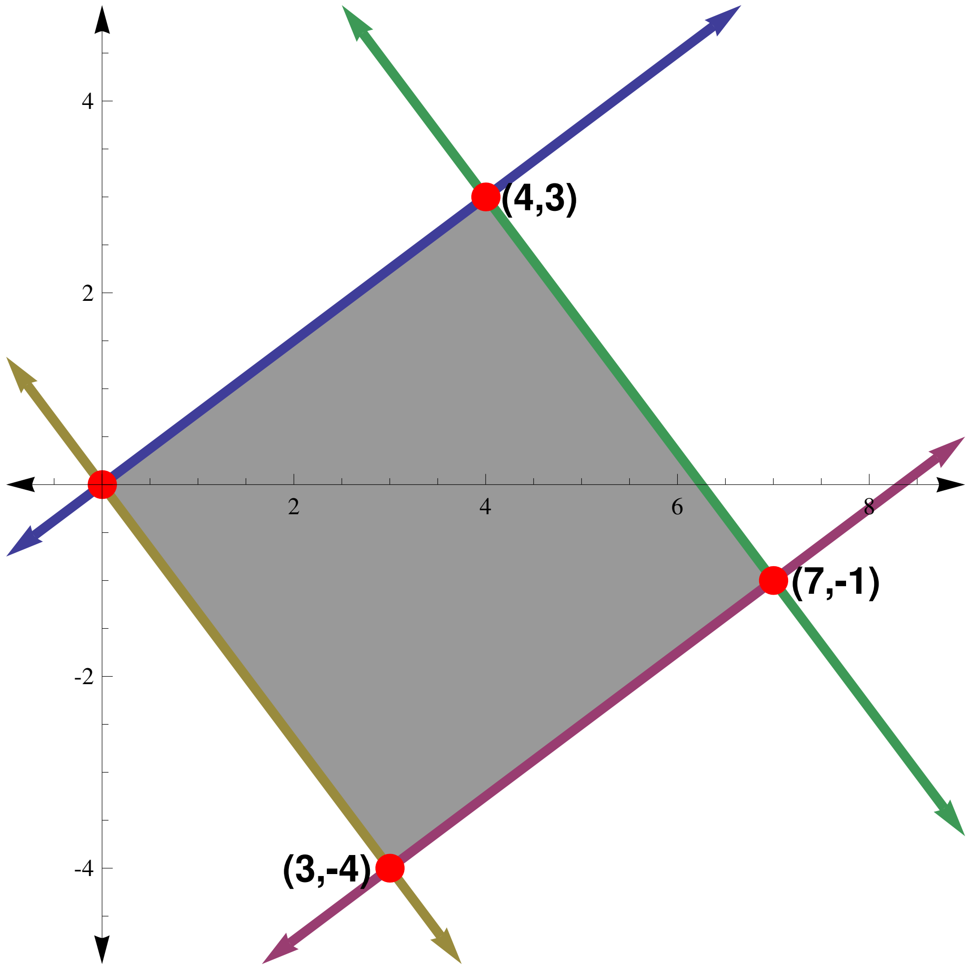
7. Normally, two conditions define a line. Each of the following problems gives a different way of expressing two conditions a line could satisfy. For each one, find the equation of the line that satisfies the given conditions. Write the line in slope intercept form.
   a. The line contains the point $(1, 2)$ and has y-intercept $b = -1$.

   b. The line contains the point $(3, 4)$ and has slope $m = \frac{1}{3}$.
   c. The line contains the points $(0, 0)$ and $(-1, \frac{1}{2})$.
   d. The line contains the points $(2, 4)$ and $(-1, 3)$.
   e. The line contains the point $(0, 3)$ and is parallel to the line given by $y = 5x + 2$.
8. Each of the following problems gives a different way of expressing two conditions a line could satisfy. For each one, find the equation of the line that satisfies the given conditions. Write the equation is standard form.
   a. The line is vertical and contains the point $(3, 2)$.
   b. The line is horizontal and contains the point $(-1, 4)$.
   c. The line contains the point $(1, \frac{1}{2})$ and is perpendicular to the line given by $y = 4$.
   d. The line contains the point $(1, \frac{1}{2})$ and is perpendicular to the line given by $x = -1$.
   e. The line contains the point $(-2, -5)$ and is parallel to the line given by $y = 2$.
   f. The line contains the point $(3, -2)$ and is parallel to the line given by $x = 4$.
9. Each of the following problems gives a different way of expressing two conditions a line could satisfy. For each one, find the equation of the line that satisfies the given conditions. Write the equation point slope form.
   a. The line contains the point $(2, 1)$ and is parallel to the line given by $y = 5x + 2$.
   b. The line contains the point $(-2, 3)$ and is perpendicular to the line given by $y = -x + 2$.
   c. The line contains the point $(-2, 3)$ and is perpendicular to the line given by $y = 4x + 2$.

10. **Investigation:**
In the graph below the four lines form a boundary of a square, shaded in gray. Find the equations of each of the lines.



11. **Investigation:**
The figure on the next page shows the graph of the lines given by the equations $-3x + 4y = 0$, $-3x + 4y = -25$, $4x + 3y = 0$, and $4x + 3y = 25$. Prove that the region that is shaded gray is a square.



12. **Ingenuity:**
Suppose L is a line given by the equation $y = 2x - 1$. Find the nearest point on the line L to the point $(7, 3)$. What is the distance from $(7, 3)$ to this nearest point?

SECTION 3.8 CHAPTER 3 REVIEW

Key Terms

continuous
cost
discrete
equation of the horizontal line
equation of the vertical line
fixed cost
initial value
linear function
parallel lines
perpendicular lines
profit
proportional
rate of change
revenue
rise
run
slope
terminal value
unit value / unit rate
variable cost
x-intercept
y-intercept

Properties and Theorems

Slopes of Parallel Lines: The line given by the equation $y = mx + b$ is parallel to any line with slope m whose y-intercept is not b.

Slopes of Perpendicular Lines: The line given by the equation $y = mx + b$, $m \neq 0$ is perpendicular to any line with slope $-\frac{1}{m}$.

Formulas

Slope of line containing the points (x_1, y_1) and (x_2, y_2):

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Forms of a Line

Slope-Intercept:
$y = mx + b$

Point Slope:
$(y - y_1) = m(x - x_1)$

Standard:
$Ax + By = C$

Practice Problems

1. Write the equations of the vertical line and the horizontal line that pass through the point $(3, -4)$.
2. a. Find the equations of two lines that are parallel to the line $x = 0$.
   b. Are the lines in part 2a horizontal or vertical?
   c. What is another name for the line $x = 0$?
3. Find the slopes of the lines that pass through the given points:
   a. $(2, 6)$ and $(8, 1)$
   b. $(-6, 3)$ and $(-5, -4)$
   c. $(-4, 6)$ and $(7, 6)$
   d. $(8, 2)$ and $(8, -4)$
4. For each of the equations below, determine the slope and y-intercept of the line given by the equation.
   a. $y = -\frac{1}{3}x + 8$
   b. $y + 2 = -5x + 4$
   c. $y = 3$
   d. $4x + 8y = 16$
   e. $y - 16.4 = \frac{1}{5}(x + 2.9)$
5. Use the values in the table to determine if the relationship between x and y is proportional.

x	1	3	5	7
y	3	13	23	33

6. Find the equations of the lines that satisfy the given information. Put the line in slope-intercept form.
   a. The line passes through $(6, 10)$ and $(-3, 7)$.
   b. The line has slope $m = 4$ and contains the point $(6, 2)$.

c. The line contains the point $(2, 10)$ and has y-intercept $b = 8$.

7. Martha rides her bike at a constant rate from her house to a park 36 miles away. After $1\frac{1}{2}$ hours she is half-way there.
   a. Determine the rule or formula for the function $d(x)$, where $d(x)$ is the distance the bike travels in x hours. What is the domain? 7
   b. What is the slope of the line from 7a.? What is the meaning of the slope?

8. Jorge sells roses for Valentine's Day. He purchases the ribbon and tissue paper he needs for all the arrangements for $20.00. He spends $0.75 per rose.
   a. Determine the formula for the function $C(x)$ where $C(x)$ is the total cost for making x flower arrangements. What is the domain?
   b. What is the slope of the line represented in the above function and what is the meaning of the slope in this context?

   c. Is the total cost proportional to the number of flower arrangements x? Explain.

9. For the following equations find the x and y-intercepts. Rewrite each equation in slope-intercept form. Graph the line and confirm the slope.
   a. $6x - 2y = 12$
   b. $3x + 5y = -30$

10. Write the following equations in standard form using only integer coefficients.
    a. $y = \frac{1}{3}(x - 4)$
    b. $2x - 4 = \frac{1}{2}(y - 6)$

11. Find the equation in slope-intercept form of the line that contains the point $(6, 4)$ and is parallel to the line given by $y = 3x + 8$.

12. Find the equation in slope-intercept form of the line that contains the point $(-3, 9)$ and is perpendicular to the line given by $y = 3x + 8$.

13. Find the equations of four lines that intersect to form a rectangle but where none of the lines are horizontal.

SYSTEMS OF EQUATIONS

4

SECTION 4.1 A GRAPHICAL APPROACH

EXPLORATION 1

1. A pile of nickels and pennies is on a table. We are told the coins are worth 25 cents in total.
   a. How many of each coin could be in the pile? How did you organize all your possibilities?
   b. Let $n =$ the number of nickels, and $p =$ the number of pennies. How are n and p related? Write an equation that describes this relationship.
   c. Graph the possible pairs (n, p) on a coordinate plane.
2. For another pile of nickels and pennies, we are told that there are 13 coins in all.
   a. How many of each coin could be in this pile?
   b. Let $n =$ the number of nickels, and $p =$ the number of pennies. How are n and p related? Write an equation that describes this relationship.
   c. Graph the possible pairs (n, p) on the same coordinate plane as above.

3. Is it possible that the two piles have the same number of number of nickels? *Hint: look at the graph you made above.*
   a. What must n and p be?
   b. Is there more than one choice for n and p?

In Exploration 1 we wanted two different conditions to be true at the same time. We expressed these conditions using the pair of linear equations:

$$5n + p = 25$$
$$n + p = 13$$

We call such problems *systems of linear equations*. The *solution* to the system is the set of ordered pairs that satisfies both conditions. In the this case, we mean the set of all pairs (n, p) that make both equations true. For the problem in Exploration 1, we used the graph to determine that the only solution was $n = 3$ and $p = 10$.

PROBLEM 1

A pile of coins consisting of quarters and dimes are on a table. We are told that there are 12 coins in all. We are also told that the pile of quarters and dimes is worth $2.55. Use the graphing method developed above to determine how many quarters and how many dimes there are in the pile.

EXAMPLE 1

Julie is four years older than her sister Katie. The sum of their ages is 24. Find the age of each girl.

SOLUTION

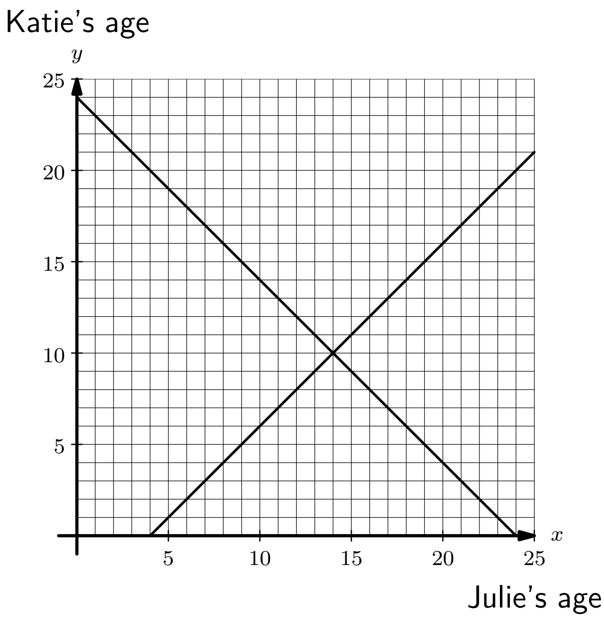
One approach is to use two variables x and y to represent the ages of the girls. Let x be Julie's age and let y be Katie's age. We can then write two algebraic equations with these variables.

The first relation, Julie is four years older than Katie, can be expressed as $x = y + 4$. The second relation, the sum of their ages, can be written as $x + y = 24$.

Our system is:

$$x = y + 4$$
$$x + y = 4$$

Our problem then is to solve these two equations simultaneously. That is, find values of x and y that satisfy both equations. Let's graph the lines given by the equations.



Notice that the two lines intersect at the point $(14, 10)$. Let's check that the pair $x = 14$ and $y = 10$ satisfies both the equations $x = y + 4$ and $x + y = 24$.

$$14 = 10 + 4 \text{ True}$$
$$14 + 10 = 24 \text{ True}$$

Hence $(14, 10)$ is a solution to the system of equations.

PROBLEM 2

Consider the two conditions about two numbers: The difference between the two numbers is 13. The sum of the two numbers is 23.

1. Write two equations that relate the two numbers as given by the conditions. Specify your variables by which is larger and which is smaller. (Notice they cannot be the same value since their difference is 13).
2. Graph the two equations on the same coordinate system.
3. Write down observations about the graphs.
4. Determine the two numbers using information from the graph.
5. Verify that the two numbers satisfy the two initial conditions.

As you will see, there are many different ways in which systems of linear equations can be solved. We call the method in this section the graphical method for solving a system of two linear equations in two unknowns. You solved these systems with pencil and paper. In the next exploration, you will use graphing calculators to solve these systems graphically.

EXPLORATION 2

The system in the previous problem consisted of the two equations: $x - y = 13$ and $x + y = 23$, where x is the larger of the two numbers and y the smaller. Write these equations in slope-intercept form. Use a graphing calculator to graph the two equations. Then determine the intersection of the two lines. Look also at the table and see if you can find a common point. Check your answer against what you got in Problem 2.

EXPLORATION 3

1. Consider the system of equations:

$$5x + 7y = 18$$
$$3x + 2y = 22$$

   a. Write the equations in slope-intercept form.
   b. Are the two lines given by the equations parallel? Explain.
   c. Will the two lines intersect? How many times?

2. Consider the system of equations:

$$2x + 7y = 18$$
$$2x + 7y = 10$$

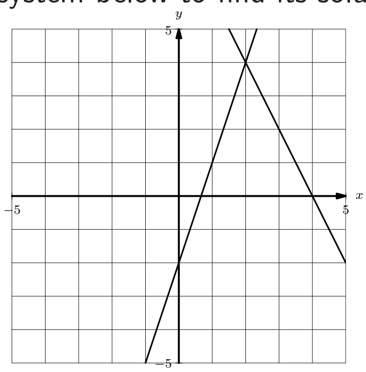
   a. Are the two lines given by the equations parallel? Explain.
   b. Will the two lines intersect? How many times?

In the next section, we explore another method for solving systems of equations called substitution.

EXERCISES

1. For each pair of equations below, graph the lines associated with these equations on the same coordinate system. Do the lines intersect each other? What point belongs to both lines? Verify that the point you choose satisfies both equations.
   a. $y = -x + 10$ and $y = 2x + 1$
   b. $2x + 2y = 11$ and $x - 2y = -2$
   c. $x + y = 5$ and $-2x + y = -1$
   d. $y = x$ and $y = 2x$
   e. $y = x$ and $y = -x$
   f. $4x + 2y = 5$ and $2x + y = -2$
   g. $4x + 2y = 5$ and $8x + 4y = 10$

2. For each pair of equations below, graph the lines associated with these equations on the same coordinate system. Do the lines intersect each other? What point belongs to both lines? Verify that the point you choose satisfies both equations.
   a. $y = 4$ and $y = 1$
   b. $x = 2$ and $x = -2$
   c. $y = 5$ and $x = 2$
   d. $x = -3$ and $y = 1$
3. For each pair of equations below, graph the lines associated with these equations on the same coordinate system. Do the lines intersect each other? Do you see a pattern that explains your answer?
   a. $x + y = 10$ and $x + y = 4$
   b. $2x + y = 6$ and $2x + y = 2$
   c. $3x - 2y = 4$ and $3x - 2y = 8$
4. Use the graph of a system below to find its solution.



5. A sportswriter observes a foot race between Alan and Bob. Alan is 36 feet ahead of Bob on the track and is running at 6 ft/s. Use the information given about Bob's speed to determine how long Bob must run before he catches up to Alan.
   a. Bob's speed is 9 ft/s
   b. Bob's speed is 6 ft/s
   c. Bob's speed is 4 ft/s
6. Diann bought gumballs and candy canes. Each gumball cost 50 cents and each candy cane cost \$1.25. She spent \$4.50 on her purchase and bought exactly 6 items. How many gumballs and candy canes did she buy? Show your answer as a point of intersection of two lines

on a coordinate system.

7. Suppose there are 42 fish in a tank and there are two kinds of fish, gold fish and perch. If you know that there are twice as many gold fish as perch, how many of each kind of fish is in the tank?

8. For each pair equations below, graph the lines associated with these equations on the same coordinate system. Do the lines intersect each other? What point belongs to both lines? Verify that the point you choose satisfies both equations.
   a. $x + y = 4$ and $x - y = 1$
   b. $x + y = 1$ and $y = x$
   c. $y = -x$ and $y = x + 1$

9. **Ingenuity:**
   Juanita walks briskly in the morning at the rate of 4 feet per second. After 40 seconds, she stops and stretches for 20 seconds. Let $d(t)$ be the distance she has walked after t seconds. Sketch a graph of her walking and stretching for the first 3 minutes. Use this graph to find the value of $d(45)$, $d(60)$ and $d(150)$. At what time or times will she be a distance of 500 feet from her starting point?

10. **Investigation:**
    For each pair of equations below, graph the lines associated with these equations on the same coordinate system. Do the lines intersect? What point belongs to both lines? Verify that the point you choose satisfies both equations.
    a. $2x + y = 5$ and $x + y = 3$
    b. $4x + 2y = 10$ and $x + y = 3$
    c. $2x + y = 5$ and $2x + 2y = 6$
    d. $4x + 2y = 10$ and $2x + 2y = 6$
    e. $4x + 2y = 10$ and $3x + 3y = 9$

11. Consider the investigation in problem 10. How are the pairs of equations related? How does this effect the point of intersection?

12. **Ingenuity:**
    Consider the investigation in problem 10. Let a be a positive number. Consider the lines given by the pair of equations $2x + y = 5$ and $ax + ay = 3a$. What point will belong to both lines? Describe what a graph of the two lines on the same coordinate system will look like.

SECTION 4.2 SUBSTITUTION METHOD

Suppose you were asked to find two numbers that had two conditions:

1. The sum of two numbers is 72
2. One number is 3 times as large as the other number

We write a system of equations corresponding to these conditions:

$$x + y = 72$$

and

$$y = 3x.$$

We have called this pair of equations a system of equations. We will now use an algebraic approach to compute the values of x and y of the point of intersection (x, y) and check that it is the same point we got with our graphing approach. We now present the *Substitution Method* for solving this system. Let's go through each step of the process.

- **Step 1:** We start with the system of two equations in two unknowns:

$$x + y = 72$$
$$y = 3x$$

- **Step2:** Substitute $3x$ for y into the first equation:

$$x + y = 72 \text{ becomes}$$
$$x + 3x = 72$$

- **Step 3:** Simplify the equation and solve for the value of x.

$$4x = 72 \text{ becomes}$$
$$x = \frac{72}{4} = 18$$

So, this must be the value of the first coordinate of our point of intersection. In fact, it is the only value of x that satisfies both of our equations.

- **Step 4:** Substitute this value of $x = 18$ into the equation $y = 3x$ and solve for y. $y = 3x$

$$y = 3(18) \text{ becomes}$$
$$y = 54$$

This is the value of the second coordinate of the point of intersection.

Using the x and y values we found above, the solution to the system is $(18, 54)$.

PROBLEM 1

Solve the following system of equations using the substitution method. Write each step carefully and show your work clearly. Check your answer by solving the system graphically.

$$x + y = 5$$
$$x = y - 3$$

EXAMPLE 1

Solve the following system of equations using the substitution method.

$$4x - y = 15$$
$$-2x + 3y = 12$$

SOLUTION It is always tempting to solve these systems by graphing. If you quickly graph these lines, you will notice that the point of intersection is not easy to determine visually. So let's begin by labeling the equations for ease of reference.

- **Step 1:**

$$4x - y = 15 \qquad \text{(Equation 1)}$$
$$-2x + 3y = 12 \qquad \text{(Equation 2)}$$

Unlike the previous examples, we do not have an equation that gives us a variable explicitly in terms of the other. For example, in the initial problem, we were given that $y = 3x$, with y given explicitly in terms of x. In Problem 1, you were given $x = y - 3$ with x explicitly in terms of y. In each case, we were able to substitute the equivalent expression for x or y into the other equation. In examining our current system of two equations, neither expresses one variable explicitly in terms of the other. How can we express one of the variables in terms of the other? Which variable would you solve for first and which equation would you use to solve for that variable? Does the choice matter? As it turns out, you may use either variable x or y but experience tells us that using Equation 1 and solving for the variable y turns out to be "easier." Here's why.
Equation 1 states: $4x - y = 15$. Solve for y to get $4x - 15 = y$. Since this is equivalent to the original equation, we restate our system with this equivalent equation and the original Equation 2. Our system now looks like this:

$$y = 4x - 15$$
$$-2x + 3y = 12$$

- **Step 2:** By substituting $4x - 15$ for y in Equation 2, we get

$$-2x + 3(4x - 15) = 12.$$

- **Step 3:** Solving for x, we get

$$-2x + 12x - 45 = 12$$
$$10x = 57$$
$$x = 5.7$$

- **Step 4:** Use this value of x in Equation 1 to solve for y.

$$y = 4x - 15$$
$$y = 4(5.7) - 15$$
$$y = 22.8 - 15$$
$$y = 7.8$$

Our solution is $(5.7, 7.8)$. Check to see if this solution satisfies both equations. Equation 1: $4x - y = 15$. Using our values we see $4(5.7) - 7.8 = 15$. Equation 2: $-2x + 3y = 12$. We get

$$-2(5.7) + 3(7.8) = -11.4 + 23.4 = 12.$$

We see that both equations are satisfied by the values 5.7 for x and 7.8 for y. Why would this solution be hard to see on a graph?

PROBLEM 2

In a right triangle, we know that one of the angles measures 90 degrees. The other two angles are acute and must add up to 90 degrees. Suppose the larger acute angle is 26 degrees more than twice the smaller. Write these conditions as a system of two equations in two unknowns, using A and B as the measures of the two acute angles. Use the substitution method to solve the system. Check your answers to see if the measures add up.

In the following section, you will learn a third method for solving a system of equations. Soon, you will see that each method has its advantages and disadvantages.

EXERCISES

1. Solve each of the following systems of linear equation using the method of substitution.

   a.

$$2x + 3y = 24$$
$$y = 2x$$

   b.

$$3x + 2y = 11$$
$$y = x - 2$$

   c.

$$x + 2y = 4$$
$$3x + 5y = 12$$

2. **Investigation:**
   How substitution relates to graphical method.

   a. Graph the lines associated with the equations $x + 2y = 4$ and $x+y = 3$ on the same coordinate system. Do the lines intersect? What point belongs to both lines? Verify that the point you choose satisfies both equations.

   b. Using substitution, we can rewrite the second equation as $x = 3-y$ and substitute into the first equation to yield the equation: $3 - y + 2y = 4$ which is equivalent to $y = 1$. Graph the line $y = 1$ on same coordinate system you made in part a. What do you notice?

   c. Substituting $y = 1$, into $x + 2y = 4$ gives $x + 2 = 4$ which is equivalent to $x = 2$. Graph the line $x = 2$ on the same coordinate system you made in part a. What do you notice?

Solve the systems in exercises 3 through 14 using substitution.

3. $y - 17 = 3x$ and $x + 2y = 41$

4. $5x - y = 11$ and $4x - 3y = 11$
5. $3x + 6y = 15$ and $5x - 4y = -3$
6. $x + 2(y - 1) = 7$ and $-4x + 3y = -14$
7. $x + 6y = 1$ and $5x + 3y = 2$
8. $y + 7 = 3x$ and $2x = 3y$
9. $4x - 2y = 20$ and $3x + y = 20$
10. $5\left(y - \frac{9}{5}\right) = x$ and $3x = 2\left(y - \frac{1}{2}\right)$
11. $y = 3x - 7$ and $3x - 4y = 15$
12. $x + y = 12$ and $3x + 3y = 4$
13. $y = 2x - 1$ and $6x - 3y = 3$
14. $y + 3(x - 7) = 0$ and $x - (y - 15) = -6$
15. Pat bought pies for a large party. Each deluxe lemon pie costs \$8 and each regular lemon pie costs \$4. She spent \$88 on her purchase (before tax) and bought a total of 16 pies. How many of each type of lemon pie did she buy?
16. **Ingenuity:**
Locate points using line intersections. Recall in Example 1 that the solution to the systems of equations was (5.7, 7.8). We decided to use the substitution method since this point of intersection would be hard to determine by simply looking at the graph. If you were asked to graph this point, you could simply graph the lines given by these two equations. You could trust that this process would give us an accurate plot of the point (5.7, 7.8). For each of the following points, find two equations whose lines intersect at the point. Do not use horizontal or vertical lines.
    a. $(\frac{1}{2}, 1)$
    b. $(\frac{3}{4}, 2)$
    c. $(5, \frac{1}{4})$
    d. $(3, \frac{5}{4})$
    e. $(\frac{1}{2}, \frac{1}{2})$
    f. $(\frac{3}{4}, \frac{1}{4})$
    g. $(\frac{1}{5}, \frac{2}{3})$
    h. $(\frac{5}{6}, \frac{10}{7})$

SECTION 4.3 METHOD OF ELIMINATION

In this section, we will solve linear systems using a different method called the *Elimination Method*. Beginning with two equations, we will use several properties to create equivalent equations. Recall that you can add or subtract the same quantity from each side of an equation and you can multiply or divide each side of an equation by the same non-zero number. We will first investigate how this effects an equation and the line that is describes.

Recall in 4.1, we looked at the problem with a pile of dimes and quarters that equaled 25 cents and modeled it as an equation. In 4.1 you used the variable n to represent the number of nickels and p to represent the number of pennies. What would the equation look like if we used $x =$ number of nickels and $y =$ number of pennies?

We shall write these equations using the variables x and y. Suppose (x, y) is a solution to this system. This means that the following of two equations are true:

Equation 1: $5x + y = 25$

Equation 2: $x + y = 13$

If we use the substitution method on this system, we subtract x from both sides of the equation $x + y = 13$ to obtain an equivalent equation, $y = 13 - x$. In our new method, we take the equation $x + y = 13$ and multiply both sides of the equation by -1. The resulting Equation 3 is $-x + -y = -13$ is equivalent to Equation 2. So we can write an equivalent system of equations as

Equation 1: $5x + y = 25$

Equation 3: $-x + -y = -13$

The idea behind the Elimination Method is to combine these two equations into one equation that involves only one variable. The strategy is to add each side of one equation to the corresponding sides of the other equation. In this case, add the equivalent quantities of $-x + -y$

and -13 to each side of the equation $5x + y = 25$, respectively.

$$\begin{aligned}
\text{Equation 4: } (5x + y) + (-x - y) &= 25 + (-13) \\
5x + -x + y + -y &= 25 + -13 \\
4x + (y + -y) &= 12 \\
4x &= 12 \\
x &= 3
\end{aligned}$$

Equation 1 and Equation 3 are combined to form Equation 4, which contains only the unknown x and has "eliminated" the variable y. Solving for x, we get $x = 3$. In order to solve for y, use either of the two original equations. Using Equation 2 and the fact that $x = 3$, you get

$$\begin{aligned}
3 + y &= 13 \\
y &= 10
\end{aligned}$$

Or, if you chose to use Equation 1 and the fact that $x = 3$, then the equation becomes

$$\begin{aligned}
5(3) + y &= 25 \\
15 + y &= 25
\end{aligned}$$

You only need to use one of the equations to find the value of the variable y. As expected, the solution to the system using the elimination method is the same as the solution obtained by graphing: 3 nickels and 10 pennies.

PROBLEM 1

Consider the following systems and discuss which variable you would choose to eliminate in each case and why. Do not solve the system.

a. $3x + 2y = 10$
$4x - 2y = 4$

c. $x + y = 9$
$-x + 2y = -3$

b. $-6x + 5y = 3$
$2x + 5y = 8$

d. $4x + y = 17$
$-4x - 5y = -5$

PROBLEM 2

Use elimination to solve the system:

$$2x + 3y = 11$$
$$x - 2y = -2$$

Check that the solution satisfies both equations.

When using elimination to solve a system of equations, you must decide which variable to eliminate. The coefficients of one of the variables must be the same or the negative of each other. For example, in Problem 1, each pair of terms involving the variable x in parts c and d and the variable y in parts a and b are ready for elimination. What do we do if neither variable is ready for elimination? For example, suppose you have a system of the form:

$$3x + 5y = 4$$
$$6x + 2y = -2$$

How can we transform the equations so we can eliminate the variable x in the new equation? We can multiply both sides of the first equation by 2 to get the system below:

$$6x + 10y = 8$$
$$6x + 2y = -2$$

or we can multiply both sides of the first equation by -2 to get the new system below:

$$-6x + -10y = -8$$
$$6x + 2y = -2$$

We could then "subtract" or "add" the equations, respectively, to eliminate the x variable. What would you do to the following system of equations in order to eliminate the variable y?

$$7x + 4y = 10$$
$$3x - 8y = 5$$

EXPLORATION 1

What is the effect on the graph of the line when we multiply sides of its equation by a number? In the table below, graph the Line 1 for each initial equation and then graph the Line 2 that is given by the new equation, which is derived from the initial equation by multiplying by the indicated factor. Use a new coordinate system for each pair of lines.

Initial Equation	Multiply by	New Equation
Line 1: $2x + y = 4$	Factor of 2	Line 2: $4x + 2y = 8$
Line 1: $3x + 2y = 5$	Factor of 3	Line 2: $9x + 6y = 15$
Line 1: $-2x + 5y = 3$	Factor of -2	Line 2: $4x - 10y = -6$

1. What do you notice about the graphs of Line 1 and Line 2 for each pair of equations?
2. Make a rule about what the effect is on the graph of a line if you multiply its equation by a non-zero factor.

In each of these examples, just one equation of the system needed to be modified. Next, you will explore systems in which the coefficients of the neither variable are multiples of each other. For example, you might have $3x$ and $7x$ or $-4y$ and $5y$. You will need to multiply both equations by factors so the equivalent equations have one variable with compatible coefficients.

PROBLEM 3

Solve the following system of linear equations by multiplying each equation by an appropriate factor in order to set up the elimination method:

Equation 1: $2x + 3y = 5$

Equation 2: $3x + 5y = 7$

EXPLORATION 2

What if we simply add the corresponding sides to form a new equation that does not eliminate one of the variables? The new equation will still have two variables. How will the line relate to the lines of the initial equations? To explore this question, graph the two lines from each initial system and then add these two equations to form a new equation. Then graph the line satisfying this new equation.

Initial Equation	New Equation
Equation 1: $2x + y = 4$ Equation 2: $x + 2y = -1$	Add the equations to get: $3x + 3y = 3$
Equation 1: $3x + y = 5$ Equation 2: $5x + 2y = 6$	Subtract Eq. 1 from Eq. 2 to get: $2x + y = 1$
Equation 1: $5x + -2y = 3$ Equation 2: $2x + y = -1$	Subtract Eq. 2 from Eq. 1 to get: $3x - 3y = 4$

Did you observe the following patterns:

1. If you multiply each side of an equation to form a new equation, the line it describes is the same.

2. If you add or subtract the corresponding sides of two equations that have different lines, the new equation is not equivalent to either of the first two equations.
3. The new line also intersects each of the first two lines at the same point.

PROBLEM 4

Solve the system of equations by elimination.

$$5x + 4y = -39$$
$$3x + 2y = -29$$

EXAMPLE 1

Suppose a summer camp orders 4 large pizzas and 2 small pizzas on Monday at a cost of \$57.14 before taxes and 3 large pizzas and 3 small pizzas on Tuesday at a cost of \$52.74. What is the cost of each of the large pizzas? What is the cost of each of the small pizzas?

SOLUTION We define each of the quantities from the problem.

$$x = \text{the cost of each of the large pizzas}$$
$$y = \text{the cost of each of the small pizzas}$$
$$57.14 = \text{cost of pizzas on Monday}$$
$$52.74 = \text{cost of pizzas on Tuesday}$$
$$4x + 2y = \text{cost in dollars of all pizzas on Monday}$$
$$3x + 3y = \text{cost in dollars of all pizzas on Tuesday}$$

Thus,

$$4x + 2y = 57.14$$
$$3x + 3y = 52.74$$

What method should we use to solve this system? The decimals numbers would make graphing difficult because of accuracy. Substitution would require solving for either x or y, and would require the use of fractions. However, the elimination method, as we shall see, is an easier way to solve the system. Again, the first step is to decide which variable to eliminate. Either variable is a candidate. Let's make a decision to eliminate the variable x. In order to do this, the coefficients for the x variable must add up to 0. At present, the coefficients add up to 7. Do you see why? Since the least common multiple for 4 and 3 is 12, what can we multiply the first equation by to get -12? What can we multiply the second equation by to get 12? We multiply each side of the first equation by -3 to get $-12x - 6y = -171.42$. We multiply both sides of the second equation by 4 to get

$$12x + 12y = 210.96$$
$$-12x - 6y = -171.42$$

Adding the left sides of the equations, we get $6y$. Adding the right sides, we get 39.54. We eliminated the x variable and can now solve for y by dividing the equation by 6 to get

$$6y = 39.54$$
$$y = 6.59$$

This is the cost of each small pizza. We must also find the cost of each large pizza. We know from the original system that $4x + 2y = 57.14$.

Substituting $y = 6.59$, we have

$$4x + 2(6.59) = 57.14$$
$$4x + 13.18 = 57.14$$

Solving for x, we have

$$4x = 43.96 \text{ or}$$
$$x = 10.99$$

This is the cost of each large pizza.

Systems of equations often model real world problems and provide a systematic way of solving these types of problems. Modeling the system is often the most challenging part of the problem. The choice of the method you may wish to use depends on this system that you identify. In order to make an efficient choice, you should feel confident about each method. For practice, do the following so that the procedural steps and the reasoning behind it are clear to you.

In the next section, you will explore applications of systems of linear equations in two variables. These problems are useful in math, science, business, and engineering.

EXERCISES

1. Solve each of these systems of linear equations by elimination:
   a.

$$2x + 3y = 17$$
$$3x - 4y = -17$$

   b.

$$3x + y = -10$$

$$5x + 3y = -14$$

c.

$$x + 2y = 11$$
$$4x - 5y = -8$$

d.

$$x + 2y = 1$$
$$4x - 5y = 30$$

e.

$$5x + 3y = 7$$
$$2x - 7y = 11$$

f.

$$5x + 3y = 7$$
$$3x - 7y = -9$$

2. Solve the systems of equations in Problem 1 on page 252 using elimination.
3. Solve each of these systems of linear equations by the method of elimination:

a.

$$3x + 5y = 9$$
$$2x + 6y = 10$$

b.

$$3x + y = 9$$
$$6x - 3y = -2$$

c.

$$\begin{aligned} -3x + 2y &= 2 \\ 4x - 5y &= -1 \end{aligned}$$

d.

$$\begin{aligned} 2x + 2y &= 3 \\ 3x + 5y &= 2 \end{aligned}$$

e.

$$\begin{aligned} 5x + 4y &= -2 \\ 8x + 7y &= -4 \end{aligned}$$

f.

$$\begin{aligned} 7x - 5y &= -7 \\ 5x - 3y &= -6 \end{aligned}$$

4. Suppose Michael and Leo together have $85. Three-fourths of Michael's amount is equal to two-thirds of Leo's amount. How much money do Michael and Leo each have? First use the information to model a system of two equations and two unknowns. Then solve the system using any two methods you would like to use and confirm that your solution makes sense.
5. Edna claims that she has 35 old gold and silver coins. The gold coins weigh 5 ounces each and the silver coins weigh 3 ounces each. If the total weight of the coins is 139 ounces, how many of each kind of coin does she have? First use the information to model a system of two equations and two unknowns. Then solve the system using any two methods you would like to use and confirm that your solution makes sense.

6. **Investigation:**
Consider the following systems of equations:

Equation A: $y = x + 2$ Equation B: $y = 2x - 5$
Equation A: $y = 2x + 1$ Equation B: $y = -3x + 6$

In Exploration 2 on page 255, you explored the effects of adding two equations. Use a graphing calculator to help visualize these effects. Graph and compare the initial lines and the new lines you get from generating new equations by

- adding the equations
- multiplying equation A by 2 and adding it to equation B
- multiplying equation B by 3 and adding it to equation A
- now make up your own pattern.

Solve problems 7 to 12 using elimination.

7.

$$6(x - y) = 18$$
$$4(x + y) = 20$$

8.

$$2x + 5y = 2$$
$$3x - y = 3$$

9.

$$3x - 7y = 1$$
$$x + 4y = 1$$

10.

$$3x = 4y - 1$$
$$2x + y = -1$$

11.

$$3x + 4y = 6$$
$$x = -\frac{4}{3}y + 1$$

12.

$$y = 3x - 11$$
$$3x - y = 11$$

13. $2x - 7y = 3$ and $3x = 2y - 4$
14. $\frac{1}{2}x + \frac{1}{3}y = 4$ and $x - y = 5$
15. $\frac{2}{5}x - y = 6$ and $x + \frac{1}{3}y = \frac{5}{6}$
16. $3.1x + 2.5y = 1$ and $4x - y = 52$
17. **Ingenuity:**
    a. Mayte was asked to solve the system of equations:

$$2x + 3y = 15$$
$$5x + 2y = 21$$

"That's easy," Mayte said. "Look at the pattern." Then she wrote down:

$$8x + y = 27$$
$$11x + 0y = 33$$
$$x = 3$$

Is Mayte correct? What pattern do you think she is seeing? Why does it work?

b. Use the number line below to visualize Mayte's method.
    - Simplify $((5x + 2y) - (2x + 3y))$. What does this represent on the number line?

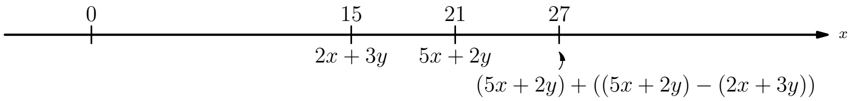
- Plot and label the point:

$$(5x + 2y) + (x - y)$$

- Plot and label the point:

$$(8x + y) + (x - y)$$

- Finally, plot and label the point x.



c. Use the same method to solve the following system of equations.

$$3x + 4y = 10$$
$$-3x + 2y = 23$$

SECTION 4.4 APPLICATIONS

So far we have practiced three methods for solving systems of linear equations: graphing, substitution and elimination. In this section we will explore several different kinds of word problems that can be modeled by a system of linear equations. We begin with a type of mixture problem from science class.

EXAMPLE 1

In a chemistry laboratory, Jeremy has a 75% acid solution and a 50% acid solution. How much of each solution should he mix to get 100 liters of a 60% acid solution?

SOLUTION We begin by identifying each quantity from the given information. The amount of the two solutions should add up to 100 liters.

let x = liters of the 75% solution needed
let y = liters of the 50% solution needed
$x + y$ = liters of total solution
100 = liters of total solution
So, $x + y = 100$
.75 = rate of acid in x liters of 75% solution
$.75x$ = liters of pure acid in x liters of 75% solution
.50 = rate of acid in y liters of 50% solution
$.50y$ = liters of pure acid in y liters of 50% solution
$.75x + .5y$ = liters of pure acid in the mixture of $x + y$ liters acid solution
.60 = rate of acid in 100 liters of mixture
.60(100) = 60 = liters of pure acid of mixture
So, $.75x + .5y = 60$

We must solve the system:

$$
\begin{aligned}
x + y &= 100 \\
.75x + .5y &= 60
\end{aligned}
$$

Let's take the first equation and solve for y, the equation becomes

$$y = 100 - x$$

By substituting this expression for y in the second equation we get

$$.75x + .5(100 - x) = 60$$

By simplifying the left hand side, we get

$$
\begin{aligned}
.75x + 50 - .5x &= 60 \\
.25x &= 10 \\
x &= \frac{10}{.25} = 40 \text{ liters}
\end{aligned}
$$

Since we need 40 liters of 75% acid, the amount of 50% acid is $y = 100 - 40 = 60$ liters.

For this problem, we chose to solve the system using substitution instead of graphically or using elimination. Why do you think we did this?

PROBLEM 1

Suppose a store sells two kinds of candies. Candy A is sold for \$6 per pound and candy B for \$4 per pound. The manager wants to make a 100 pounds of a mixture that she could sell for \$4.50 per pound. How much of candy A and how much of candy B should she mix together to make the new \$4.50 per pound mixture?

Another type of "mixture" occurs in banking. Interest rates are used in savings accounts, certificates of deposits or loans.

EXAMPLE 2

Investment. Anna decides to invest \$500 in two different accounts. One account offers an interest rate of 5% per year and the other account offers an interest rate of 8% per year. If Anna earns \$35.50 in interest in one year, how much money did she invest in each account? First model the problem as two equations in two unknowns and then solve.

SOLUTION Simple interest, denoted by the variable I, is computed by taking the initial amount invested, called the principal, P, and multiplying it by the annual interest rate, r, and the number of years the money is invested t. The formula is:

$$I = Prt$$

We want to know how much of the \$500 is invested in each account. Let's list all of the given information:

$$\begin{array}{lcl} x & = & \text{amount invested at 5\%} \\ y & = & \text{amount invested at 8\%} \\ x+y & = & \text{total invested} \\ 500 & = & \text{total invested} \end{array}$$

So, $x + y = 500$.

Using $I = Prt$, we can also represent the interest earned.

$$\begin{array}{lcl} .05x & = & \text{interest earned at 5\%} \\ .08y & = & \text{interest earned at 8\%} \\ .05x + .08y & = & \text{total interest earned} \\ 35.50 & = & \text{total interest earned} \end{array}$$

Thus $.05x + .08y = 35.50$. So we must solve the system:

$$x + y = 500$$
$$.05x + .08y = 35.50$$

You use either substitution or elimination to solve this equation. Solving the first equation for y, we get

$$y = 500 - x$$

By substituting this into the second equation, we get

$$.05x + .08(500 - x) = 35.50$$

By simplifying the left side gives us

$$.05x + 40 - .08x = 35.50$$
$$-.03x = -4.5$$

By dividing both sides by -.03, we get

$$x = \frac{-.45}{-0.3} = 150.$$

This means Anna's investment at 5% is in the amount of $150. To determine Anna's investment at 8%, we use the expression,

$$y = 500 - x$$
$$y = 500 - 150 = 350$$

Therefore, Anna's investment at 8% is $350.

Now let's explore a distance, rate and time problem.

EXAMPLE 3

A boat on a river travels 96 miles upstream in 8 hours. The return trip takes the boat 6 hours. The rowers on the boat are doing the same amount of work at all times and travel at a constant speed in still water. Find the rate of the boat and the rate of the current.

SOLUTION The formula that relates distance, rate and time is $d = rt$. The distance upstream and the trip downstream is 96 miles. We know the trip upstream takes 8 hours and the trip downstream takes 6 hours. The times are different because of the current of the river. Write the information as follows:

96 = miles of each trip
8 = hours of trip upstream
6 = hours of trip downstream
x = rate of the boat in still water in miles per hour
(if there were no current)
y = rate of the current

Can we combine some of these to form new quantities? We could apply the distance formula to both trips if we knew the rate of speed of the boat traveling upstream and the rate of the boat traveling downstream. The current slows the rate of boat going upstream. The upstream rate is $x - y$. The rate down stream is increased with the help of the current. The downstream rate is $x + y$.

$x - y$ = rate of boat going against the current traveling upstream
$x + y$= rate of boat going with the current traveling downstream
$(x - y)\frac{\text{mi}}{\text{hr}} \cdot 8$ hr = miles of distance traveled upstream
$8(x - y) = 96$

or

$$\text{Equation 1: } 8x - 8y = 96$$

The calculations for the trip downstream gives us

$$(x+y)\frac{\text{mi}}{\text{hr}} \cdot 6 \text{ hr} = \text{ miles of distance traveled down stream}$$
$$6(x+y) = 96$$

or

$$\text{Equation 2: } 6x + 6y = 96$$

Let's rewrite the system in the more familiar format and solve using elimination:

$$8x - 8y = 96$$
$$6x + 6y = 96$$

If possible, it is a good idea to simplify the equations before deciding how to eliminate variables. In this case, it is possible to simplify equation 1 by dividing each side by 8 and equation 2 by dividing each side by 6 to get the following equivalent system:

$$x - y = 12$$
$$x + y = 16$$

Add these new equations to get

$$2x = 28$$
$$x = 14$$

Now solve for y to get

$$14 + y = 16$$
$$y = 2$$

This tells us that the rate of the boat in still water is 14 miles per hour and the rate of the current is 2 mph. Why do you think we used elimination here instead of substitution or graphing? Since we already had the opposite coefficients on the y, elimination was easy to use here.

PROBLEM 2

A driver averages 40 mph going from town A to town B. On the return trip the driver averages 56 mph and takes 2 hours less. What is the distance between towns A and B? How many hours did the driver spend going from town A to town B? Model the problem as a system of equations and then solve using an appropriate method.

EXAMPLE 4

A company sells DVDs for \$10 each. The company manufactures these DVDs at a cost of \$2 each with a fixed set-up cost of \$240. If the company manufactures and sells x DVDs:

1. Write a function that expresses the cost to the company for making x DVDs to sell.
2. Write a function that expresses the revenue the company makes when they sell x DVDs.
3. Graph the two functions on the same coordinate system noting the domain of the functions.

SOLUTION The formula for the cost is $C(x) = 2x + 240$ and the formula for the revenue is $R(x) = 10x$ where x is the number of DVDs produced and sold. Since the graphs of these functions are lines, you can also describe these using the slope-intercept form. The two equations are written as

$$y = 10x$$
$$y = 2x + 240$$

You should note that the two y's are amounts in dollars but one y is cost and the other is revenue. What does the intersection of the two lines represent?

This point has a very special name in business and economics. The point is called the Break-Even point; where the cost equals revenue. We can determine the Break-Even point for the DVD problem using substitution:

$$\begin{aligned} y &= 2x + 240 \\ 10x &= 2x + 240 \\ 8x &= 240 \\ x &= 30 \end{aligned}$$

To find y substitute $x = 30$ into $y = 10x = 10 \cdot 30 = 300$. If the company sells 30 DVD's, the revenue and cost will be \$300. And the profit will be 0. What happens if the company sells less than 30 DVDs? What happens if the company sells more than 30 DVDs?

There are many other problems that can be modeled as a system of two equations in two unknowns. Read the problems below carefully and you should be able to set up a system of linear equations that can help you solve the problem efficiently. The method that you choose is now up to you! The method you choose can simplify the solution process greatly. Knowing which method to choose comes with practice.

EXERCISES

1. Jack has 35 hamsters and gerbils. He has 4 times as many gerbils as hamsters. How many of each animal does he have?
2. A shop owner sells organic pecans for \$8 per lb. and organic walnuts

for \$5 per lb. He wants to make 60 pounds of a mixture to sell for \$6.80 per pound. How many pounds of pecans and how many pounds of walnuts should he put in the mixture?

3. A chemist wants to mix a 25% acid solution with a 50% acid solution to produce 1000 liters of a 34% acid mixture. How many liters of the 25% acid solution and how many liters of the 50% acid mixture should the chemist use?
4. Linda had a phenomenal game in basketball. She scored a total 41 points and made 17 shots. If she did not make any free throws, how many 2 point shots and how many 3 point shots did she make?
5. A dairy farmer mixes 4% milk with 1% milk to make 2% milk. If he wants to make 600 gallons of 2% milk, how many gallons of 4% milk and 1% milk does he need to mix?
6. A teacher and some parents took a class to a concert. The tickets cost \$5.50 for students and \$7.50 for adults. They bought 30 tickets and the total cost of the tickets was \$177. How many student tickets and how many adult tickets did they buy?
7. Vanessa invested \$400 for a year in two funds. Fund A earned 4% per year and Fund B earned 5% per year. If her investments yielded \$18.50 in interest, how much did she invest in each fund?
8. Two snails were 244 feet apart. They started moving toward each other. The big snail starts at point A and travels at the speed of 7 ft/min and the small snail starts at point B and travels at the speed of 5 ft/min. How long did it take the two snails to meet?
9. For the class picnic, Mr. Garza bought two kinds of hot dogs. The beef hot dogs cost \$3 per pound and the turkey hot dogs cost \$5 per pound. He bought 10 pounds in all and spent \$35. How many pounds of each kind of hot dog did he buy?
10. Rent A Car Enterprises (RACE) charges \$60 to rent a car plus \$0.50 per mile driven. Tony's Rental and Custom Kars (TRACK) charges \$20 to rent a car plus \$1.25 per mile driven. When is it cheaper to rent from RACE instead of TRACK?
11. Each cell phone text messaging plan charges a flat monthly fee for having text-messaging service plus an additional cost per text

message.

Plan	Service Cost per Month	Cost per Msg
A	$15	$0.05
B	$10	$0.10
C	$25	unlimited texting

In terms of the number of text messages, when is A the best plan? When is B the best plan? When is C the best plan?

12. Fred wants to make a fruit punch with 26% pure grape juice. He has a large supply of both 20% grape juice and 40% grape juice. How much of each should he mix together to make 1 gallon of 26% mixture?

13. Jack and Jill are 90 miles apart. Jack is in city A and Jill is in city B. They ride bikes to meet. Jack rides at 10 mi/hr and Jill rides at 15 mi/hr. When will they meet? How far does each ride? Let $f(t)$ $=$ the distance that Jack is from city A after t hours and $g(t) =$ the distance that Jill is from city A after t hours. Find formulas for f and g. For what value of t is $f(t) = g(t)$?

14. Train station A is 400 miles west of Train Station B. Train X travels towards station B at 40 mph and train Y travels toward station A at 60 mph.
    a. If both trains leave their respective stations at 6AM, when will they meet?
    b. If the two trains both travel in opposite directions when will they be 1000 miles apart?
    c. If both trains travel westward, when will train Y overtake train X?
    d. If both trains travel eastward, when will the distance between the trains be exactly 600 miles?

15. A boat can travel 10 mph in still water. In a river with a current of 4 mph, it travels downstream for an unknown distance, then turns upstream and returns to its starting point in 5 hours. How far did it travel before turning?

16. Due to a river's current, a boat travels twice as fast going downstream as it does traveling upstream. It makes a round trip

of 60 miles in $7\frac{1}{2}$ hours. What is the boat's rate in still water? What is the rate of the current?

17. A boat traveling downstream passes a floating log at noon. Two hours later the boat turns and proceeds upstream. At what time will the boat meet the log?

18. Reversing the digits of a two-digit number increases its value by 36. If the sum of its digits is 10, what is the original number?

19. Reversing the digits of a two-digit number forms a new two-digit number and decreases its value by 27. Find all the two-digit numbers for which this is true.

20. Reversing the digits of a three-digit number increases its value by 693. For how many numbers is this true?

21. Danny has 17 coins; all either nickels or dimes. if the coins have a total value of $1.35, how many nickels does he have?

22. Billy is three times as old as his son. In 13 years, he will be twice as old as his son. How old is Billy now?

23. Harry can paint a house in 6 days and Mike can paint the same house in 8 days. Harry begins the job working alone. After 2 days, Mike joins him and they work together until the job is completed. How many days do they work together?

24. The numerator and denominator of a fraction are positive integers. If 3 is added to both the numerator and denominator, the fraction's value is $\frac{4}{5}$. If 6 is subtracted from both the numerator and denominator, its value is $\frac{2}{3}$. What is the original fraction?

25. **Ingenuity:**
    Albert leaves at noon for a long hike, walking at a pace of 3 miles per hour. Maria leaves at 2:00 PM, walking at a pace of 5 miles per hour in the same direction. When will she catch Albert?

26. **Ingenuity:**
    Suppose we have the following two equations that relate the variables x, y and z:

$$\begin{aligned} 3x + 6y - 2z &= 4 \\ x + 2y + z &= 3 \end{aligned}$$

What values of x, y and z make both of these equations true? If you check the values of $x = 4$, $y = -1$, and $z = 1$, then both of the equations are true. However, these equations are also true if $x = 2$, $y = 0$, and $z = 1$. This is different from the systems of equations that we have seen in this section because there is more than one solution to the set. Are there any other solutions to these two equations? Can you find all of them?

SECTION 4.5 CONSISTENT AND INCONSISTENT SYSTEMS

EXPLORATION 1

Take two uncooked spaghetti noodles and place them on a coordinate grid. How many ways can the two noodles intersect? Classify how many cases there can be. Illustrate the pair(s) in different coordinate grids.

If we think of these pairs of uncooked noodles as parts of lines, then this exercise can be viewed as modeling graphs of systems of two linear equations. We have seen many examples where two lines intersect. In what type of a problem might two equations in the system produce parallel lines? In what type of problem might two equations actually be the same line when graphed? In this section we will classify these 3 types so that you can predict the outcome of the respective solutions.

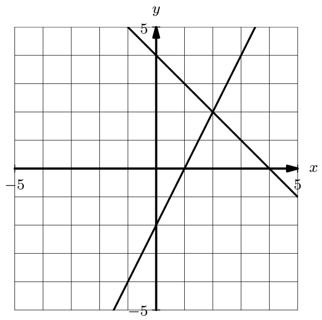
CONSISTENT SYSTEM OF EQUATIONS
A system with one or more solutions is called *consistent*. A system with no solutions is called *inconsistent*.

In Exploration 1, the three different possible systems were:

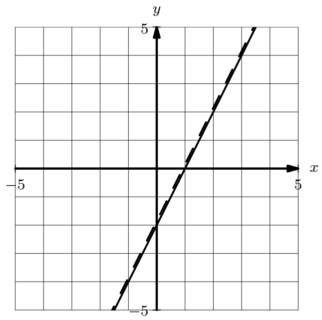
1. The lines intersect at one point.
2. The lines are the same.
3. The lines are parallel.

We can use the definition of consistent and inconsistent systems to determine which category the three systems fit. When two lines intersect in exactly one point, the intersection point is the solution to the system. Therefore, the system is consistent.

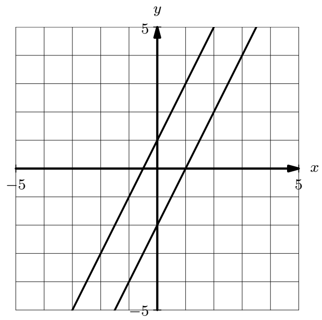
Notice that when two equations of the system represent the same line, the system has infinitely many solutions and is consistent. All the points on the line are part of the solution set. Lastly, if the lines are parallel, they share no points in common and hence have no solution. This is an inconsistent system.



Consistent
$y = 2x - 2$
$y = -x + 4$



Consistent
$y = 2x - 2$
$2x - y = 2$



Inconsistent
$y = 2x - 2$
$y = 2x + 1$

Graphing the lines is very helpful for determining the kinds of solutions you may obtain. Of course, finding the solution still requires some work and you have used several techniques for solving a system from sections 4.1-4.3. Suppose you do not have a graph and approach the problem directly with either the substitution or elimination method. What would happen for each of the three categories? Let us do a little investigating.

EXPLORATION 2

1. Write a system with two equations of parallel lines. Give reasons for the equations that you chose. Use the substitution method to solve the system as in section 4.2. What happened? What does your result suggest? Does this happen anytime you start with two equations of parallel lines? Discuss the possible reasons for this.
2. Write two equivalent but not identical equations that give the same line as their graph. Use substitution to solve the system. What happened? What does this result suggest? Does this happen anytime you start with two equations that are equivalent? Discuss the possible reasons for this.

In the next example, let's see what happens if you try to use the algebraic methods to solve an inconsistent system.

EXAMPLE 1

Consider the system: $2x + 3y = 6$ and $2x + 3y = -6$.

1. Explain why the lines given by the equations are parallel.
2. Try to solve the system using the substitution method. What happens? What does this mean?
3. Try to solve the system using the elimination method. What happens? What does this mean?

SOLUTION

1. Writing the equations in slope-intercept form gives $y = -\frac{2}{3}x + 2$ and $y = -\frac{2}{3}x - 2$. The lines have the same slope but different y-intercepts, so they are parallel.
2. **Substitution Method:** From above, we know that $2x + 3y = 6$ is equivalent to $y = -\frac{2}{3}x+2$. If we substitute into the second equation, we get

$$\begin{aligned} 2x + 3(-\frac{2}{3}x + 2) &= -6 \\ 6 &= -6 \end{aligned}$$

   This is impossible. What does this mean?
3. **Elimination Method:** Subtracting the original second equation from the first,

$$\begin{array}{rrcl} & 2x + 3y & = & 6 \\ - & (2x + 3y & = & -6) \\ \hline & 0 & = & 12 \end{array}$$

   This is also impossible.

From Example 1, we see that if we try to solve a system of linear equations using one of the algebraic methods and the process leads to a false or impossible statement like $6 = -6$ or $0 = 12$, it doesn't mean we have made an error, it just means that the system was inconsistent.

Let's see what happens when the system is made of two equivalent equations.

EXAMPLE 2

Consider the system $2x + 4y = 2$ and $3x + 6y = 3$.

1. Explain why the lines given by the equations are the same.
2. Try to solve the system using the substitution method. What happens? What does this mean?
3. Try to solve the system using the elimination method. What happens? What does this mean?

SOLUTION

1. Writing the equations in slope-intercept form gives $y = -\frac{1}{2}x + 1$ for both equations.
2. **Substitution Method:** From above, we know that $2x + 4y = 2$ is equivalent to $y = -\frac{1}{2}x + 1$. If we substitute into the second equation,

$$\begin{aligned} 3x + 6(-\frac{1}{2}x + 1) &= 3 \\ 3 &= 3 \end{aligned}$$

   which is always true. Can you name two ordered pairs that would satisfy both equations? Do all ordered pairs satisfy both equations? Can you name an ordered pair that does **not** satisfy both equations?
3. **Elimination Method:** Stack the two equations,

$$\begin{aligned} 2x + 4y &= 2 \\ 3x + 6y &= 3 \end{aligned}$$

Multiply the first equation by 3 and the second equation by -2. The system now looks like

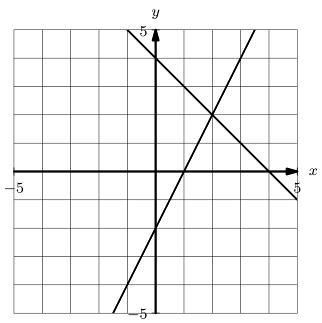
$$
\begin{aligned}
6x + 12y &= 6 \\
-6x - 12y &= -6
\end{aligned}
$$

If we add the second equation to the first, we get $0 = 0$. This statement is always true.

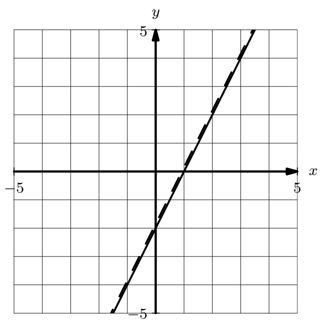
From Example 1, we know that if we try to solve a system of linear equations using one of the algebraic methods and the process leads to a true statement not involving either of the variables like, $6 = 6$ or $0 = 0$, it means that the system is consistent but made of two equivalent equations.

To distinguish between the two consistent cases, we define the new terms "independent" and "dependent".

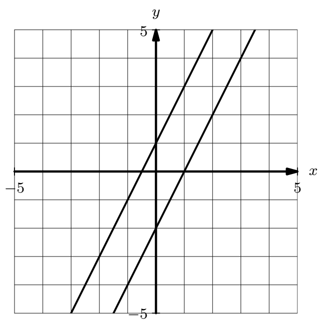
DEPENDENT AND INDEPENDENT
A consistent system with exactly one solution is called *independent*. A consistent system with more than one solution is called *dependent*.



Consistent and **Independent**
$y = 2x - 2$
$y = -x + 4$



Consistent and **Dependent**
$y = 2x - 2$
$2x - y = 2$



Inconsistent
$y = 2x - 2$
$y = 2x + 1$

EXERCISES

1. Use the terms consistent, inconsistent, dependent and independent to answer the following questions. Explain your reasoning.
   a. If the graph of a system of equations is parallel lines, then what type of system is this?
   b. If the system of equations has a unique solution $(1, -2)$, then what type of system is this?
   c. If the system of equations has the same slope but different intercepts, then describe the graph of this system. What type of system is this?
   d. If the system of equations has the same slope and same intercepts, then describe the graph of this system. What type of system is this?
2. Consider the following systems. Without solving the system, classify them as consistent or inconsistent and further identify any consistent system as either dependent or independent. Explain your reasoning.
   a.

$$\begin{aligned} y &= 3x - 1 \\ y &= 3x + 1 \end{aligned}$$

   b.

$$\begin{aligned} y &= 3x - 1 \\ 3y &= 9x - 3 \end{aligned}$$

   c.

$$\begin{aligned} 7x - y &= 1 \\ 2x + y &= 1 \end{aligned}$$

3. Solve each system. Identify them as consistent or inconsistent. If consistent, identify them as dependent or independent.
   a.

$$x + 2y = 3$$

$$2x + 4y = 6$$

b.

$$x + 2y = 4$$
$$x = 8 - 2y$$

c.

$$2 + 9x = 3$$
$$y = 3x + 8$$

d.

$$x - y = 7$$
$$x + y = 13$$

e.

$$2x - y = 4$$
$$6x = 3y + 12$$

f.

$$3x - y = 8$$
$$9x + 2 = 3y$$

4. Write an inconsistent system by introducing another equation to pair with the given equation. Justify your choice.
   a. $y = 3$
   b. $x = 2$
5. The sum of the digits of a two-digit number is 10. If the digits are reversed the new number is 18 more than the original number. Use a system of two equations in two unknowns to find the original number. Show all your work. Is this an example of a consistent or inconsistent system?
6. Consider the following points: Point A $(5, -2)$, Point B $(0, 0)$, Point C $(2, 1)$ and Point D $(8, 20)$. For each system below, identify it as consistent or inconsistent. If consistent, identify it as dependent or independent. Then determine which of the points (if any) is a solution to the system. Explain why or why not.
   a.
$$x + y = 3$$
$$x + y = 7$$
   b.
$$x + y = 3$$
$$y = 3 - x$$
   c.
$$y = 4 + 2x$$
$$y = 3x - 4$$

For questions 7 to 16, determine whether the following systems of equations are consistent or inconsistent. If the system is consistent, determine if it is dependent or independent. Explian your reasoning.

7. $x - 3y = 1$ and $4x = 11y + 2$
8. $y = 3x - 7.7$ and $3y - 9x = 23.1$
9. $y = 4x + 17$ and $y - x = 11$

10. $3x + 7y = 51$ and $x = -\frac{7}{3}y + 17$
11. $3x - 5y = -10$ and $x = 2y + 1$
12. $4x - 10y = 22$ and $6x = 15y + 33$
13. $x - 7y = 100$ and $x + 7y = 0$
14. $y = \frac{2}{3}x + 6$ and $2x - 3y = -18$
15. $4x - 11y = 22$ and $11x = 4y - 22$
16. $7 + 2x = y$ and $3y - 6x = 28$

SECTION 4.6 CHAPTER 5 REVIEW

Key Terms

break-even point
coefficient
consistent system of equations
dependent system
elimination method
graphing method
inconsistent system of equations
independent system
intersect
least common multiple
linear equation
parallel lines
simultaneously
solution to the system
substitution method
system of linear equations

Formulas

distance = rate · time   interest = principle · rate · time

Practice Problems

1. Solve by graphing.
   a. $3x + 5y = 3$, $y = 4x - 2$
   b. $y = -x + 3$, $2x = y + 6$
   c. $4x + 2y = 1$, $-2x + 3 = y$
2. Solve by graphing. Write the system of equations before solving. Kayla bought shin guards and jerseys at *Sports R Us*. She spent \$12.00 per set of shin guards and \$20.00 for each shirt. She spent \$108.00 and bought 7 items. How many of each did she buy?
3. Solve by substitution.
   a. $5x - y = 11$, $4x - 3y = 11$
   b. $3x - 7 = y$, $2x + 9 = y$
   c. $3x + 2y = 1$, $y = -1.5x + 2$
   d. $y + 7 = 3x$, $2x - 3y = 4$

4. Solve by elimination.

   a. $6x - 4y = 10$
 $6x + 8y = 10$

   b. $2x - 6y = 4$
 $-x - 3y = 12$

   c. $3x + 3y = 21$
 $2x + y = 40$

5. Set up a system of equations and solve.

   a. One week Pedro buys 3 baseballs and 2 gloves for \$40.50. The next week, he spends \$41.25 for 5 balls and 1 glove. Assuming he bought the same type of glove and ball for each week, what is the cost of each?

   b. A class of 30 students has 4 times as many girls as boys. How many girls and boys are there in the class?

   c. The bark-a-roni Dog Food company sells dog food for \$40 per bag. It manufactures the dog food for \$10 per bag and a flat fee of \$300. Find the break-even point.

   d. Marie is three times as old as her daughter. In 15 years she will be twice as old as her daughter. How old is Marie?

6. Solve each system. Identify each as consistent or inconsistent. If consistent, identify them as dependent or independent.

   a. $2x + 4y = 6$
 $4x + 8y = 12$

   b. $3x + 6y = 12$
 $24 - 6y = 3x$

   c. $6 + 27x = 9$
 $27x + 72 = 9y$

   d. $3x - 4y = 8$
 $9x + 6 = 3y$

LINEAR INEQUALITIES

5

SECTION 5.1 PROPERTIES OF INEQUALITIES

In Chapter 1, we introduced *inequalities*, statements which compare two given quantities. We can use inequalities to represent statements such as the following:

- Lawrence is taller than Molly
- A duck is lighter than an elephant
- Corinna made a B on her math test

Let's look closer at how the first two statements correspond to inequalities. If we let L be Lawrence's height and let M be Molly's height, then the first statement is simply saying that $L > M$. If we let d and e be the weight of a duck and the weight of an elephant, respectively, then the second statement is saying that $d < e$. It is not as clear how we might represent the third statement as an inequality; let's examine what this statement is saying. When we say Corinna made a B on a test, we are saying that she earned enough points on the test to earn a B, but not enough points to earn an A. If we assume that the test was graded on a standard grading scale, this means that Corinna earned at least 80 points, but she did not earn 90 points. So if x is Corinna's grade, then we can say $80 \leq x$ **and** $x < 90$. To save space, we write this as the *compound* *inequality*: $80 \leq x < 90$.

If a and b are two numbers, what exactly does it mean to say that $a < b$? On a number line $a < b$ if and only if a is to the left of b. But what exactly does it mean for a to be to the left of b on the number line? We can define the statement $a < b$ more formally as follows:

LESS THAN, GREATER THAN

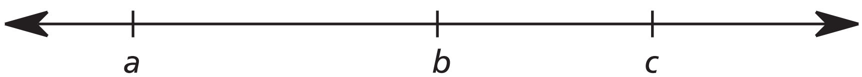
The statement that the number a is *less than* the number b, written $a < b$, means that there is a positive number x such that $b = a + x$. This number x must be $b - a$.

If a is less than b, we also say that b is *greater than* a; this is written $b > a$.

To illustrate if $a = 3$ and $b = 10$, we say that $3 < 10$ because $10 = 3+7$ and 7 is a positive number. When we say that a duck's weight is less than an elephant's weight, or $d < e$, we are saying that there is a positive weight w such that $d+w = e$. That is, there is an amount of weight that we could strap to the duck's back so that the duck plus the extra weight would weigh the same as the elephant (though we probably wouldn't actually do it for fear of harming the duck).

In this section, we will explore properties of inequalities that allow us to take known inequalities and transform them into other inequalities that are also true.

First, suppose that a, b, and c are numbers such that $a < b$ and $b < c$. Then this means that a is to the left of b on the number line, and b is to the left of c. This means that a is to the left of c:



In a numerical example, since $1 < 7$ and $7 < 10$, then we know that $1 < 10$. We formally state this as:

TRANSITIVE PROPERTY OF INEQUALITY
Suppose that a, b, and c are numbers such that $a < b$ and $b < c$. Then $a < c$.

The transitive property of inequality tells us that if we have several numbers, then we can put these numbers in order even if we are given only a few comparisons between pairs of the numbers. For example, if we know that $a < b$, $a > c$, and $c > d$, then the four numbers, in increasing order, are d, c, a, and b. We can summarize this by writing several inequalities in a row: $d < c < a < b$.

In section 1.4 we studied properties of equality, for example, we found that when you add the same amount to both sides of an equation, you obtain an equivalent equation. In the next two explorations we examine what happens when you add the same amount to both sides of an inequality.

EXPLORATION 1

1. Draw a number line. Be sure to locate and label 0 and 1.
2. Label two points on the number line a and b with a to the left of b. These can be any points you like as long as $a < b$.
3. Locate $a + 1$ and $b + 1$ on the number line. Is $a + 1 < b + 1$? How can you tell? Do you think it depends on your choice of a and b?
4. Locate $a + 2$ and $b + 2$ on the number line. Is $a + 2 < b + 2$?

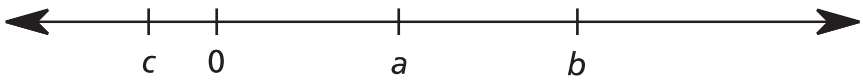
EXPLORATION 2

Suppose we know that $a < b$. If c is a real number, then can we say that $a + c < b + c$? Is this statement always true? Is it sometimes true and sometimes false? Does it depend on the value of c?

1. Each of the following number lines depicts numbers a and b such that $a < b$. A number c is also shown. For each number line, locate $a + c$ and $b + c$ on the number line, and determine whether $a + c < b + c$.
   a.


0
c
a
b

b.



2. If we know that $a < b$, is it always true that $a + c < b + c$? Does this depend on whether c is positive or negative?
3. What do you notice about the distance between $a + c$ and $b + c$?

Our observations about the distance between $a + c$ and $b + c$ suggest a way to prove that $a + c < b + c$: the fact that $a < b$ means that there is a positive number x such that $a + x = b$. But then we have $(a+c)+x = b+c$, and thus $a+c < b+c$. The number x here represents the distance between a and b, which is the same as the distance between $a + c$ and $b + c$. So we arrive at the following property:

ADDITION PROPERTY OF INEQUALITY
Suppose that a, b, and c are numbers. If $a < b$, then $a + c < b + c$.

PROBLEM 1

Draw a number line. Locate and label 0, 1, -5 and 12. Use the number line to illustrate why if $-5 < 12$ then $-5 + 7 < 12 + 7$. Explain.

PROBLEM 2

Suppose that a, b, and c are numbers such that $a > b$. Is it true that $a + c > b + c$? Explain.

Suppose we are given numbers a, b and c such that $a < b$. Is it also true that $a - c < b - c$? We can rewrite this inequality as $a + (-c) < b + (-c)$; we know this inequality is true by the addition property of inequality. So it is indeed true that $a - c < b - c$. We can make a similar argument if $a > b$, so we have

SUBTRACTION PROPERTY OF INEQUALITY
Suppose that a, b, and c are numbers. If $a < b$, then $a - c < b - c$.

PROBLEM 3

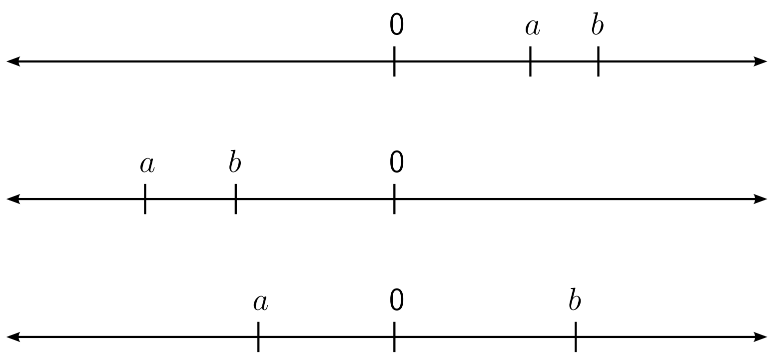
Draw a number line. Locate and label 0, 1, -5 and 12. Use the number line to illustrate why if $-5 < 12$ then $-5 - 7 < 12 - 7$. Explain.

We now explore the effect of opposites (or taking the negative of each side) on inequalities. You will see that you must be very careful when working with negatives in an inequality.

EXPLORATION 3

Suppose a and b are numbers and $a < b$.

1. Using the three possible locations for a and b below, plot the locations of the points representing $-a$ and $-b$ in each case:



2. If $a < b$, make a rule for the relationship between $-a$ and $-b$. Justify your rule.

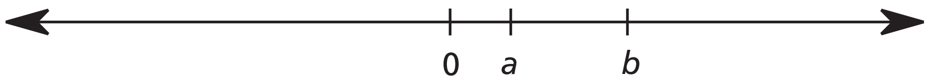
PROBLEM 4

Find an example of a pair of numbers a and b for which each of the following is true: $a < b$ and $a + b < a$. Illustrate that your answer is correct on the number line using exact numerical values. Don't forget to mark 0 on the number line.

We have addition and subtraction properties of inequality. Are there also multiplication and division properties of inequality? Exploration 3 tells we must be careful when dealing with negatives.

EXPLORATION 4

The following number line depicts a situation in which $a < b$:



1. Trace this number line on your paper four times, so that you have four copies of the number line.
2. On the first copy of the number line, locate the numbers $2a$ and $2b$. Is it true that $2a < 2b$?
3. On the second copy of the number line, locate the numbers $\frac{a}{2}$ and $\frac{b}{2}$. Is it true that $\frac{a}{2} < \frac{b}{2}$?
4. On the third copy of the number line, locate the numbers $-2a$ and $-2b$. Is it true that $-2a < -2b$?
5. On the fourth copy of the number line, locate the numbers $\frac{a}{-2}$ and $\frac{b}{-2}$. Is it true that $\frac{a}{-2} < \frac{b}{-2}$?
6. Repeat steps (1) through (5) for the following number line:

7. What do you notice about the distance between $2a$ and $2b$? How about the distance between $\frac{a}{2}$ and $\frac{b}{2}$? The distance between $-2a$

and $-2b$? The distance between $\frac{a}{-2}$ and $\frac{b}{-2}$?

Recall that if we know that $a < b$, this means that there is a positive number x such that $a + x = b$. If we keep in mind that x represents the distance between a and b on the number line, we can come to a better understanding of what happens when we multiply a and b by the same number. If c is a positive number, then the distance between ca and cb is cx:

$$c(a + x) = cb$$
$$ca + cx = cb$$

Since c is positive and x is positive, the product cx is positive. So by definition, $ca < cb$. We can use a similar argument to prove that if c is positive, then $\frac{a}{c} < \frac{b}{c}$.

What happens if c is negative? Let's see what happens if we try the same argument:

$$c(a + x) = cb$$
$$ca + cx = cb$$

This time, c is negative and x is positive, so cx is negative. So let's try adding $-cx$ to both sides:

$$ca = cb + (-cx)$$

If we switch the two sides of this equation, we get $cb + (-cx) = ca$. Since $-cx$ is positive, this means that $cb < ca$; that is, $ca > cb$. So if we multiply a and b by a negative number c, then the relationship between a and b gets reversed! The same happens if we divide by a negative number c: $\frac{a}{c} > \frac{b}{c}$.

We can summarize our findings as follows:

MULTIPLICATION PROPERTY OF INEQUALITY

Suppose that a, b, and c are numbers such that $a < b$. Then

1. If $c > 0$, then $ac < bc$.
2. If $c < 0$, then $ac > bc$.

DIVISION PROPERTY OF INEQUALITY

Suppose that a, b, and c are numbers such that $a < b$. Then

1. If $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.
2. If $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.

Note that all of the properties we have discussed so far can be adapted to work with inequalities such as $a \leq b$ and $a \geq b$, where the two quantities are allowed to be equal. This is because the transitive property and the addition, subtraction, multiplication, and division properties work for equalities as well as inequalities.

Now let's try applying the properties we have discovered.

EXAMPLE 1

Suppose that a and b are numbers such that $a < b$. For each of the following statements, decide if the statement is *always true*, *always false* or *could be true or false, depending on the values of a and b*. Explain using the properties of inequality.

1. $a < b + 1$
2. $a + 1 < b$
3. $a + 5 < b + 5$
4. $3a < 3b$

5. $-a < -b$
6. $a + b < 2b$
7. $7a - 10 < 7b - 10$

SOLUTION

1. This is always true. We know that $a < b$, and $b < b + 1$. By the transitive property, $a < b + 1$.
2. This is sometimes true, sometimes false. This is true if (for example) $a = 2$ and $b = 5$, but false if $a = 1.1$ and $b = 1.3$.
3. This is always true, by the addition property of inequality. Add 5 to both sides of the inequality $a < b$.
4. This is always true, by the multiplication property of inequality. Multiply both sides of the inequality $a < b$ by 3.
5. This is always false, by the multiplication property of inequality. This property states that if we multiply both sides of $a < b$ by -1, then the inequality is reversed; that is, $(-1)a > (-1)b$. So $-a > -b$.
6. This is always true. By the addition property of inequality, we can add b to both sides of $a < b$ to obtain $a + b < 2b$.
7. This is always true; use the multiplication property of inequality, followed by the subtraction property of inequality.

PROBLEM 5

Suppose that a and b are numbers and $a < b$. For each of the following statements, determine whether the statement is *always true*, *always false* or *could be true or false, depending on the values of a and b*.

1. $-3a > -3b$
2. $2a < 4b$
3. $10 - a < 10 - b$
4. $ab < b^2$

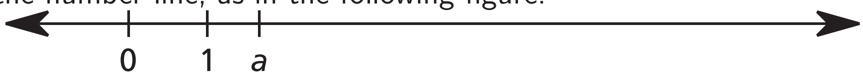
We end this section with an observation that will be useful later, when we study polynomial functions:

SQUARES ARE NONNEGATIVE
If x is a real number, then $x^2 \geq 0$.

Why is this true? We know that if x is a real number, then by trichotomy, x must be either positive, negative, or zero. If $x = 0$, then $x^2 = 0$, and thus x satisfies $x^2 \geq 0$. If $x > 0$, then by the multiplication property of inequality, we can multiply both sides of the inequality by x to obtain $x \cdot x > 0 \cdot x$, or $x^2 > 0$. Finally, if $x < 0$, then by the multiplication property of inequality, multiplying both sides by x reverses the inequality, since x is negative. So in this case, we have $x \cdot x > 0 \cdot x$; that is, $x^2 > 0$. So in each case, we have $x^2 \geq 0$.

EXERCISES

1. Suppose that a and b are numbers and $a \leq b$. For each of the following statements, determine whether the statement is *always true*, *always false* or *could be true or false, depending on the values of a and b*.
   If always true or always false,explain using the properties of inequalities. If it could be true or false, give an example of each.
   a. $a < b$
   b. $a + 2 \leq b + 22$
   c. $-4a \geq -4b$
   d. $\frac{a}{6} - 3 \leq \frac{b}{6} - 3$
   e. $|a| \leq |b|$
2. Suppose that a and b are numbers and $a < b$. The multiplication property of inequality tells us that if we multiply both sides of this inequality by a positive number, then the inequality is preserved. For example, we can say that $3a < 3b$. It also tells us that if we multiply both sides by a negative number, then the inequality is reversed. For example, $-3a > -3b$. What happens if we multiply both sides of the inequality by zero?
3. Suppose you are given the locations of the numbers 0, 1 and a on the number line, as in the following figure:



   a. Explain how you would locate the number $2a + 3$. First explain your procedure in words, then demonstrate how to do it using the number line above.
   b. Find a number b on the number line such that $b > 2b+3$. Show the locations of 0, 1, b and $2b + 3$ on your number line.
4. Find a number a such that $a > 5a + 100$. This time, rather than locating a on a number line, give a numerical value for a. (You may want to use a number line and the ideas from the previous exercise to investigate this problem.)

5. Suppose that a and b are numbers and $a < b$. Is it always true that $a^2 < b^2$? Explain.

6. Find an example of a pair of numbers a and b for which each of the following is true where $a < b$ and $a < a + b < b$. Illustrate your answer on the number line.

7. **Investigation:**
Suppose that we are given that $3x + 8 > 35$. Use the addition, subtraction, multiplication, and/or division properties of inequality to manipulate both sides of this inequality so that the left side of the inequality is simply x. What number is on the right side of the inequality? What does this mean?

8. **Ingenuity:**
In this exercise, we will take a look at a famous classical inequality. Suppose that x and y are positive numbers. Then the *arithmetic mean* of x and y is defined to be $\frac{x+y}{2}$; that is, the average of the two numbers. The *geometric mean* of x and y is defined to be $\sqrt{xy}$. We call this the "geometric mean" because it is a number between x and y that arises naturally in certain geometry problems.
   a. Explain why $(\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{y}) \geq 0$.
   b. Use the distributive property to multiply the expression on the left of this inequality. (*Hint:* One possible first step is to use the distributive property to separate the terms on the left: $\sqrt{x}(\sqrt{x} - \sqrt{y}) - \sqrt{y}(\sqrt{x} - \sqrt{y}) \geq 0$.)
   c. Use properties of inequalities on the inequality you obtained in (b) to show that $\sqrt{xy} \leq \frac{x+y}{2}$.

This elegant inequality is called the *AM-GM Inequality*. It tells us that the geometric mean of two positive numbers is always less than or equal to their arithmetic mean.

SECTION 5.2 SOLVING LINEAR INEQUALITIES

In the last section, we discovered some properties of inequalities that allow us to transform an inequality into other true inequalities. We will now use these properties to solve some interesting problems that involve inequalities.

EXPLORATION 1

In her algebra class, Olivia's homework grade is 94, and her quiz grade is 91. She has a test coming up. Her average in the class will be the average of her grade on the test, her homework grade and her quiz grade. Olivia wants to earn an average of at least 90 so that she will make an A in the class.

1. Let T represent Olivia's grade on the upcoming test. Write an expression in terms of T for Olivia's average in her algebra class.
2. Write the statement "Olivia's average in her algebra class is at least 90" as an inequality in terms of T.
3. Use the properties of inequalities from 5.1 to rewrite this inequality so that T alone is on the left side of the inequality.
4. What grade does Olivia need to earn in order to make an A in the class?

The steps we went through In Exploration 1 are similar to those we go through when we solve a linear equation. It turns out that these processes are very similar, with one significant difference. When we solve a linear equation in one variable, we usually find that there is only one value for the variable that makes the equation true. When we solve a linear inequality in one variable, we are looking for a range of values that make the inequality true. In this case, we found that we need $T \geq 85$; that is, any test grade greater than or equal to 85 will give Olivia an A in the class.

The process we used in Exploration 1 is called *solving an inequality*. We have solved equations in earlier chapters, for example, equations like $2x = 8$. When we solve this equation, we are looking for all the values of x that make the statement $2x = 8$ true. Since the equation $2x = 8$ is equivalent to the equation $x = 4$, we know that 4 is the only value of x that makes the statement true. So the solution set – the set of all values of x that make the statement true – is $\{4\}$. Recall that we use the symbols $\{$ and $\}$ to enclose the elements of a set. It may seem silly to use this notation to represent a single number, but this notation becomes much more important when we begin solving inequalities.

EXAMPLE 1

Solve the inequality $2x < 8$.

SOLUTION What does it mean to solve this inequality? We are trying to find the set of all values of x that make this statement true. In order to do this, we'll use properties of inequalities to isolate the variable on the left side. By the division property of inequality, we can divide both sides by 2 to obtain $x < 4$. This tells us that the solution set is the set of all numbers x less than 4. We can represent this as $\{x \mid x < 4\}$. We can graph this solution set on the number line:

Our argument here actually leaves out a subtle but important point. The division property of inequality tells us that if $2x < 8$, then $x < 4$. That is, if x is a number satisfying $2x < 8$, then x must be less than 4. But does this mean that *all* numbers less than 4 satisfy this inequality? It turns out that this is the case. If we let $x < 4$, then we can use the multiplication property of inequality to multiply both sides by 2 and obtain $2x < 8$. So, all numbers less than 4 satisfy the inequality $2x < 8$.

The fact that we can "reverse the steps" of our solution process tells us something very useful: when we use the addition, subtraction, multiplication, and division properties of inequality, we actually get *equivalent inequalities*. That is, if we apply one of these properties to a given inequality, we will get an inequality with the same solution set. So in situations like these, we don't have to look back and make sure that we can reverse all of our steps each time. However, it is always a good idea to take a second look at your solution and check that it makes sense! One way to do this is to test some values that are in your solution set, make sure that they satisfy the inequality, and try some values that are *not* in your solution set to make sure that they do not satisfy the inequality.

EXPLORATION 2

One day in Weatherford, Texas, the temperature at 12:00 PM was 45 degrees Fahrenheit. A cold front came, and the temperature dropped at a steady rate of 2 degrees per hour until midnight. In this exploration, we'll determine what hours of the day the temperature was below freezing (32 degrees Fahrenheit).

1. Let t be the number of hours that have passed since noon. Write an expression, in terms of t, for the temperature at time t.
2. What inequality do we need to solve to determine when the temperature was below freezing?
3. Use the properties of inequality to solve this inequality. Be sure to state what the solution set of the inequality is.
4. During what part of the day was the temperature in Weatherford below freezing?

PROBLEM 1

Solve the following inequalities. In each case, give the solution set of the inequality.

1. $2x < 5$

2. $2x + 3 < 13$
3. $1 - 3x < 10$
4. $4x + 8 < 2x + 2$

An interesting thing about inequalities is that we can combine them to form statements involving more than one inequality.

EXAMPLE 2

A bakery makes apple pies and sells them for \$8 each. Sales from the pies range between \$544 and \$664. How many pies does it take to produce this range of sales?

SOLUTION As in previous word problems, we start by defining a variable for the number of pies.

$$x = \text{number of pies}$$
$$8x = \text{dollars of sales of } x \text{ pies}$$

We want to know what values of x will produce the amount of sales, $8x$, between \$544 and \$664. This means we have two inequalities which must **both** be true: $544 \leq 8x$ **and** $8x \leq 664$. We solve each of the inequalities separately and then see if there are any solutions which satisfy both of the inequalities.

$$544 \leq 8x \qquad\qquad 8x \leq 644$$
$$\frac{544}{8} \leq \frac{8x}{8} \qquad\qquad \frac{8x}{8} \leq \frac{664}{8}$$
$$68 \leq x \qquad\qquad x \leq 83$$

Therefore, we see that the number of pies that produces between \$544 and \$664 in sales must be $\geq$ 68 **and** $\leq$ 83. Since x is the number of pies and must be a whole number, we see that the solution set is $\{\text{all whole numbers } x | 68 \le x \le 83\}$.

Once you are familiar with these kinds of inequalities, you can express the original relationship as a compound inequality:

$$544 \le 8x \le 664$$

and solve the two inequalities at the same time. As above, to solve this inequality for x, we divide each quantity by 8 to get the equivalent compound inequality:

$$\frac{544}{8} \le \frac{8x}{8} \le \frac{664}{8}$$
$$68 \le x \le 83$$

PROBLEM 2

Jacob has between \$3.85 and \$4.55 in nickels in his piggy bank. How many nickels are there in his bank?

EXPLORATION 3

Graph the three lines given by equations below and find the points of intersection. Then solve the inequality: $1 < 2x + 3 < 11$. Discuss the connection between the graphs and the solution set for the inequality.

$$y = 10$$
$$y = 2x + 3$$
$$y = 1$$

EXAMPLE 3

Charlie is hauling 70 identical pieces of steel pipe from Dallas to Austin on a trailer truck that weighs 18500 pounds when it is empty. When he weighs the his truck; the scale says the truck weighs 29075 pounds. Charlie knows that his weight is 194 pounds, and the scale is accurate to within a margin of ten pounds. What is the range of possible values for the weight of one of the pieces of pipe?

SOLUTION Let w be the weight (in pounds) of one piece of pipe. The total weight of the pipes, the truck, and the driver is given by the expression $70w + 18500 + 194$. According to the scale the total weight is within ten pounds of 29075 pounds. So somewhere between $29075 - 10 = 29065$ and $20975 + 10 = 29085$ pounds. Putting this together, we know that

$$29065 < 70w + 18500 + 194 < 29085.$$

Now let's solve for w. How do we do this when we have two different inequalities? One way is to treat these as two separate inequalities. That is, we could say that $29065 < 70w + 18694$, and $70w + 18694 < 29085$. We could then manipulate these inequalities separately. However, we will be performing the same steps in each case – subtracting 18694 and then dividing by 70 – so why not perform these steps on both inequalities at the same time?

First we subtract 18694 from each "side" of the inequality, keeping in mind that there are now three "sides":

$$29065 - 18694 < 70w < 29085 - 18694$$

$$10371 < 70w < 10391$$

Now we divide each "side" by 70.

$$\frac{10371}{70} < w < \frac{10391}{70}$$

$$148.1 < w < 148.5$$

So each piece of pipe must weigh between 148.1 and 148.5 pounds. It is interesting that we can use a scale that is only accurate to within ten pounds to obtain a measurement that is within about 0.2 pounds of the actual weight!

In the following exploration, we let's solve another problem that involves multiple inequalities.

EXPLORATION 4

Suppose we want to find the solution set of the inequality

$$4 - 3x < 2x + 5 \leq 12 - x.$$

1. Use the properties of inequalities to isolate x on one "side" of the inequality, so that one "side" is simply x and the other two "sides" are numbers.
2. Write this inequality as two separate inequalities.
3. Find the solution set for each of these inequalities. Graph each solution set on a number line. (Use a different number line for each solution set.)
4. Find the set of all numbers x on the number line that satisfy the inequality $4 - 3x < 2x + 5 \leq 12 - x$. Graph this set of numbers on the number line, and write the set using set notation.

PROBLEM 3

Solve the following inequalities.

1. $4 < 2x - 2 < 9$
2. $x \leq 3x - 5 < x + 20$
3. $8 - x \leq 3x \leq 5x - 2$

4. $5x - 9 < 2x < 4x - 12$

Just like with compound inequalities of the form $f(x) < g(x) < h(x)$, where f, g, and h are functions, we can solve other kinds of compound inequalities.

EXAMPLE 4

Find all numbers x such that $5(x - 1) < -5$ or $3(x + 2) > 12$.

SOLUTION The word "or" is important here. This means that we are trying to find all values of x that satisfy *at least one* of the two inequalities. So we'll solve each inequality separately, and then decide how to combine our solutions to them.

To solve the first inequality, we can divide both sides by 5 and get $x-1 < -1$. We can then add 1 to each side to get $x < 0$. So the solution set for the first inequality is $\{x \mid x < 0\}$:

To solve the second inequality, we divide both sides by 3 to get $x+2 > 4$, and then subtract 2 from both sides to get $x > 2$. So the solution set for the second inequality is $\{x \mid x > 2\}$:

Now we need to figure out which values of x satisfy at least one of the two inequalities. We have shaded two different regions of the number line; each region corresponds to the solution set of one inequality. So for x to satisfy at least one of the inequalities, it needs to be in at least one of the regions we graphed. So the solution set for the compound inequality

"$5(x-1) < -5$ or $3(x+2) > 12$" is

This set is $\{x \mid x < 0 \text{ or } x > 2\}$.

We see here that graphing solution sets of inequalities on the number line is a useful way to understand how solutions to compound inequalities "fit together", as well as how to represent solution sets for these problems.

EXERCISES

1. Solve the following inequalities. Choose one point in your solution set, substitute into the original inequality and check to see if the inequality holds.
   a. $4x < 32$
   b. $3x + 2 > 32$
   c. $18 > 7x - 3$
2. Solve the following inequalities. Choose one point in your solution set, substitute into the original inequality and check to see if the inequality holds.
   a. $3x < 12$
   b. $-3x < 12$
   c. $3x > -12$
3. Solve the following inequalities.
   a. $2x - 1 > x + 4$
   b. $3(x-2) < x - 14$
   c. $4x > 6x + 10$
   d. $x - 3(x+3) < 2x - 11$
   e. $4x - 0.52 > 2x + 1.48$
   f. $6x + 3.72 < 2x - 1.44$

4. In a coffee shop, the cost of making a large mug of coffee is \$2 per mug. If the daily set-up costs is \$100, then the total cost $C(x)$ of making x mugs per day is given by $C(x) = 2x + 100$. If the coffee shop has between \$2000 and \$3500 a day to spend on making coffee, how many mugs of coffee can they make each day?

5. Suppose Ann is selling rag dolls she makes for \$6 each. It costs her \$3 dollars to make each doll. It also cost her \$48 to set up her little business. How many dolls does she need to make and sell before she starts earning a profit?

6. Solve the following compound inequalities. For each inequality, graph the solution set, and write it in set notation.
   a. $-3 < x + 10 \leq 6$
   b. $x \leq 3x + 4 < x + 8$
   c. $2x > 4$ and $3x < 9$
   d. $x \leq 4x + 9 < 5x + 6$
   e. $2x - 1 < 5$ or $3x + 1 > 10$
   f. $3x < 5x + 6 < 4x + 1$

7. Solve the inequality $x^2 > 4$. Graph the solution set of this inequality on the number line, and write it using set notation.

8. Let A be the set $\{x \mid -2 < x \leq 7\}$, and let B be the set $\{x \mid x \leq 0 \text{ or } x > 3\}$. Let S be the set of all numbers x that are in both A and B. What is the set S? Graph this set on the number line, and write it in set notation.

9. Suppose that the function $C(x) = 100 + 2x$ represents the cost, in dollars, of producing x items, and the function $I(x) = 10x$ represents the income, in dollars, obtained by selling x items.
   a. Write a function $P(x)$ for the profit, in dollars, obtained by producing and selling x items. (Recall that profit is defined as income minus cost.)
   b. What is the smallest number of items we can produce and sell to make a profit?
   c. What is the smallest number of items we can produce and sell to make a profit of at least \$500?

10. Katie is $4x - 3$ years old, and her older brother Richard is $x + 10$ years old. Both ages are expressed in whole numbers of years, and

they are not the same. Find all the possible values of Katie's age.

11. Donny wants to find out how much a single U.S. penny weighs, the only scale he has is a postal scale that is accurate to within a margin of error of a quarter of an ounce. (In other words, the weight reported by the scale may be up to a quarter of an ounce more or less than the actual weight of the object.) Donny puts 140 pennies on the scale, and the scale reports a weight of $12\frac{1}{4}$ ounces. Find the range of all possible values for the weight of a single penny.
12. Maverick is in a fighter jet, which he wants to land on an aircraft carrier with a runway that is 600 feet long. The beginning of the runway is 5500 feet away, and the jet's current speed is 220 feet per second. From the time he begins his descent to the time he touches the runway, Maverick will travel 3200 feet. In addition, he needs 300 feet of runway to come to a stop. How long can Maverick wait before beginning his descent?
13. Shannon, Cedric, and Heather took a test that contained true-false questions and multiple-choice questions. Each multiple-choice question was worth 5 points, and each true-false question was worth the same (positive integer) number of points. Shannon got 7 true-false questions and 9 multiple-choice questions correct; Cedric got 10 true-false questions and 7 multiple-choice questions correct; and Heather got 6 true-false questions and 9 multiple-choice questions correct. Shannon got a higher score on the test than Cedric, who got a higher score than Heather. How many points was each true-false question worth?
14. **Investigation:** Examine some inequalities from Exercise 6 graphically. Compare with your answers from 6b and 6f.
    a. Using different colors, graph the functions $f(x) = x$, $g(x) = 3x + 4$, and $h(x) = x + 8$ on the same coordinate grid with $-4 < x < 4$. Looking at the graph, determine for which x values the graph of $f(x) < g(x) < h(x)$. Mark this interval on the x-axis.
    b. Graph the functions $f(x) = 3x$, $g(x) = 5x + 6$, and $h(x) = 4x + 1$ on the same coordinate grid with $-10 < x < 5$. For what values of x is $f(x) < g(x)$? When is $g(x) < h(x)$? Are there any values of x for which $f(x) < g(x)$ **and** $g(x) < h(x)$?

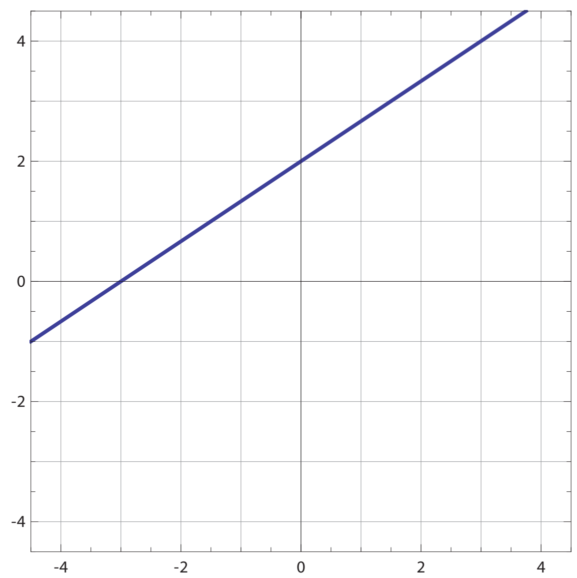
15. **Ingenuity:**
Solve the following inequalities. *Hint:* $|x - c|$ *is the distance between* x *and* c *on the number line.*
a. $|x - 5| < 15$
b. $|2x - 9| > 19$
c. $|x| - |x - 6| \leq 2$

SECTION 5.3 SYSTEMS OF LINEAR INEQUALITIES

In the previous section, we saw how to work with linear inequalities in one variable. What if we are given a linear inequality in two variables, such as $3y - 2x < 6$? As we saw in Chapter 3, an equation in two variables, such as $3y - 2x = 6$, does not have a unique solution. However, we can still understand such an equation by graphing it on a coordinate plane. In this section, we will see that the same is true for inequalities in two variables.

EXPLORATION 1

Recall that the graph of the equation $3y - 2x = 6$ is a line:



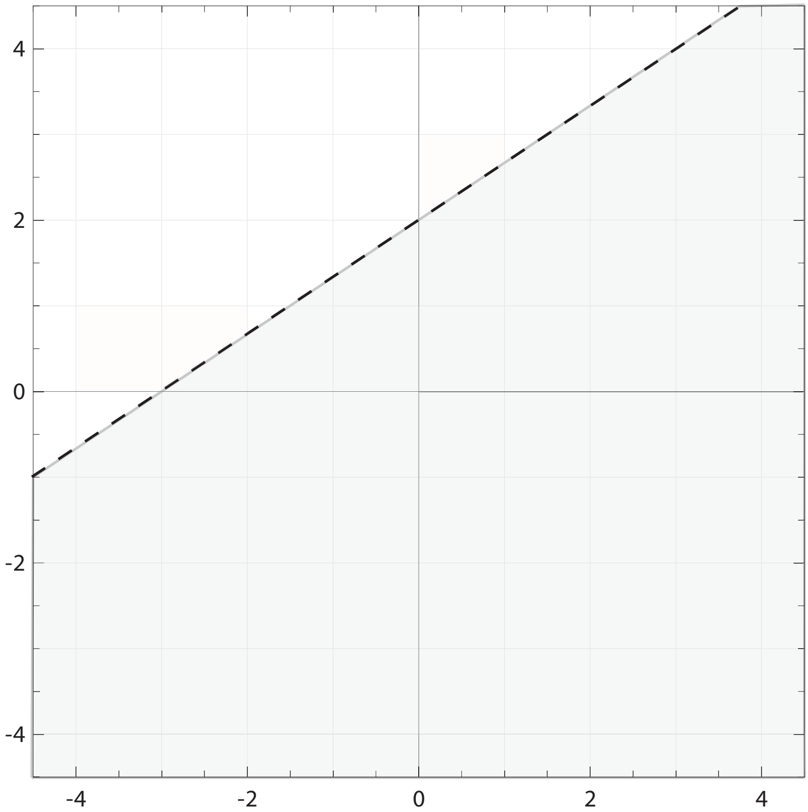
1. Identify the x-intercept and the y-intercept of this line. What is the slope of the line?
2. Identify three points on this line other than the x-intercept and y-intercept. Explain how you found these points.
3. There are many points on this line. There are also many other points

in the coordinate plane that are not on this line. Identify three points that are not on the line. Are these points above or below the line?

4. If we write the equation $3y - 2x = 6$ in slope-intercept form, we get $y = \frac{2}{3}x + 2$. How can we use this form of the equation to determine whether the point $(0, 2)$ is on, above, or below the line?
5. If we change the equality in $y = \frac{2}{3}x + 2$ into an inequality, we get $y < \frac{2}{3}x + 2$ or $y > \frac{2}{3}x + 2$. Does the point $(0, 2)$ satisfy either of these inequalities? How about the point $(0, 0)$?
6. The point $(0, 0)$ is on which side of the line $3y - 2x = 6$? Pick another point on this side of the line, and see if it satisfies either of the inequalities $y < \frac{2}{3}x + 2$ or $y > \frac{2}{3}x + 2$. Then try it with another point on the same side of the line.

Exploration 1 shows us that not only is $(0, 0)$ a solution to the inequality $y < \frac{2}{3}x + 2$, but in fact, all the points on the side of the line $y = \frac{2}{3}x + 2$ containing $(0, 0)$ are also solutions to this inequality. Points below the line have y-values less than the y-values for points on the line with the same x-values. For example, $(0, 0)$ is below the line because $(0, 2)$ is on the line, and the point $(0, 0)$ is below $(0, 2)$. It makes sense to use the words "above" and "below" here because we are comparing the y-coordinates of these points, and the y-coordinate of a point tells us how high or low the point is in the coordinate plane.

A point satisfies the inequality $y < \frac{2}{3}x + 2$ if and only if it lies below the line $y = \frac{2}{3}x + 2$. If we want to represent the solution set of the inequality $y < \frac{2}{3}x + 2$ graphically, we can shade the region below the line $y = \frac{2}{3}x + 2$:

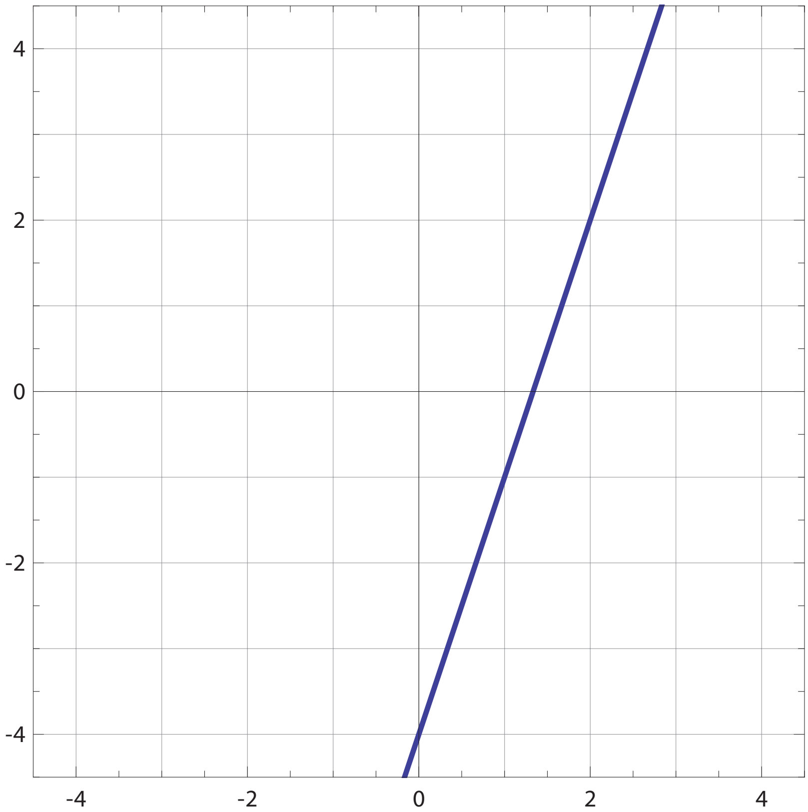


Notice that we have marked the line $y = \frac{2}{3}x + 2$ with a dotted line in this picture. We do this because the line itself is not part of the solution set. However, if we want to represent the solution set of the inequality $y \leq \frac{2}{3}x + 2$ graphically, we mark the line $y = \frac{2}{3}x + 2$ with a solid line.

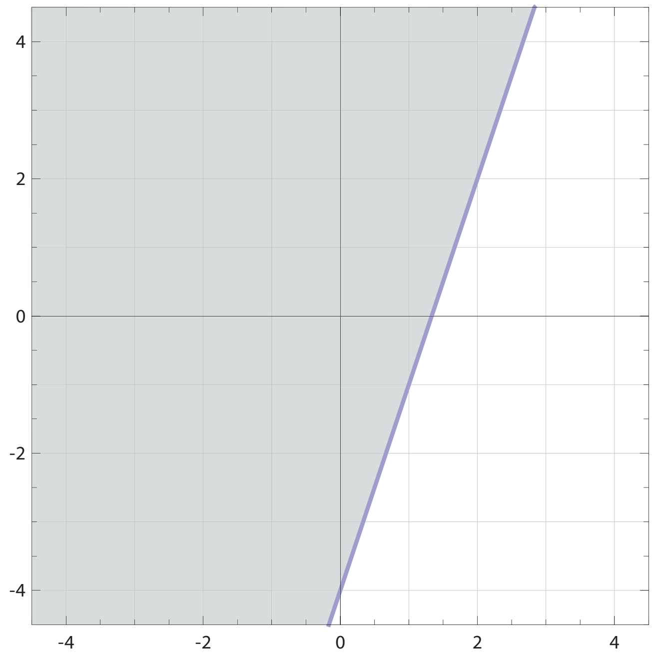
EXAMPLE 1

Graph the inequality $3x - y \leq 4$.

SOLUTION As in Exploration 1, we will start by solving the inequality for y. If we subtract $3x$ from both sides, we get $-y \leq -3x + 4$. We can then multiply both sides by -1 and reverse the inequality; we get $y \geq 3x - 4$. So first we graph the line $y = 3x - 4$:



We need to decide which side of the line to shade. If a point (x, y) satisfies the inequality $y \geq 3x - 4$, that means that the point (x, y) is at or above the point $(x, 3x - 4)$. But the point $(x, 3x - 4)$ is on the line $y = 3x - 4$. This means that (x, y) is on or above the line $y = 3x - 4$. So we shade the region above the line:



Note that this time, the line $y = 3x - 4$ is marked with a solid line, since this line is part of our solution set.

SYSTEMS OF INEQUALITIES

Recall that a system of equations consists of at least two equations. The solution set for a system of linear equations in two variables consists of all the ordered pairs of numbers that satisfy both equations. When we studied systems of linear equations in Chapter 4, the solution set usually consisted of only one ordered pair.

A *system of linear inequalities* consists of at least two inequalities. The solution set of a system of linear inequalities consists of all the ordered pairs of numbers that satisfy all the inequalities in the system. In this section, we will solve systems of linear inequalities graphically.

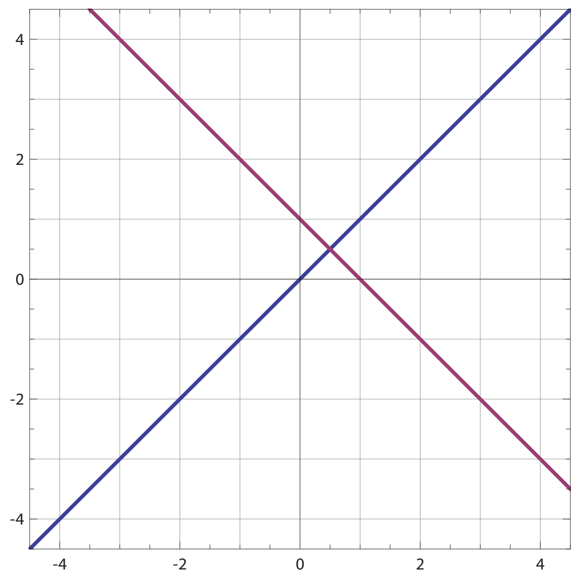
First, let's review the process of solving a system of equations graphically.

EXAMPLE 2

Solve the following system of equations by graphing:

$$y = x$$
$$y = 1 - x$$

SOLUTION Begin by graphing the two equations:



We want to find all ordered pairs (x, y) that satisfy both equations. We know that (x, y) satisfies the first equation if it is on the line $y = x$, and it satisfies the second equation if it is on the line $y = 1 - x$. So the ordered pair (x, y) satisfies both equations if and only if the point (x, y) lies on both lines. So to find the solution of this system of equations, we look at the intersection of the two lines. This intersection is $(\frac{1}{2}, \frac{1}{2})$. So the solution set of this system is $\{(\frac{1}{2}, \frac{1}{2})\}$.

Can we use a similar technique to solve a system of inequalities?

EXPLORATION 2

Consider the system of inequalities

$$y < x$$
$$y < 1 - x$$

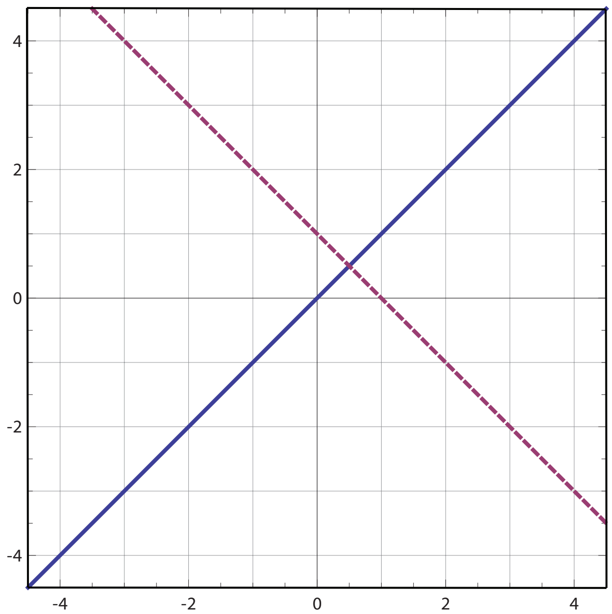
1. Graph the equations $y = x$ and $y = 1 - x$ on the same set of axes. Use dotted lines for these, since the given inequalities are strict inequalities.
2. On this set of axes, graph the solution set of the inequality $y < x$.
3. On another set of axes, draw the lines $y = x$ and $y = 1 - x$ using dotted lines again, and graph the solution set of the inequality $y < 1 - x$.
4. On a third set of axes, graph the set of points that satisfy both inequalities.
5. Pick some points in the region you shaded, and check that they satisfy both inequalities.

EXAMPLE 3

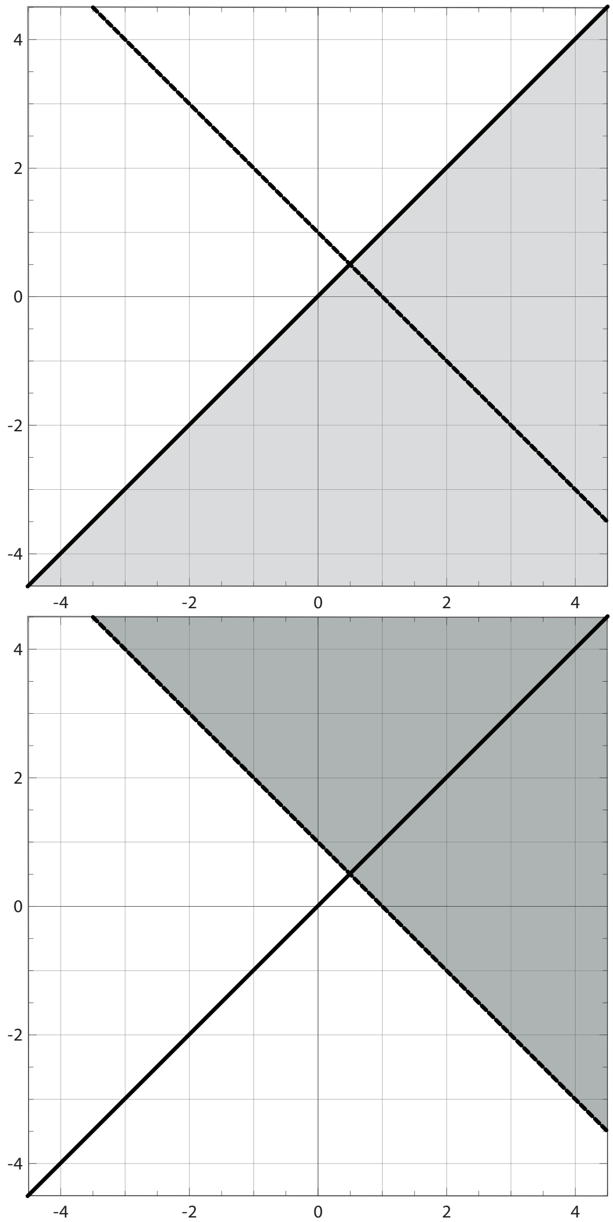
Graph the solution set for the following system of inequalities:

$$y \leq x$$
$$y > 1 - x$$

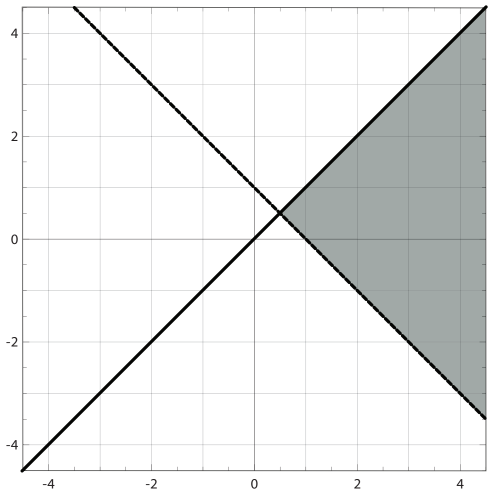
SOLUTION Start by graphing the lines $y = x$ and $y = 1 - x$. This time, we use a solid line for $y = x$, and a dashed line for $y = 1 - x$:



Now we need to find all the points in the plane that are in the region $y \leq x$ and in the region $y > 1 - x$. We start by graphing these two regions separately:



Now we find the intersection of these two regions; that is, the set of all points that are in both regions.



This is the solution set of the given system of inequalities.

This technique of solving systems of inequalities graphically comes in handy when we encounter problems that involve multiple constraints. One example of this kind of problem is a situation where we have limited resources and want to use them as effectively as possible. Here is an example:

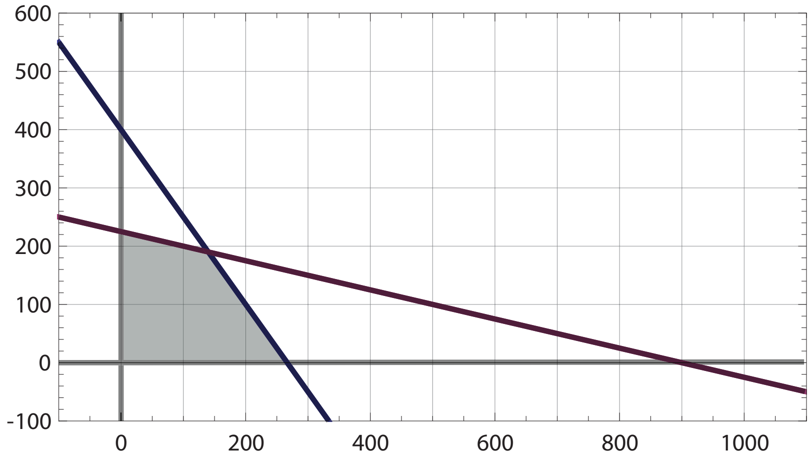
EXAMPLE 4

Acme Robotworks, a manufacturer of household robots, produces two kinds of robots: "chorebots," which do household chores, and "funbots," which provide entertainment. Acme sells chorebots for \$600 each, and funbots for \$1000 each. Each chorebot requires 3 hours to build and 1 hour to program. Each funbot requires 2 hours to build and 4 hours to program. Acme has enough employees to devote 800 hours to building and 900 hours to programming per month. If Acme wants to make as much money as possible, how many of each type of robot should it produce each month?

SOLUTION We will first determine which combinations of chorebots and funbots Acme can produce with the resources it has. Let x be the number of chorebots Acme produces, and let y be the number of funbots Acme produces. There are certain "obvious" conditions that Acme is bound to. We know that Acme cannot produce a negative number of chorebots or funbots, so we know that $x \geq 0$ and $y \geq 0$.

We also know that each chorebot requires 3 hours to build, and each funbot requires 2 hours to build. Since Acme can only devote 800 hours to building, we must have $3x + 2y \leq 800$. Similarly, since each chorebot requires 1 hour to program, and each funbot requires 4 hours to program, and Acme can only devote 900 hours to programming, we must have $x + 4y \leq 900$.

There are no other conditions that Acme must satisfy, so let's graph this system of inequalities:



This time, we get a quadrilateral-shaped region as our solution set. Now we want to figure out which ordered pair (x, y) in this solution set – that is, which feasible combination of robots – will give Acme the greatest profit. We want to maximize the quantity $600x + 1000y$, since Acme makes \$600 for each chorebot and \$1000 for each funbot. The function $600x + 1000y$ is called an *objective function*, since our objective is to maximize this quantity. In order to figure out where in our quadrilateral region this objective function has its maximum, we will use the following theorem:

THEOREM 5.4: THEOREM OF LINEAR PROGRAMMING

If R is a closed, convex polygon in the coordinate plane, and f is a linear objective function in x and y, then the maximum value of f on the region R is attained at at least one vertex of R. Similarly, the minimum value of f on R is attained at at least one vertex of R.

This theorem tells us that the best possible combination of chorebots and funbots will be one of the vertices of the region we have shaded. So we need to find the vertices of this region. We can do this by finding the intersection points of the lines we have graphed.

Our region is a quadrilateral with four vertices. One vertex is the origin $(0, 0)$. We know that if Acme does not produce any chorebots or funbots, then it won't make any money, so this cannot be the best combination for Acme.

Another vertex is the x-intercept of the line $3x + 2y = 800$. The x-intercept of this line is $\left(\frac{800}{3}, 0\right)$; at this point, we have $600x + 1000y = 600 \cdot \frac{800}{3} = 160000$. So with this combination, Acme makes \$160000. We must be careful here: we know that Acme cannot produce a fraction of a robot, so this combination is not exactly possible. If it turns out that this is the best possible combination, we will return to this and decide how to adjust this combination so that it gives us a whole number of each type of robot.

Another vertex is the y-intercept of the line $x+4y = 900$. This y-intercept is $(0, 225)$; at this point, we have $600x + 1000y = 1000 \cdot 225 = 225000$. So in this case, Acme makes \$225000.

The other vertex that we must find is the intersection of the lines $3x + 2y = 800$ and $x+4y = 900$. We can find this intersection point by solving this system of equations. When we do this, we get $x = 140$ and $y = 190$. In this case, we have $600x + 1000y = 600 \cdot 140 + 1000 \cdot 190 = 274000$.

So in this case, Acme makes \$274000. This is the best combination of chorebots and funbots for Acme to make.

Therefore, Acme should make 140 chorebots and 190 funbots.

The method we used to solve this problem is called *linear programming*. Though the process of graphing the inequalities and finding the vertices is arduous, once we know the vertices, we need only check them to determine the "best" point in an entire region.

PROBLEM 1

Acme changes the prices of its robots so that they make \$800 for each chorebot and \$500 for each funbot. What is the most profitable combination they can produce?

EXERCISES

1. Graph the solution set of each of the following inequalities.
   a. $y > 2x$
   b. $y \leq x + 3$
   c. $2x + y < 6$
   d. $x \geq 4$
   e. $3x - 2y > 1$
   f. $-5 < 4x + 3y < 5$
2. Solve each of the following systems of inequalities by graphing.
   a. $y > x - 3$ and $y < -2x + 9$
   b. $y \geq x + 4$ and $y \geq 5$
   c. $x + 2y \leq 15$ and $3x \geq y - 4$
   d. $x > 4$, $y > 2$, and $x + 2y \leq 10$
   e. $x + y > 3$, $3x + y < 7$, and $x - y \geq 3$
   f. $x + y < 8$ and $x + y > 2$
   g. $x - y < 4$ and $x - y < 1$
   h. $x - 2y < 2$ and $2x - 4y > 10$
3. The Widget Company produces both large and small widgets. They sell the large widgets for $15 each and the small widgets for $9 each. Their production is limited by various factors so that the maximum number of large widgets produced in one day is 50, and the maximum number of small widgets is 80. Also, the factory cannot produce more than 100 widgets per day. Make a graph of all the possible combinations of large and small widgets the factory can produce. How many widgets of each size should the Widget Company produce so that they obtain the greatest possible daily income? What will this daily income be?
4. Trevor's home was damaged during a severe windstorm, and he wants to hire people to repair his fence, his roof, and the shed in his backyard. He calls two repair companies, A-Affordable Home Repair and Helping Hand Home Services. After talking to both companies, Trevor knows A-Affordable will take them 8 days to repair the fence, 6 days to repair the roof, and 4 days to repair the shed. Helping Hand will take 3 days to repair the fence, 5 days to repair the roof, and 7 days to repair the shed. Each company offers a crew of workers

that is large enough to repair the fence, the roof and the shed simultaneously. Since each repair company has different strengths and weaknesses, Trevor wonders if it might be more cost-effective to hire one company to do part of the work, and then hire the other company to do the rest of the work. He can hire each company for as long as he wants, even if the amount of time each company spends is not a whole number of days. However, the two companies cannot work simultaneously, and Trevor wants all the repairs completed in 10 days. If A-Affordable charges \$400 per day, and Helping Hand charges \$600 per day, how many days should Trevor hire each company for in order to minimize the cost of the repairs?

5. Suppose that Trevor wants his home repaired, as in Exercise 4, but his only priority is to have everything fixed as quickly as possible, regardless of the cost. How many days should he hire each company for?

6. **Ingenuity:**
What is the area of the solution set of the following system of inequalities?

$$0 \leq x \leq 20 \qquad 0 \leq y \leq 30 \qquad x + y \leq 45 \qquad 2y - x \geq 10$$

SECTION 5.4 CHAPTER REVIEW

Key Terms

compound inequalities
equivalent inequalities
inequalities
linear programming
objective function
solving an inequality
system of linear inequalities

Properties of Inequalities

Addition Property:
If $a < b$, $a + c < b + c$.

Subtraction Property:
If $a < b$, $a - c < b - c$.

Multiplication Property:
For $a < b$
1. If $c > 0$, $ac < bc$.
2. If $c < 0$, $ac > bc$.

Division Property:
For $a < b$
1. If $c > 0$, $\frac{a}{c} < \frac{b}{c}$.
2. If $c < 0$, $\frac{a}{c} > \frac{b}{c}$.

Transitive Property:
If $a < b$, $b < c$ then $a < c$.

Squares are Nonnegative:
For real number x, $x^2 \geq 0$.

Practice Problems

1. Find a number x such that $-x > x + 2$.
2. Consider the statement "If $a > b$, then $ka > kb$."
   a. For what values of k is the statement true?
   b. For what values of k is the statement false?
3. Solve the following inequalities. Graph the solution set on a number line.
   a. $6x - 8 > 24$
   b. $5 - x \leq 20$
   c. $5x - 1 > 9$ or $\frac{1}{3}x + 6 < 5$
   d. $12x + 7 > 14x - 8$
   e. $2x \leq 4x + 6 \leq x + 20$
4. Chris wants to start a business making book bags. She bought a sewing machine for \$250 and spends \$3.00 on thread, fabric and trim per bag. She decides to sell the bags for \$12.00 each. Write an

inequality and solve it to determine how many bags she must make and sell in order to start making a profit.

5. Carl is making salad dressing for his restaurant. To have enough for the week, he must make 10 gallons. He will be mixing oil and vinegar to make the dressing. However, he only has 7 gallons of oil and a minimum of 5 gallons of vinegar.
   a. Represent this situation as a system of inequalities using $x =$ number of gallons of oil and $y =$ number of gallons of vinegar. Graph this system to show all possible combinations of the two ingredients.
   b. Of the possible solutions, which would be the oiliest recipe? Which would be the most vinegary?

EXPONENTS 6

SECTION 6.1 EXPONENTS

When you think about buying a MP3 player or a computer, the amount of memory is one of the most things to consider. If you think about the memory sizes which are available, you will notice that the numbers $32, 64, 128, 256$ and 512 are often used. What do these numbers have in common? They can each be written as a power of 2. For example, $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$ and $128 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$. The exponents represent the number of times 2 is used as a factor in each product. Computers run on a system of 0's and 1's that are used as instructions for switches to be off or on. At each stage in a process, there are two choices on or off. At each new stage the number of possible paths doubles and so increases by a factor of 2. In this chapter we will explore how to compute with numbers such as 2^5 and 2^7. We will look for patterns and develop some rules.

EXPLORATION 1

Show that each of the following products can be written as 2^n for some positive integer n. Use a calculator to check that various answers for each product are equivalent.

1. $(8)(32) = (2^3)(2^5) =$
2. $(16)(64) = (2^4)(2^6) =$
3. $(2^7)(2^{10}) =$
4. Is each answer above a power of 2? Why?
5. Make a rule for the product: $(2^n)(2^m) =$ __________. Explain your reasoning.

EXAMPLE 1

Compute the products $(3^2)(3^4)$ and $(3^6)(3^8)$. Are both of these products a power of 3? Explain.

SOLUTION One way to compute the product $(3^2)(3^4)$ is to first find that $3^2 = 9$ and $3^4 = 81$. Then the product $(3^2)(3^4) = (9)(81) = 729$.

Another way is write out each factor and count the total number 3's in the product: $(3^2)(3^4) = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^6$. The number of factors of 3 in the product is the sum of the number of factors in each of 3^2 and 3^4. Thus the $(3^2)(3^4) = 3^{2+4} = 3^6$.

Similarly, the product $(3^4)(3^7)$ can be written as

$$\underbrace{(3 \cdot 3 \cdot 3 \cdot 3)}_{\text{4 factors}} \; + \; \underbrace{(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)}_{\text{7 factors}} = \underbrace{(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)}_{\text{11 factors}} = 3^{11}$$

If you calculated that $3^4 = 81$ and $3^7 = 2187$, then the product of these two numbers is 177147. This is the same as 3^{11}. Each form of the answer is useful.

We can form similar products with a variable. For example, if x is a number, then

$$(x^2)(x^4) = (x \cdot x)(x \cdot x \cdot x \cdot x) = (x \cdot x \cdot x \cdot x \cdot x \cdot x) = x^6$$

In general, we have the rule:

PRODUCT OF POWERS RULE
If x is a number and each of m and n is a natural number, then the product $(x^m)(x^n) = x^{m+n}$

This rule is sometimes called the "first law of exponents".

PROBLEM 1

Use this rule to compute the following products. Write the answer as a power of a natural number.

1. $(5^4)(5^3)$
2. $(7^5)(7^8)$
3. $(10^4)(10^3)$

EXPLORATION 2

Compute each pair of products.

1. $(2^4)(3^4) =$ __________ and $6^4 =$ __________
2. $(2^3)(5^3) =$ __________ and $10^3 =$ __________
3. $(x^6)(y^6) =$ __________ and $(xy)^6 =$ __________
4. Is there a pattern to these answers. Write a rule to explain the pattern.

In computing the product $(a^3)(b^3)$, you write out the factors and then reorganize them as groups of the product (ab):

$$(aaa)(bbb) = (ab)(ab)(ab) = (ab)^3$$

This brings us to another rule:

DISTRIBUTIVE PROPERTY OF EXPONENTS
For each pair of numbers a and b and natural number n, $(a^n)(b^n) = (ab)^n$.

EXPLORATION 3

Compute each of the following. Write the answer as a power of a natural number if you can.

1. $(3^2)^4 =$ __________
2. $(2^3)^3 =$ __________
3. $(x^4)^2 =$ __________
4. Make a conjecture about a rule that would explain the results of each pair of products.

In computing the product $(a^3)^4$, you write a^3 as a factor 4 times. You can they write each of these factors a^3 as $a \cdot a \cdot a$:

$$\begin{aligned}(a^3)^4 &= (a^3)(a^3)(a^3)(a^3) \\ &= (a \cdot a \cdot a)(a \cdot a \cdot a)(a \cdot a \cdot a)(a \cdot a \cdot a) \\ &= a^{12}\end{aligned}$$

There are 4 groups of factors containing 3 a's in each group. The outer exponent 4 gives us the number of groups of factors and the inner exponent 3 gives us the number of times a is used as a factor in each group.

POWER OF A POWER RULE
For each number a and natural numbers m and n, $$(a^m)^n = a^{mn}$$

We often need to find the product of several expressions involving powers of one or more variables.

EXAMPLE 2

Compute the following products and write the answer in as simple a form as possible:

1. $(5x^3)(4x^6)$
2. $(x^2y^3)(x^4y^2)$
3. $(7x^4y^6)(3x^2y)$

SOLUTION

1. $(5x^3)(4x^6) = (5)(4)(x^3)(x^6) = 20x^9$
2. $(x^2y^3)(x^4y^2) = (x^2)(x^4)(y^3)(y^2) = x^6y^5$
3. $(7x^4y^6)(3x^2y) = 21x^6y^7$

In an earlier class, you learned how to find the *greatest common factor* or *GCF* of a pair of numbers. The rules for exponents can help in computing a GCF. One way to find the GCF of 288 and 648, for example, is to write each number as the product of prime factors:

$$288 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = (2^5)(3^2) = (2^3)(2^2)(3^2)$$

$$648 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = (2^3)(3^4) = (2^3)(3^2)(3^2)$$

So the GCF of 288 and 648 is $(2^3)(3^2) = 72$. We can also use the prime factorization to find the *least common multiple* or *LCM* of a pair of numbers. To find the LCM for 288 and 648, we look at the prime factorization above and see that any multiple should contain at least the product of 2^5 and 3^4. So the LCM of 288 and 648 is $(2^5)(3^4) = 2592$.

PROBLEM 2

Find the GCF and LCM of x^8y^3 and x^4y^{10}.

EXERCISES

1. Use one of the rules developed in this section to compute the following products. Write the answer as a power of a natural number. Use a calculator to compute the answer, if you can.

a. $(4^2)(4^3)$
b. $(5^2)(5^2)$
c. $(4^3)(6^3)$
d. $(2^4)(5^4)$
e. $(2^3)^5$
f. $(2^{12})(2^{14})$
g. $(x^5)(x^9)$
h. $(y^3)(y^4)$
i. $(x^4)(y^4)$
j. $(a^5)^2$
k. $(5)(5^3)$
l. $(2)(2^9)$
m. $(x)(x^5)$

2. In each of the following pairs of numbers, write each number as a power of a natural number and then determine which number is greater.
   a. $(2^4)(2^6)$ or $(2^4)^3$
   b. $(3^6)(3^7)$ or $(3^4)^4$
   c. $(4^3)^7$ or $(4^7)^3$
3. Compute the following products and write the answer in the simplest form possible:
   a. $(2a^3)(3a^4)$
   b. $(5x^6y^4)(x^2y^5)$
   c. $(x^5y)(3x^2y^3)$
   d. $(10ab)(12a^2b^4)$
   e. $(xy)(x^3y^4)$
   f. $(5x^6y^4)(x^2y^5)$
   g. $(ab)(ab)(ab)$
   h. $(3a^2b^4)^2$
   i. $(ab^2)^3(a^3b)^4$
4. Find the GCF of each of the following pairs of numbers:

a. $(2^4)(3^3)$ and $(2^3)(3^2)$

b. $(5^4)(7^6)$ and $(5^8)(7^3)$

c. $(2^5)(3^2)(5^3)$ and $(2^3)(3^4)(5)$

d. (x^5y^2) and (x^2y^6)

e. $(a^8b^3c^{1}0)$ and $(a^5b^6c^4)$

5. Find the LCM of each of the following pairs of numbers:

   a. $(2^3)(3^2)$ and $(2^5)(3)$

   b. $(5^5)(7^2)$ and $(5^3)(7^3)$

   c. $(2^5)(3)(5^3)$ and $(2^3)(3^4)(5^2)$

   d. (x^5y) and (x^2y^6)

   e. $(a^8b^3c^{10})$ and $(a^5b^6c^4)$

6. **Investigation:**
   Compare the answers to the following questions:

   a. How much greater is 2^6 than 2^5? An equivalent question is: What is the difference between 2^6 and 2^5?

   b. How many times greater is 2^6 than 2^5? In other words what is $\frac{2^6}{2^5}$?

   c. What do these two questions and answers say about the relationship between 2^5 and 2^6?

7. **Ingenuity:**
   On the first day of the month, we put \$2 in a jar. Each day we add enough to double the amount in the jar. How much money will be in the jar after 5 days? 8 days? On what day we will have at least 10 times as much money as we have on day 8?

SECTION 6.2 NEGATIVE EXPONENTS

Suppose on the first day of the month, we put \$2 in a jar. Each day we add enough to double the amount in the jar. In order to see the pattern, let's make a table of days and dollars.

Day	1	2	3	4	5	6	7	8	9	10	n
Dollars	2	4	8	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^n

Notice the pattern here: the money in the jar doubles each day and so on the 7th day, there is 2^7 dollars. On what day will the jar have more than 10 times the amount on day 8. On day 8 there would be $2^8 = 256$ dollars. So we start by comparing the amount on day 9 with the amount on day 8. On day 9 there would be $2^9 = 512$ dollars in the jar. We form the ratio of the amount on day 9 divided by the amount on day 8: $\frac{512}{256} = 2$. So, there are twice as many dollars on day 9 than on day 8. We form these ratios in the following chart:

Ratio of Day 9 to Day 8	Ratio of Day 10 to Day 8	Ratio of Day 11 to Day 8	Ratio of Day 12 to Day 8
$\frac{2^9}{2^8} = 2$	$\frac{2^{10}}{2^8} = 4 = 2^2$	$\frac{2^{11}}{2^8} = 8 = 2^3$	$\frac{2^{12}}{2^8} = 16 = 2^4$

This chart shows that on day 12, there will be 16 times as many dollars as on day 8. You might have noticed a pattern in these fractions. What would you speculate would be the ratio of number of dollars on day 15 to the number of dollars on day 8?

The ratio written as the fraction and simplified as follows:

$$\frac{2^{15}}{2^8} = \frac{2^8 2^7}{2^8} = \frac{2^7}{1} = 2^7$$

If each of m and n is a natural number so that $m < n$, then

$$\frac{2^n}{2^m} = \frac{2^{(n-m)}2^m}{2^m} = 2^{(n-m)}.$$

This brings us to another rule for exponents:

QUOTIENT OF POWERS RULE

If x is not 0 and m and n are natural numbers so that $m < n$, then

$$\frac{x^n}{x^m} = x^{(n-m)}.$$

PROBLEM 1

Simplify each of the following fractions using the Quotient of Powers Rule:

1. $\frac{3^5}{3^2}$
2. $\frac{4^7}{4^3}$
3. $\frac{5^8}{5^7}$
4. $\frac{10^6}{10}$

Question: Can we extend the definition of exponents to include 0 and negative integers such as -1 or -2?

First, we will explore what number 2^0 represent. Simplifying the following fraction, we get that

$$\frac{2^4}{2^4} = \frac{16}{16} = 1$$

If we extend the Quotient of Powers Rule to this case so that $n = m = 4$, we get the result that

$$\frac{2^4}{2^4} = 2^{(4-4)} = 2^0$$

Thus, it is natural to define 2^0 to be the number 1.

What about 3^0? Again, for any natural number n, $\frac{3^n}{3^n} = 1$ and applying Quotient of Powers Rule, we would predict that $1 = \frac{3^n}{3^n} = 3^{(n-n)} = 3^0$. Thus, $3^0 = 1$. We define x^0 so that it is consistent with our rules of exponents.

ZERO POWER
For any number x, given that $x \neq 0$, $x^0 = 1$. For example, $5^0 = 1$.

Let's explore the possible meaning of 2^{-1}. Recall that the Product of Powers Rule states that for natural numbers n and m, $2^n 2^m = 2^{(n+m)}$. Apply this law of exponents to the following product:

$$(2^1)(2^{-1}) = 2^{(1+(^-1))} = 2^0 = 1.$$

So, when you multiply 2 by 2^{-1}, you get the product of 1. Do you know of a number that when multiplied by 2 gives you 1? This number is called the multiplicative inverse of 2. The number $\frac{1}{2}$ has the property that $2(\frac{1}{2}) = 1$. Therefore, the number 2^{-1} must be equivalent to $\frac{1}{2}$. We end up with two ways to write the multiplicative inverse.

What about 2^{-3}? Again, we can compute

$$(2^3)(2^{-3}) = 2^{(3+(-3))} = 2^0 = 1.$$

So 2^{-3} is the multiplicative inverse of $2^3 = 8$. But we know that the multiplicative inverse of 8 is $\frac{1}{8} = \frac{1}{2^3}$. Thus, $2^{-3} = \frac{1}{2^3}$. We can generalize this idea as follows:

NEGATIVE POWER RULE
If x is a non-zero number, then x^{-1} is the number $\frac{1}{x}$. For example, $5^{-1} = \frac{1}{5}$.

We can extend this pattern to see that for each $x \neq 0$ and for each natural number n,

$$x^{-n} = \frac{1}{x^n}.$$

PROBLEM 2

Convert each of the following into a fraction:

a. 2^{-4} b. 3^{-2} c. 5^{-1} d. 10^{-2}

PROBLEM 3

Write each of the following fractions as a natural number to a negative power:

a. $\frac{1}{2^5}$ b. $\frac{1}{3^6}$ c. $\frac{1}{4^3}$ d. $\frac{1}{10^4}$

When simplifying fractions, there can be several ways to write the answer. For example,

$$\frac{x^3}{x^7} = x^{(3-7)} = x^{-4} = \frac{1}{x^4}$$

or

$$\frac{x^3}{x^7} = \frac{x^3}{x^4 x^3} = \frac{1}{x^4}$$

EXERCISES

1. Simplify each of the following fractions using the rules of exponents:
   a. $\frac{3^{12}}{3^7}$
   b. $\frac{7^{10}}{7^4}$
   c. $\frac{9^6}{9^5}$
   d. $\frac{10^6}{10^2}$
   e. $\frac{x^8}{x^5}$
   f. $\frac{(x^5y^9)}{(xy^3)}$
   g. $\frac{(x^3y^6)}{(x^3y^5)}$
   h. $\frac{(x^5y^{10})}{(x^3y^2)}$
2. Write each of the following fractions as a natural number to a negative power:
   a. $\frac{1}{2^{12}}$
   b. $\frac{1}{3^5}$
   c. $\frac{4^2}{4^5}$
   d. $\frac{10^4}{10^6}$
   e. $\frac{3}{2^4}$
   f. $\frac{7}{2^5}$
3. Convert each of the following into a fraction:
   a. 2^{-7}

b. 3^{-6}

c. 4^{-1}

d. 10^{-2}

e. x^{-4}

f. y^{-3}

4. Simplify the following fractions using only positive exponents:

a. $\frac{(2^7)(3^5)}{(2^3)(3^2)}$

b. $\frac{(3^8)(4^2)}{(3^5)(4^6)}$

c. $\frac{4x^2}{8x^5}$

d. $\frac{3a^7}{6a^5}$

e. $\frac{6(x^8)(y^3)}{2(x^5)(y^8)}$

f. $\frac{(x^4)(y^7)}{(x^6)(y^3)}$

5. How can you simplify the fractions $\frac{1}{2^{-3}}$ and $\frac{1}{x^{-3}}$?

6. What is 6^0? What is 1^0? What is π^0?

SECTION 6.3 EXPONENTIAL FUNCTIONS

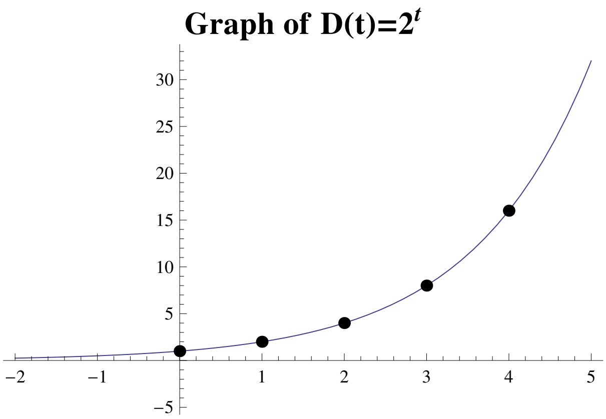
In earlier chapters, we studied functions that modeled repeated addition. For example, the function f given by the rule: $f(x) = 4x$, produces the sequence of outputs

$$4, 8, 12, 16, \dots \text{ from the inputs of } 1, 2, 3, 4, \dots$$

In this section we will study functions whose rules are based on repeated multiplication. Recall the example where we doubled the money in a jar each day. Let $D(t)$ be the number of dollars after t days. The table of inputs and outputs can be written as follows:

$$D(1) = 2$$
$$D(2) = 4 = 2^2$$
$$D(3) = 8 = 2^3$$
$$D(4) = 16 = 2^4$$

Looking at the pattern, we see that $D(t) = 2^t$. $D(0)$ represents the output on the 0^{th} day, the day before day 1, is $D(0) = 2^0 = 1$. If we graph this function, we get



What does this graph indicate the value of $D(-1)$ is? The pattern can be described as "the amount of money on any day is double that of the previous day." Conversely, the amount of money in the jar on any day is $\frac{1}{2}$ of what will be in the jar the next day. Using this pattern, we can extend the function D to include negative inputs. Since $D(0)$ is \$1, then:

$$D(-1) = \tfrac{1}{2}(1) = \tfrac{1}{2} = 2^{-1}.$$

Using this pattern also gives us that

$$D(-2) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4} = 2^{-2}$$

EXPLORATION 1

Suppose a generous aunt decides to contribute to the college funds for her nephews, Billy and Ted. One summer she asks that them to help around the house for a month (30 days). She gives them two options:

- Option 1: Put 50 dollars into the bank each day
- Option 2: Put 2 cents into the bank on the first day and double the amount each day

Billy chose Option 1, and Ted chose Option 2. Who made the wiser choice? Explain.

EXPLORATION 2

Consider the graph of $D(t) = 2^t$ above.

1. If you draw any horizontal line above the x-axis, at how many points will this line intersect the graph of D?
2. Pick two numbers a and b, $a \neq b$. Label the points $(a, D(a))$ and $(b, D(b))$ on the graph. Can $D(a) = D(b)$? Explain.
3. Suppose $D(3x) = D(5)$. What does this tell you about x?

PROBLEM 1

For each of the following find the value of k which makes the equation true.

1. $2^5 = 2^k$
2. $8 = 2^k$
3. $\frac{1}{2} = 2^k$
4. $2^{3t} = 2^{kt}$

Population Growth

One of the most important social issues is population growth. The way population growth is modeled usually involves doubling time, which is the time it takes for the population to double.

EXAMPLE 1

The population of gerbils doubles every month. If we start with 10 gerbils, how many gerbils will there be after 6 months?

SOLUTION We begin by making a table. The first column is the number of months that have passed from the present time, and the second column is the corresponding population after this much time has passed.

months	0	1	2	3	4	5	6	t
Population	10	20	40	80	160	320	640	?

One way to think of the population of gerbils is as a function of the time. For each input of t months, let $P(t)$ be the number of gerbils. So:

$$P(0) = 10 = \text{ initial population}$$

$$P(1) = 20 = \text{ population after 1 month} = 10 \cdot 2$$

$$P(2) = 40 = \text{ population after 2 months} = 10 \cdot 2 \cdot 2$$

$$P(3) = 80 = \text{ population after 3 months} = 10 \cdot 2 \cdot 2 \cdot 2$$

How many gerbils are there after 12 months?

In general, the population $P(t)$ after t months is given by the formula $P(t) = 10 * 2^t$.

PROBLEM 2

A lab discovers that the amount of new bacteria triples every week. Let $A(t)$ be the amount of bacteria after t weeks. At the beginning of the first week, there is 4 grams of bacteria. How much bacteria is there after 1 week? Fill in the table below. On what week will the amount of bacteria exceed 1 kilogram?

week	0	1	2	3	4	5	12	t
Bacteria in grams								$A(t) =$

EXAMPLE 2

Suppose a population of gerbils quadruples every month. If the initial population is 5 gerbils, how many gerbils will there be after 6 months?

SOLUTION During the first months, we get:

$$P(0) = 5 = \text{ initial population}$$

$$P(1) = 20 = \text{ population after 1 month} = 5 \cdot 4$$

$$P(2) = 80 = \text{ population after 2 months} = 5 \cdot 4 \cdot 4$$

$$P(3) = 320 = \text{ population after 3 months} = 5 \cdot 4 \cdot 4 \cdot 4$$

In general, the population $P(t)$ after t months is given by the formula:

$$P(t) = 5 \cdot 4^t.$$

So after 6 months there will be $P(6) = 5 \cdot 4^6 = 20480$ gerbils.

Here again, we have an exponential function. We say that 4 is the base of the exponential function and t is the exponent. We call the coefficient 6 the initial value, or $P(0)$.

EXAMPLE 3

Suppose the population of gerbils doubles every three months. If the initial population is 10 gerbils, how many gerbils will there be after 6 months? How many gerbils will there be after t months?

SOLUTION Again, we can make a table of the population at time t.

months	0	3	6	9	12	t
Population	10	20	40	80	160	$P(t) =$

How is this relationship different from our previous two examples? This second row of the table looks the same as that for Example 1, BUT the period of time it takes to double the gerbil population is three months instead of one month. Will this affect the equation?

Since we want the same outputs, Let's see how $P(t)$ from Example 1 works for out new problem.

$P(0) = 10 * 2^0 = 10 * 1 = 10$ This function works for t = 0.
$P(3) = 10 * 2^3 = 10 * 8 = 80$ But for t = 3, it fails.

So changing the doubling period to 3 months does require a change in the function.

Since the rest of the pattern is the same, is there a way we can alter the exponent to make this function work? Let us suppose that the rule for this population function is an exponential function of the form: $Q(t) =$

$10 \cdot 2^{\left(\frac{t}{k}\right)}$ for some number k. What value of k will give us the outputs given in the table above? To find k, we use the given information with our rule:

$$Q(0) = 10 \cdot 2^{\left(\frac{0}{k}\right)} = 10 = \text{ initial population}$$

$$Q(3) = 10 \cdot 2^{\left(\frac{3}{k}\right)} = 20 = \text{ population after 3 months}$$

Let's try to solve for k.

$$\begin{aligned} 10 \cdot 2^{\left(\frac{3}{k}\right)} &= 20 \\ 2^{\left(\frac{3}{k}\right)} &= 2 \\ 2^{\left(\frac{3}{k}\right)} &= 2^1 \end{aligned}$$

In Exploration 1 we discovered that if $2^a = 2^b$, then $a = b$. Since $2^{\left(\frac{3}{k}\right)} = 2^1$, the exponents must be the same. So, $\frac{3}{k} = 1$. Now we can solve for k, $k = 3$.

The rule for Q is

$$Q(t) = 10 \cdot 2^{\frac{t}{3}} = 10 \cdot 2^{\left(\frac{1}{3}t\right)}$$

Use a graphing calculator to check that this rule for Q gives us the outputs that we expect for $Q(0)$, $Q(1)$, $Q(2)$ and $Q(3)$.

PROBLEM 3

Consider the same set up as in Example 3.

1. Use a calculator to find $Q(1)$ and $Q(2)$. Interpret what they mean.
2. Use the table function of a graphing calculator to estimate when there will be 100 gerbils and 1000 gerbils.

General Exponential Function

We now define a general exponential function.

EXPONENTIAL FUNCTION
For numbers $a > 0$, k and $b > 0$, the function $f(t) = ab^{\frac{t}{k}}$ is called an *exponential function*. The number $a = f(0)$ is the *initial value*, b is the *base* and k is the *growth period*.

PROBLEM 4

The population of gerbils from Examples 1 and 3 can be summarized as:

Example 1: Start with 10 gerbils and double every month.

Example 3: Start with 10 gerbils and double every 3 months.

Look at the exponential functions used to model the populations. For each example:

1. Identify a, b and k.
2. Explain how your choice for a, b and k are related to the description of the population given above.

EXPLORATION 3

Explore the effect of changing a, b and k.

1. Graph each of the following functions:
   a. $f(t) = 2^t$
   b. $f(t) = 2 \cdot 2^t$
   c. $f(t) = 6 \cdot 2^t$
   d. $f(t) = -4 \cdot 2^t$
2. Each function in part 1 has the form $f(t) = a2^t$. What is the effect of changing a? Predict what the graph of $f(t) = 5 \cdot 2^t$ would look like.
3. Graph each of the following functions:
   a. $g(t) = 2^t$
   b. $g(t) = 3^t$

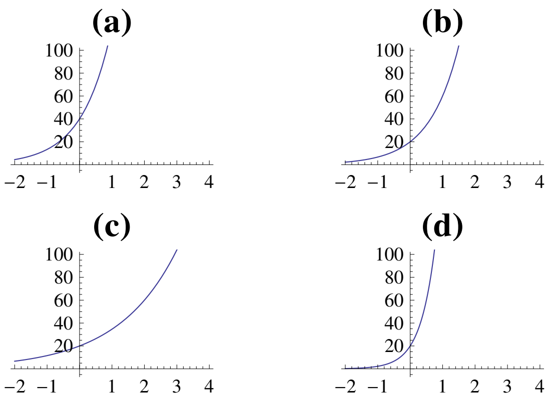
c. $g(t) = 4^t$
d. $g(t) = (2.5)^t$

4. Each function in part 3 has the form $f(t) = b^t$. What is the effect of changing b? Predict what the graph of $g(t) = 5^t$ would look like.
5. Graph each of the following functions:
   a. $h(t) = 2^t$
   b. $h(t) = 2^{(\frac{t}{2})}$
   c. $h(t) = 2^{\frac{t}{3}}$
   d. $h(t) = 2^{2t}$
6. Each function in part 5 has the form $h(t) = 2^{(\frac{t}{k})}$. What is the effect of changing k? Predict what the graph of $f(t) = 2^{(\frac{t}{4})}$ looks like.
7. Graph $h(t) = 4^t$ and $g(t) = 2^{2t}$. What do you notice? Explain why this happens.

PROBLEM 5

The following exponential functions are shown in the figure below. Match the function to its graph.

1. $f(t) = 20 \cdot 3^t$
2. $f(t) = 40 \cdot 3^t$
3. $f(t) = 20 \cdot 3^{2t}$
4. $f(t) = 20 \cdot 3^{(\frac{1}{2}t)}$



EXAMPLE 4

For each of the following conditions write the exponential function which describes the size of the population of bacteria after t weeks.

1. The initial population is 3 grams, and the population doubles every four weeks.
2. The initial population is 4 grams and the population doubles every 3 weeks.
3. The initial population is 2 grams and the population triples every four weeks.

SOLUTION Compare these to the general definition $f(t) = a \cdot b^{\left(\frac{t}{k}\right)}$.

1. Since the initial population is 3 grams, $a = 3$, the population is doubling, so $b = 2$. Hence $f(t) = 3 \cdot 2^{\left(\frac{t}{k}\right)}$. We find k using the same method we used in Example 3. Since the population doubles every 4 weeks, $f(4) = 3 \cdot 2^{\left(\frac{4}{k}\right)} = f(0) \cdot 2 = 3 \cdot 2$. Hence $2^{\left(\frac{4}{k}\right)} = 2$ and $k = 4$. Thus, $f(t) = 3 \cdot 2^{\left(\frac{t}{4}\right)}$.
2. $f(t) = 4 \cdot 2^{\left(\frac{t}{3}\right)}$
3. $f(t) = 2 \cdot 3^{\left(\frac{t}{4}\right)}$

In review, a is the initial value. The base b indicates how we are describing the growth. If we talk about doubling $b = 2$, if we talk about tripling $b = 3$, etc. The constant k tells us how much time it takes to grow by that amount. If it takes 3 weeks to double $k = 3$ and $b = 2$.

Expressing Growth as a Percentage

In many situations, the growth of population will be described in percentage change instead of doubling time. In this case, we can write the base in terms of the growth rate.

EXAMPLE 5

The current population at Miller Middle School is 1000 students. The school building is large enough for 1400 students. School officials predict that the school population will increase 10% each year.

1. What will the population be next year? In two years?
2. Write an exponential function that represents the population t years from now.
3. Estimate when the population will be too large for the current building? Use the graph of the function to justify your answer.

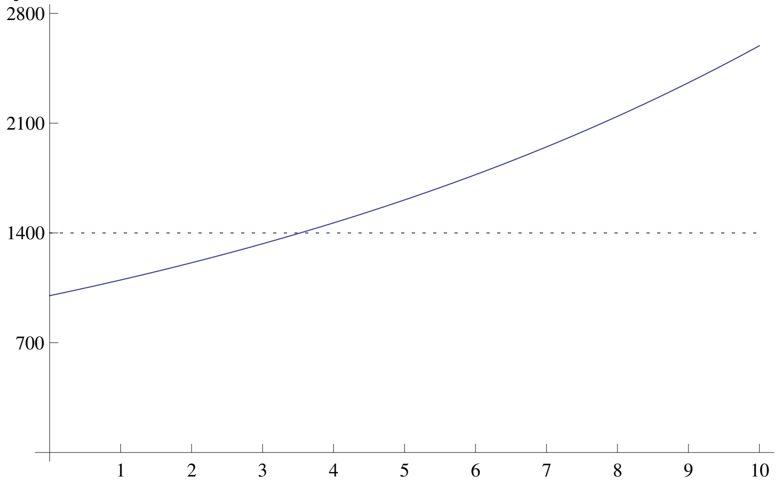
SOLUTION

1. The current population is 1000 students. Ten percent of 1000 is $.1 \cdot 1000 = 100$. So next year, the school's population will be $1000 + 100 = 1100$. The following year, the increase is 10% of 1100 and so on. So in year two, the population is:
$$1100 + .1 \cdot 1100 = 1210.$$
2. To see how the pattern continues over time, notice that $1100 = 1000(1 + .1)$. Let $t =$ the number years from now and make a table:

Year t	Population $P(t)$
0	1000
1	$1000(1 + .1)$
2	$1100(1 + .1) = 1000(1 + .1)(1 + .1) = 1000(1 + .1)^2$
3	$1000(1 + .1)^2(1 + .1) = 1000(1 + .1)^3$
t	$1000(1 + .1)^t$

Let $r = .1$ represent the 10% growth rate in the school. We did not replace $1 + .1$ with 1.1 above, so that it is easier to see the growth rate in the final formula $P(t) = 1000(1 + r)^t$.

3. In the graph below, we see the population will be greater than 1400 four years from now.



When the growth rate is expressed in percentage change r, the base of the exponential function will be $1+r$. In the last example, the population growth model had the form $P(t) = a(1 + r)^t$, where $a = 1000$ was the initial population, and $r = .1$ was the growth rate.

PROBLEM 6

For each of the following conditions, write the exponential function which describes the size of the population after t years:

1. The initial population is 10000, and the population grows 8% each year.
2. The initial population is 7 billion and the population grows by 2% each year.

EXERCISES

1. For each of the following functions, make a table with 5 points including both positive and negative values for x. Then graph the functions and compare. What do you notice?
   a. $f(t) = 2^x$
   b. $g(x) = 3^x$
   c. $h(x) = (\frac{1}{2})^x$
   d. $m(x) = (\frac{1}{3})^x$
2. Abel puts $100 in his safe every week. Bertha puts $1 in the first week, and then doubles the amount each week. Carlos puts 1 cent in the first week, and then triples the amount each week.
   a. Who has the most money after 4 weeks?
   b. Who has the most money after 12 weeks?
   c. Who will have the most money after 24 weeks?
3. Suppose Juan is performing an experiment with a fast growing bacteria. He notices that it doubles in population every week. If he starts with 5 grams of bacteria, how much bacteria will he have after 2 weeks? How much bacteria will he have at the end of 5 weeks? Let $A(t)$ be the weight of bacteria he has in grams after t weeks. Make a sequence of the amounts of bacteria Juan has at the end of each week until the end of week 6. Write a formula for $A(t)$ where t is the number of weeks since he started.
4. The population of bats doubles every year. If the initial population is 1000, what will the population be after t years? Suppose B(t) is the number of bats after t years and that $B(t) = B(0)2^{(\frac{t}{k})}$.
   a. What is $B(0)$?
   b. What is the value of k?
   c. How many bats are there after 2 years? 8 years?
   d. Estimate when there will be 10,000 bats.

5. For each of the following conditions, write the exponential function which describes the size of the population of bacteria after t weeks.
   a. The initial population is 2 grams, and the population quadruples every 3 weeks.
   b. The initial population is 3 grams, and the population quadruples every 2 weeks.
   c. The initial population is 4 grams and the population triples every 2 weeks.
6. Have you ever told someone a secret and the next day everyone knows it? Exponential functions are also used to model the spread of rumors. Suppose a rumor starts with one person who tells 3 people the first day. The next day each of those people tell 3 people, and so on. Let $R(t)$ be the number of people that hear the rumor on day t.
   a. Write a formula for $R(t)$.
   b. Find $R(4)$. What does this mean?
7. If the population of bacteria doubles every 5 hours, and the initial population is 1000, what will the population be after 10 hours? After 20 hours?
8. Suppose the population of rabbits triples every 6 months, and the initial population is 12 rabbits. Let $R(t)$ be the number of rabbits at the end of t months and $R(t) = R(0)3^{\left(\frac{t}{k}\right)}$. What is $R(0)$? What will the population be after 2 years? What is k? When will there be 100 rabbits? 500 rabbits?

SECTION 6.4 EXPONENTIAL DECAY

In Section 6.3 we used exponential functions to model growth of money and populations. In this section we will use exponential functions to model decay. In other words, things that are decreasing over time. So instead of putting money into the jar, we are going to take some out.

EXPLORATION 1

We have $1024 in a jar. We don't want to spend all the money at once. So we decide to spend only half of the money in the jar each week.

1. Make a table of how much money is left in the jar after 1, 2, 3 and 4 weeks.
2. Let $M(t)$ be the amount of money left after week t. How much money will be left after week $t + 1$?
3. Find a formula for $M(t)$.

Decay Expressed with Percents

In the children's game telephone, everyone sits in a circle and the first person whispers a secret message in the next person's ear, who then whispers the message into the next person's ear and so on. The idea of the game is to see how much the message changes by the time it reaches the last person. A similar process occurs when a message is sent electronically over a wire. A certain percentage of the message is lost as the message passes through the wire.

EXAMPLE 1

Suppose that as a signal is passed from your computer to your printer some of the signal is lost. For every meter of wire 1% of the signal is lost. What percent of the signal, $S(x)$, is retained after x meters?

You decide to put the printer all the way across the room, so you need to use a cable which is 3.5 meters long. What percentage of the original signal will arrive at the printer? What percentage will be lost?

SOLUTION We need a way to represent the initial signal. Since we start with the full signal we let $S(0) = 100\%$. For every meter of wire 1% of the signal is lost, so 99% is kept. As the signal passes through the wire, we get:

$$\begin{aligned} S(0) &= \text{ initial signal } = 100 \\ S(1) &= \text{ signal after 1 meter} = 100 \cdot .99 = 99 \\ S(2) &= \text{ signal after 2 meters} = 100 \cdot .99 \cdot .99 = 98.01 \\ S(3) &= \text{ population after 3 meters} = 100 \cdot .99 \cdot .99 \cdot .99 = 97.0299 \end{aligned}$$

In general, the signal $S(x)$ after x meters is given by the formula:

$$S(x) = 100 \cdot .99^x.$$

So after 3.5 meters there will be $S(3.5) = 100 \cdot .99^{3.5} = 96.5435\%$ of the original signal, and 3.34647% will be lost.

The decay rate in the signal problem is expressed in terms of percent of signal lost. In the previous section, when given growth rates in percents, the exponential function had the form $P(t) = a(1+r)^t$. If we let $r = .01$ represent the rate at which the signal is lost, then $S(t) = 100(1 - r)^t$. Because the signal is decaying instead of growing, the base is $1-r$ instead of $1 + r$.

PROBLEM 1

For each of the following conditions, write the exponential function that describes the population after t years:

1. The initial population is 10000, and the population declines 8% each year.

2. The initial population is 1.3 billion and the population declines by .2% each year.

Different Forms of the Same Model

EXPLORATION 2

For each of the following exponential functions identify the base b and the growth constant k. Letting t vary from -4 to 4 use a graphing calculator to graph f and g given below. Now graph h and m. What do you notice? Explain why this happens.

1. $f(t) = (\frac{1}{2})^t$
2. $g(t) = 2^{-t}$
3. $h(t) = 2^t$
4. $m(t) = (\frac{1}{2})^{-t}$

We observed above that there are two ways to write an exponential function that represents decay, since for $b > 0$:

$$b^{-(\frac{t}{k})} = (\frac{1}{b})^{(\frac{t}{k})}.$$

So we can either use a base $0 < b < 1$ and a positive growth constant $k > 0$, or a base $b > 1$ and a negative growth constant.

PROBLEM 2

In a biological experiment 1000000 bacteria are placed in a dish with antibiotic. After the first hour, only one tenth of the bacteria survive. This process continues, after each hour $\frac{9}{10}$ of the bacteria die and $\frac{1}{10}$ survive. Write the formula for $A(t)$ the number of bacteria surviving after t hours in two different ways. Once use a positive growth constant and one using a negative growth constant.

Radioactive Decay and Half-Life

One of the most common uses of exponential functions is to model radioactive decay. Radioactive material slowly emits radiation, and in the process the material is transformed and the amount that is still radioactive is reduced. The decay is often described in terms of its half-life or the time needed for exactly half of the radioactive material to transform. For example, the half-life of Uranium-238 is 4.5 billion years. So if we make 2 grams of Uranium-238 and wait 4.5 billion years, we will only have 1 gram left. Next let's explore this idea using a material that decays a little faster.

EXAMPLE 2

The half-life of a certain radioactive material is 25 days. If we start with 800 g, how much will be left after t days? How much is left after 10 days?

SOLUTION We begin with a table of the amount $A(t)$ at time t days. Every 25 days the amount is cut in half.

t days	0	25	50	75	100	t
grams	800	400	200	100	50	$A(t) =$

Let's suppose that the rule for this population function is an exponential function of the form: $A(t) = 800 \cdot (\frac{1}{2})^{(\frac{t}{k})}$ for some number k. What value of k will give us the outputs from the table above? To find k, we use the given information with our rule:

$$A(0) = 800 \cdot (\frac{1}{2})^{(\frac{0}{k})} = 800 = \text{ initial amount}$$

$$A(25) = 800 \cdot (\frac{1}{2})^{(\frac{25}{k})} = 400 = \text{ amount after 25 days}$$

Using information for the 25th day, we get the equation:

$$800 \cdot (\frac{1}{2})^{(\frac{t}{k})} = 400$$
$$(\frac{1}{2})^{(\frac{25}{k})} = \frac{1}{2}$$

Since $(\frac{1}{2})^{(\frac{t}{k})} = \frac{1}{2}$, then the exponents must be the same, that is, $\frac{25}{k} = 1$. Thus, $k = 25$. So the rule for A is:

$$A(t) = 800 \cdot (\frac{1}{2})^{(\frac{t}{25})}$$

Use a graphing calculator to check that that this rule for A gives us the outputs that we expect for $A(0)$, $A(50)$, $A(75)$ and $A(100)$. Note that k is the half life of the material.

Substituting $t = 10$, we find that after 10 days we have

$$A(t) = 800 \cdot (\frac{1}{2})^{(\frac{10}{25})} = 800 \cdot 2^{-.4} = 606.287g$$

PROBLEM 3

The half-life of a certain radioactive material is 4 years. If we start with 200 grams of the material, how much will be left after t years? How much is left after 1 year? How many years will it take before there is only 50 grams left?

EXERCISES

1. In each of the following compute the value of the variable.
   a. $x = 2^{-5}$
   b. $16 = 2^{-t}$
   c. $\frac{49}{25} = x^{-2}$
   d. $y = 3^{.51}$
   e. $p = 6^{-.75}$
   f. $q = (\frac{1}{4})^{-3/2}$
2. Xerxes, Yaz and Zenon are brothers. Each brother has a jar with \$1296. Xerxes plans to take \$100 out of his jar every week. Yaz plans to take out half of the money in his jar each week. Zenon plans to take out two-thirds of the money every week.
   a. Find the formulas for the amount of money each brother will have in his jar at the end of week t.
   b. How much money will Xerxes have after week 3? How about Yaz? Zenon?
   c. How many weeks will it be before Xerxes has less than \$100 in his jar? How about Yaz? Zenon?
3. A radioactive material has a half-life of 30 years. If there is 50 kilograms of this material now, how much will be left in 60 years? 300 years?
4. If you stick a hot object into a cool environment, the temperature will decrease exponentially. Suppose you take a hot can of soda (40 degrees Celsius) and place it into a refrigerator which is at 0 degrees Celsius. The temperature in Celsius of the can after t minutes is given by the exponential function $C(t) = 40 \cdot (\frac{1}{2})^{(\frac{t}{10})}$
   a. Find $C(0)$. Interpret this value and say why it makes sense.
   b. Graph $C(t)$ for $0 \leq t \leq 30$.
   c. What will be the temperature after 20 minutes?
   d. After how many minutes will the temperature be 5°Celsius?
5. A middle school has 512 students. A model for the number of people $N(t)$ in a middle school who have heard a rumor is:
$$N(t) = 512 - 512 \cdot 2^{-\frac{t}{5}}$$
   a. Make a table of the number of people who have heard the rumor after $t = 0, 1, 2, 3$ and 4 days.
   b. Find and interpret $N(0)$. Why does the value make sense?
   c. Use the table of values to graph $N(t)$.
   d. How many days will it take until half of the school has heard

the rumor?

6. The amount of medication that is in a patient's blood decays exponentially. Often, the half-life for the medication is listed in the fine print that comes with the medication. For example, the half-life of acetaminophen is approximately 4 hours. Suppose you take 2 tablets containing 500 mg each at time $t = 0$. How much acetaminophen is left in your bloodstream after t hours? The directions tell you to take "two tablets every 6 hours". How much will left in your bloodstream after 6 hours from the first dose, even before you take the next dose?
7. A 100 gallon tank contains a solution that is 70% acid and 30% water. Bill drains 40 gallons from the tank and replaces it with water. After mixing this solution he repeats the process. If this is done a total of 4 times, what percent of the final solution is acid?
8. Jessica has a "fun fund" and decides each year she will spend 10% of the available amount. She begins with $10000 in the fund. Make a table showing the amount remaining in the fund at the end of each of the first 6 years. Estimate to the nearest year when her fund will drop below $500.

SECTION 6.5 GEOMETRIC SEQUENCES

In Section 2.3 we used functions to describe the pattern in a list of numbers. Recall that a *sequence* is the list of outputs of a function whose domain is the natural numbers. So far we have focused on arithmetic sequences. In Section 3.5 we saw that arithmetic sequences are really linear functions whose domain has been restricted. Let's review arithmetic sequences.

EXPLORATION 1

Restrict the domain of $A(x) = 15 - 3x$ to the natural numbers to define a sequence $\{a_1, a_2, \ldots\}$.

1. List the first 4 numbers in the sequence: a_1, a_2, a_3, a_4.
2. Compare consecutive terms. Describe how a term is related to the previous term. Use this pattern to write out the next two terms in the sequence.
3. Compute $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$. Look at the definition of an arithmetic sequence in Section 2.3. Explain why this sequence is arithmetic.

Let's explore what happens if we start with an exponential function and restrict its domain to the natural numbers.

EXPLORATION 2

Restrict the domain of $G(x) = 3 \cdot 2^x$ to the natural numbers to define a sequence $\{g_1, g_2, \ldots\}$.

1. List the first 4 numbers in the sequence: g_1, g_2, g_3, g_4.
2. Compare consecutive terms. Describe how a term is related to the previous term. Use this pattern to write out the next two terms in the sequence.

3. Compute $g_2 - g_1$, $g_3 - g_2$, $g_4 - g_3$. How does this compare to what happened for the arithmetic sequence in Exploration 1?
4. Compute $\frac{g_2}{g_1}$, $\frac{g_3}{g_2}$, $\frac{g_4}{g_3}$. What do you notice about the ratios?
5. If we write a recursive formula for the sequence as $g_{n+1} = g_n \cdot b$, what is the value of b?

A sequence that comes from an exponential function has a special name.

GEOMETRIC SEQUENCE
A sequence $g_1, g_2, g_3, \ldots$ is a *geometric sequence* if there is a number b such that for each natural number n, $g_{n+1} = g_n \cdot b$, that is $\frac{g_{n+1}}{g_n} = b$.

PROBLEM 1

Consider the sequences:

- $13, 17, 21, 25 \ldots$
- $3, 9, 27, 81 \ldots$
- $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2} \ldots$
- $80, 40, 20, 10, \ldots$

For each sequence:

1. Determine if its arithmetic or geometric.
2. Make a table using the natural numbers as inputs and the sequence as outputs.
3. Graph the points of the table on a coordinate plane. Compare the graphs of the geometric sequences, how are they similar? Do they look like a line?

EXAMPLE 1

Psychologists have found that the amount of food people eat depends on the total amount available. In one experiment, researchers started with a large bowl of 768 candies on the table in the teacher's lunchroom. They left the bowl on the table for 4 days without adding more candy. At the beginning of each day, researchers wrote down how many candies were in the bowl: 768, 384, 192, 96.

1. Explain why the pattern in the data matches a geometric sequence.
2. If the pattern continues, how much candy will in the bowl at the start of the fifth day.
3. If the pattern continues, what is the first day when there will there be less than 10 candies in the bowl at the start of the day.
4. Write the formula for the n-th term.

SOLUTION Notice this problem is similar to those in the last section. Tables are good way to organize the information to help see the pattern.

day	g_n	Pattern
1	796	796
2	384	$= 796 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^1$
3	192	$= 384 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^2$
4	96	$= 192 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^3$
5	48	$= 192 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^4$
6	24	$= 48 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^5$
7	12	$= 24 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^6$
8	6	$= 12 \cdot \frac{1}{2} = 796\left(\frac{1}{2}\right)^7$
n	g_n	$796\left(\frac{1}{2}\right)^{n-1}$

1. Each term is one-half the previous term. So this is a geometric sequence with $b = \frac{1}{2}$.
2. At the start of the fifth day, there will be $g_5 = g_4 \cdot \frac{1}{2} = 96 \cdot \frac{1}{2} = 48$.
3. If the pattern continues, the sequence will be $\{768, 384, 192, 96, 48, 24, 12, 6\}$.

The number of candies is less than 10 on the 8th day.

4. Looking at the pattern we notice that the n-th term is 796 times $\frac{1}{2}$ raised to a power. To write the formula, you need to see the relationship between the position in the sequence (which is day in this case) and the exponent. Here the exponent is always one less than the position of the term in the sequence. So the formula is $g_n = 796\left(\frac{1}{2}\right)^{n-1}$.
5. If we continue the pattern, the sequence is $796, 384, 192, 96, 48, 24, 12, 6, \ldots$. The number of candies is less than 10 on the 8th day.

EXERCISES

1. Determine if the sequence below is arithmetic, geometric or neither.
   - $5, 10, 20, 40, 80, \ldots$
   - $.1, .01, .001, .0001, \ldots$
   - $-1, 2, 7, 14, 23, \ldots$
   - $5, 2.5, 0, -2.5, -5, \ldots$
   - $15, 5, \frac{5}{3}, \frac{5}{9}, \ldots$
   - $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
2. Restrict the domain of $g(t) = 96(\frac{1}{2})^{n-1}$ to the natural numbers to form a sequence g_n.
   a. List the fist four members of the sequence.
   b. What is the recursive formula for g_n.
3. For each of the geometric sequences below, find the recursive formula and the formula for the n-th term:
   a. $2, 6, 18, \ldots$
   b. $20, 10, 5, \ldots$
   c. $3, 2, \frac{4}{3}, \ldots$
4. **Investigation:**
   The recursive formula is not enough information to write down a sequence. You need to know the value of one of the terms as well. Typically you are given the first term. Suppose a geometric sequence satisfies the recursive formula $g_{n+1} = g_n \cdot 4$.
   a. What does the recursive formula mean in words?

b. Let $g_1 = 1$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
c. Let $g_1 = 2$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
d. Let $g_1 = 5$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
e. Compare the three sequences. How does the formula depend on the value of the first term?
f. Graph the sequences on the same coordinate plane. How are the graphs similar? How are they different?

5. **Investigation:**
Suppose an arithmetic sequence satisfies the recursive formula $g_{n+1} = g_n \cdot b$ with $g_1 = 1$.
a. What does the recursive formula mean in words?
b. Let $b = 1$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
c. Let $b = 3$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
d. Let $b = .2$. List the first 5 terms of the sequence. Find the formula for the n-th term of the sequence.
e. Compare the three sequences. How does the formula depend on the value of b?
f. Graph the sequences on the same coordinate plane. How are the graphs similar? How are they different?

6. An arithmetic sequence has recursive formula: $a_{n+1} = a_n + c$. A geometric sequence has recursive formula: $g_{n+1} = g_n \cdot b$. Does there exist a sequence that is both arithmetic and geometric? If so, what does it look like. If not, why not?

7. Two terms of an geometric sequence are given: $g_6 = 4$ and $g_8 = 1$.
a. Find g_1
b. Is there more than one possible value for g_1?

SECTION 6.6 SCIENTIFIC NOTATION

When we look at our world, we often need to talk about amounts that are quite large. For example, the distance from the earth to the sun averages about 93 million miles. The number of people on earth July 5, 2013 was estimated to have been 7.13 billion. The age of the universe is estimated to be about 13.7 billion years. What are these numbers? Why do we use the names million and billion?

We can write the numbers out as follows:

$$\begin{aligned} 93 \text{ million } &= 93{,}000{,}000 \\ 7.02 \text{ billion } &= 6{,}680{,}000{,}000 \\ 13.7 \text{ billion } &= 13{,}700{,}000{,}000 \end{aligned}$$

Large numbers are often estimations. Nevertheless, the first several digits are very important. For example, the distance that Venus is from the sun is about 68 million miles. If you compare this distance to the earth's distance from the sun, they both look like similar large numbers. However, the distance from from the earth to the sun is 1.4 times the distance from Venus to the sun. The consequences of this difference can be seen in the much higher temperatures on Venus.

Numbers that result from measurements are often accurate to only a few digits. For example, what is the difference when a football player gains 14 yards and an engineer who says he needs a steel beam of length 14.0 yards? Did the player gain exactly 14 yards? Do the numbers 14 and 14.0 represent the same distance?

The record keeper is estimating the distance gained on the play. We can say from this measurement that the number of yards gain is between 13.5 and 14.5 yards. On the other hand, the engineer needs for the steel beam to be between 13.95 and 14.05 yards in length.

When we read that the age of the universe is 13,700,000,000 years old, only the first 3 digits are considered significant or accurate. The scientist making this estimate thinks the age of the universe is between 13,650,000,000 and 13,750,000,000 years. If the measurement is more accurate, the notation must indicate this.

The second important aspect of such large numbers is their magnitude, indicated by the number of digits in the number (usually zeros). When numbers are greater than a million, we often switch to a notation that uses exponents. Recall the following pattern of the base ten system.

We can write the numbers out as follows:

$$\begin{aligned} 10 &= 10 \\ 10^2 &= 100 \\ 10^3 &= 1000 = 1 \text{ thousand} \\ 10^6 &= 1{,}000{,}000 = 1 \text{ million} \\ 10^9 &= 1{,}000{,}000{,}000 = 1 \text{ billion} \\ 10^{12} &= 1{,}000{,}000{,}000{,}000 = 1 \text{ trillion} \end{aligned}$$

Using exponential notation, we write large numbers in the form of $a \times 10^n$ where $1 \leq a < 10$. Writing a number in this form is called *scientific notation*. For example, we can rewrite the numbers from above as

$$\begin{aligned} 93 \text{ million } &= 93{,}000{,}000 = 9.3 \times 10^7 \\ 6.68 \text{ billion } &= 6{,}680{,}000{,}000 = 6.68 \times 10^9 \\ 13.7 \text{ billion } &= 13{,}700{,}000{,}000 = 1.37 \times 10^{10} \end{aligned}$$

PROBLEM 1

Write the following numbers using scientific notation.

1. 313,000,000 population of United States (June 2012)
2. 491,040,000,000 distance from earth to sun in feet
3. 590,000,000,000,000,000 diameter of the Milky Way in miles
4. 5,900,000,000,000,000 thickness of the Milky Way in miles

PROBLEM 2

Write the following numbers using standard notation.

1. 2.6×10^7 approximate population of Texas.
2. 3.8×10^7 approximate population of California.
3. 9.5×10^2 approximate population of Wink, Texas.
4. 1.1×10^8 approximate population of Mexico.

Writing numbers using scientific notation makes it easier to compare their sizes and use them in computations.

EXAMPLE 1

How much wider is the Milky Way than it is thick? (See parts 3 and 4 of Problem 1.)

SOLUTION To compare these two numbers, we form the ratio of diameter to the thickness of the Milky Way:

$$\begin{aligned}\frac{5.9 \times 10^{17}}{5.9 \times 10^{15}} &= \frac{5.9}{5.9} \times \frac{10^{17}}{10^{15}} \\ &= 1 \times 10^{(17-15)} \\ &= 10^2 = 100\end{aligned}$$

Thus, the Milky Way is 100 times as wide (diameter of the disk) as it is thick in the center. So, the Milky Way is shaped like a saucer.

PROBLEM 3

Compare the populations (as of July 2008) of the following:

China	1.33×10^9
Spain	4.05×10^7
Ecuador	1.83×10^7

EXAMPLE 2

Compute the following sums and products:

1. $(4.5 \times 10^9) + (3.2 \times 10^9)$
2. $3(4.5 \times 10^9)$
3. $(4.5 \times 10^9)(4.5 \times 10^9)$
4. $(1.3 \times 10^9) + (8.2 \times 10^7)$

SOLUTION

1. Using the pattern $a \times c + b \times c = (a + b)c$, we get:

$$(4.5 \times 10^9) + (3.2 \times 10^9) = (4.5 + 3.2)(10^9) = 8.7 \times 10^9$$

2. Using the pattern $a(b \times c) = (a \times b)c$, we get:

$$3(4.5 \times 10^9) = (3 \times 4.5)(10^9) = 13.5 \times 10^9 = 1.35 \times 10^{10}$$

3. Using the pattern: $(a \times c)(b \times c) = (a \times b)(c \times c)$, we get

$$(4.5\times10^9)(4.5\times10^9) = (4.5^2)(10^{18}) = 20.25\times10^{18} = 2.025\times10^{19}$$

4. Notice that these two numbers have exponential components that

are of different magnitude, 10^9 and 10^7. In order to add these two quantities, they must have the same power of 10 in their scientific notation. We write the sum as

$$(1.3\times10^9)+(8.2\times10^7) = (1.3\times10^9)+(.082\times10^9) = 1.382\times10^9$$

What about very small numbers? We can use scientific notation to write very small numbers as well. For example, the following fractions can be written as:

$$\begin{aligned} 10^{-1} &= .1 = \text{ one tenth} \\ 10^{-2} &= .01 = \text{ one hundredth} \\ 10^{-3} &= .001 = \text{ one thousandth} \\ 10^{-6} &= .000001 = \text{ one millionth} \end{aligned}$$

We can express small decimal numbers using these exponents. For example:

$$\begin{aligned} .0000053 &= 5.3 \times 10^{-6} \\ .000000000025 &= 2.5 \times 10^{-11}m : \text{the radius of hydrogen atom} \\ .0000000000000071 &= 7.1 \times 10^{-15}m : \text{the diameter of lead atom nucleus} \end{aligned}$$

PROBLEM 4

Write the following in scientific notation:

1. .0000123
2. $\frac{57}{10000000}$
3. .000023002
4. $\frac{64}{400000}$
5. 356.2×10^{-8}

PROBLEM 5

Write the following in standard notation:

1. 2.3×10^{-5}
2. 9.78×10^{-1}
3. 6.547×10^{-8}
4. 5.32×10^{-4}

We can use the same rules as Example 2 to combine very small numbers written in scientific notation.

PROBLEM 6

Compute the following sums and products:

1. $(6.8 \times 10^{-5}) + (7.5 \times 10^{-4})$
2. $7(3.8 \times 10^{-6})$
3. $(4.0 \times 10^{-3})(3.6 \times 10^{-4})$

EXERCISES

1. Write the following numbers in scientific notation:
   a. 6,930,000,000
   b. 25,000,000,000,000,000
   c. .00000509
   d. .000000000487
   e. 602200000000000000000000
2. Compute the following sums:
   a. $(4.5 \times 10^{9}) + (8.2 \times 10^{9})$
   b. $(4.5 \times 10^{9}) + (8.2 \times 10^{8})$
   c. $(4.5 \times 10^{9}) + (8.2 \times 10^{7})$

d. $(5.8 \times 10^{-6}) + (7.6 \times 10^{-6})$

e. $(5.8 \times 10^{-7}) + (7.6 \times 10^{-6})$

f. $(5.8 \times 10^{-8}) + (7.6 \times 10^{-6})$

3. Compute the following products:

   a. $(7 \times 10^{5})(6 \times 10^{3})$

   b. $7(8 \times 10^{12})$

   c. $(4 \times 10^{-5})(2 \times 10^{4})$

   d. $80(7 \times 10^{7})$

   e. $(5 \times 10^{8})(4 \times 10^{12})$

   f. $(27 \times 10^{8})(8.5 \times 10^{9})$

4. Convert the age of the universe from 13,700,000,000 years into days, hours, minutes and seconds.

5. The distance from the sun to each of the planets is shown below. Compare the ratio of the distance from each planet to the sun to the distance from the earth to the sun.

Planet	Distance from Sun (km)	Ratio
Mercury	5.8×10^{7}	
Venus	1.08×10^{8}	
Earth	1.496×10^{8}	
Mars	2.27×10^{8}	
Jupiter	7.78×10^{8}	
Saturn	1.427×10^{9}	
Uranus	2.871×10^{9}	
Neptune	4.497×10^{9}	

6. The number of atoms in 12 grams of oxygen is estimated to be 6.022×10^{23} and is called Avogadro's number. How many atoms are there in 1 gram of oxygen? How many atoms are there in 48 grams of oxygen?

7. The mass of the sun is 1.989×10^{30} kg. The masses of some of the planets are given in the chart below.

Planet	Mass in kg	Ratio
Mercury	3.3×10^{23}	
Venus	4.87×10^{24}	
Earth	5.98×10^{24}	
Mars	6.5×10^{23}	
Jupiter	1.9×10^{27}	
Saturn	5.6×10^{26}	

a. Compute the ratio of the mass of each of the planets to the mass of the earth.
b. Compute the sum of the masses of the planets listed in the chart.
c. What is the ratio of the mass of the sun to the total mass of these planets?

8. How big are these numbers anyway? In 2012 the total budget for the United States government was 3.729 trillion dollars. How can we visualize how big this number is?
   a. In 2012 the US population was 313,000,000. How much did the government spend per person?
   b. The one dollar bill is .010922 cm thick. Suppose we took the total US budget in one dollar bills and stacked them one on top of the other. How high would the stack be in kilometers?
   c. Compare the height you computed in part 8b to the average distance from the Earth to the moon, 384,403 km.

9. To deal with very large or small numbers scientists often use unusual units. For example, a light year is the distance that light can travel in a vacuum over the course of a year. A light year, which is abbreviated ly, is approximately equal to 9.46×10^{12} km.
   a. Using the information from Exercise 5 convert the distance from the Sun to the Earth from km to light years.
   b. Approximately how many minutes does it take for light to travel from the Sun to the Earth?
   c. The Andromeda Galaxy is approximately 2.5×10^6 light years from Earth. How many times farther away from the earth is the Andromeda Galaxy than the Sun is?

SECTION 6.7 CHAPTER REVIEW

Key Terms

arithmetic sequence
base
exponent
exponential decay
exponential function
exponential growth
factor
geometric sequence
greatest common factor (GCF)
growth period
half-life
initial value
interest rate
least common multiple (LCM)
multiple
multiplicative inverse
power
prime factorization
radioactive decay
scientific notation

Rules of Exponents

Product of Powers Rule:
$x^m \cdot x^n = x^{m+n}$

Distributive Property of Exponents:
$y^n \cdot z^n = (y \cdot z)^n$

Power of a Power Rule:
$(z^a)^b = z^{ab}$

Quotient of Powers Rule:
$\frac{x^n}{x^m} = x^{n-m}$

Zero Power:
If $x \neq 0$, $x^0 = 1$

Negative Power Rule:
If $x \neq 0$, $x^{-1} = \frac{1}{x}$

Formulas

Exponential Function:
$f(t) = ab^{\frac{t}{k}}$

Practice Problems

1. Compute the following products, writing answers in exponential form. State the rule that you applied to arrive at your answer.
   a. $(7^2)(7^6)$
   b. $(12^4)(9^4)$
   c. $(13^2)^4$
   d. $23 \cdot 23^3$
   e. $2^5 \cdot 3^4 \cdot 2^4 \cdot 3^5$
   f. $(3a^2)(2ab^3)(5ab)$
   g. $7x^3y^2 \cdot 6xy^3z$
   h. $(3ab^2)^3$
   i. $(mn)^2(m^3n^4)^2$
   j. $(a^2b^3c^{11})(a^5b^4c^9)$
2. Simplify. Express answers as a product of powers (no fractions). State the rule that you applied to arrive at your answer.
   a. $\frac{12^5}{12^2}$
   b. $\frac{9^{15}}{9^7}$
   c. $\frac{x^3y^5}{x^2y}$
   d. $\frac{m^7n^3}{m^7n}$
   e. $\frac{5kl^6}{2k^2l^5}$
3. Simplify, expressing your answer with positive exponents.
   a. 13^{-2} b. $\frac{2^7 3^5}{2^8 3^2}$ c. $\frac{15a^5bc^2}{5a^2b^3c}$
4. Simplify, expressing your answer with negative exponents.
   a. $\frac{3}{2^7}$ b. $\frac{p^{12}q^4}{p^{10}q^7}$
5. Suppose that on Sunday you see 8 mosquitoes in your room. On Monday, you count 12 mosquitoes, and Tuesday there are 18 mosquitoes. Assume that the population will continue to grow exponentially.
   a. Find the number of mosquitoes after 5 days (in other words on Friday), after 10 days and after 2 weeks.
   b. Write an equation that models the population growth.
6. Steve wrote a function that models the spreading of a rumor. He decided to conduct an experiment. He was going to spread the rumor that his math teacher, Mr. Frand, had won the Texas lottery with a $127.4 million jackpot. Suppose that the rumor starts with him telling 1 student at 8 am Friday morning. The next hour the number of people triples, and so on.
   a. How many people will have heard the rumor by 11 am? By 1 pm?
   b. The whole school has 894 students. At what time will the whole school know the rumor.

c. Write an equation that models this population growth.

7. The radioactive element Bismuth-215 has a half-life of 5 days.
   a. Find the amount of radioactive Bismuth-215 left from a 100-gram sample after 5 weeks.
   b. Write an equation that models the exponential decay.

8. Write the following in scientific notation.
   a. 0.0000457
   b. The national debt: \$12, 452, 000, 000, 000
   c. 0.00000000902
   d. 964,400

9. Write the following in standard form.
   a. Speed of light: $2.998 \times 10^8 \frac{m}{sec}$
   b. Diameter of human red blood cell: 7.6×10^{-6} m
   c. Size of 2010 gulf oil spill: 6.737×10^7 gallons
   d. Radius of the Carbon Atom: 1.34×10^{-10} m

POLYNOMIAL OPERATIONS AND FACTORING

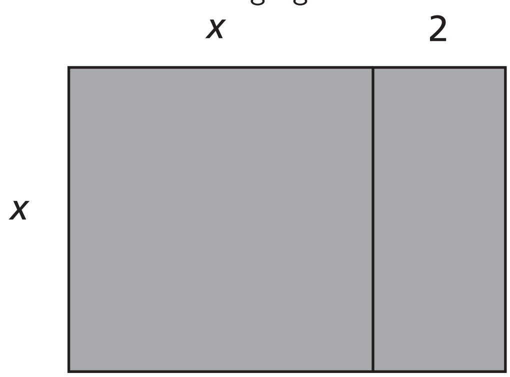
7

SECTION 7.1 POLYNOMIAL EXPRESSIONS

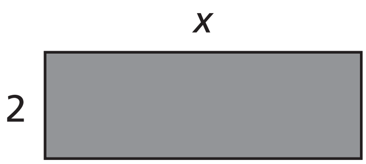
In Chapter 1, we reviewed expressions that were equivalent. Some of these had two or more variables and came from geometric models of area and length. We now make a connection to polynomial expressions involving variables to compute the area of rectangles.

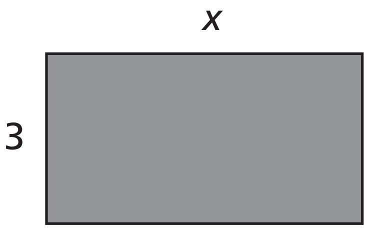
EXAMPLE 1

1. Calculate the area of the following figure:



2. Calculate the sum of the areas of the following rectangles:


x
2


x
3

SOLUTION

1. The area is calculated as the product of length and width:

$$x(x+2) = x^2 + 2x$$

   Notice that the second expression has two terms and represents the sum of the two smaller rectangles. Although each term represents area, we cannot combine the terms.

2. The area of the first rectangle is $(x)(2) = 2x$ and the area of the second rectangle is $(x)(3) = 3x$. The total area is computed as $2x + 3x$. Can this expression be written as one term? Yes, we can rewrite it as $5x$. Explain why this makes sense.

In both of these examples, we assumed that x represents a variable length. Note that we could change the length of x in the picture, but the expressions for area would remain the same. Each of these expressions is an example of a polynomial.

POLYNOMIAL
A *polynomial* is an algebraic expression obtained by adding, subtracting and/or multiplying real numbers and variables.

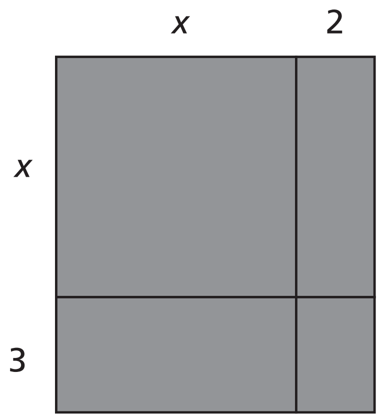
The expression x^2+2x is an example of a polynomial, because we obtain it by multiplying the variable x by itself to get x^2, multiplying x by 2 to get $2x$, and adding these two terms to get $x^2 + 2x$. The expression $x^4 - 3x^2 + 5x + 2$ is also a polynomial, since it can be formed by adding, subtracting and multiplying real numbers and variables. One-term expressions, such as x^3, -5 and 0 are also polynomials. Expressions such as $\frac{1}{x}$ and $\sqrt{x}$ are not polynomials, since we cannot form these expressions using only the operations of addition, subtraction and multiplication.

The expression x^2+2x is called a *binomial* and the second expression $5x$ is called a *monomial*. Find these two words up in the dictionary. What would you call a polynomial expression with three terms that cannot be simplified?

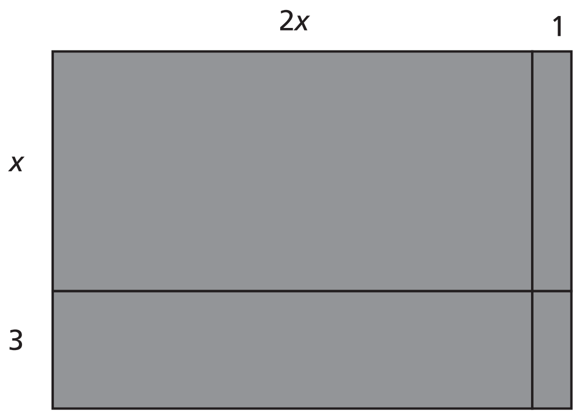
PROBLEM 1

Write polynomial expressions for the total area of each of the figures. Determine whether each expression is a monomial, binomial or trinomial.

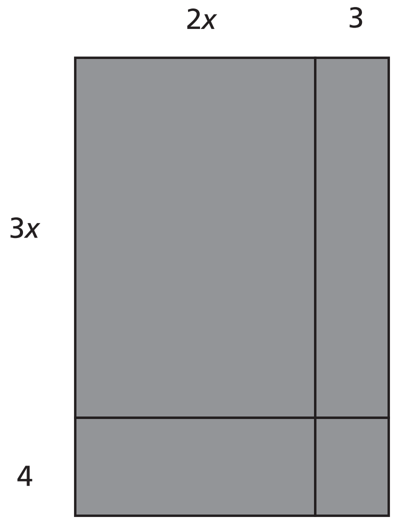
1.



2.



3.



PROBLEM 2

In mathematics, we often use a polynomial to represent the rule of a function. In fact, we have seen examples of many linear functions that are monomials or binomials. For each of the following linear functions, determine if they are monomials or binomials:

1. $f(x) = 3x + 5$
2. $f(x) = 4x$
3. $f(x) = 6$
4. $f(x) = 3 - 2x$
5. What do the graphs of these functions have in common?

We will now use the polynomial expressions from the activities above as the rules for functions. We will explore and study such functions in the next chapter. However, let's compare their graphs now with a graphing calculator.

EXPLORATION 1

For each of the following functions, determine if they are monomials, binomials, or trinomials. Then graph each function using a graphing calculator.

1. $f(x) = x^2$
2. $f(x) = x^2 + 2x$
3. $f(x) = x^2 + 5x + 6$
4. $f(x) = 2x^2 + 7x + 3$
5. $f(x) = 6x^2 + 17x + 12$
6. What do you notice that is common to each of these graphs? What else is similar about the polynomials?

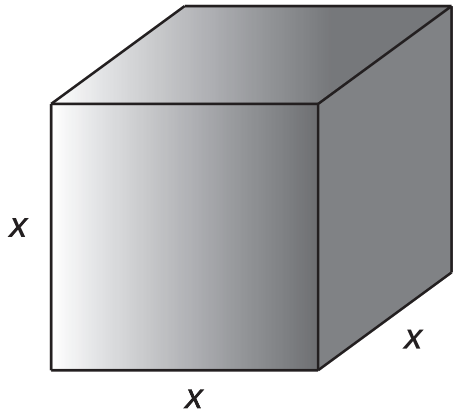
The graphs of the polynomials in Exploration 1 are similar. They also have one term involving x^2. In fact, the highest exponent for the variable x in each of these polynomials is 2. We say they are polynomials of degree 2. The linear functions in problem 2 that have a term involving x are called polynomials of degree 1 because the variable x can be written as x^1. For example, the polynomial $3x^2 + 4x + 5$ is a trinomial of degree 2 because of the term $3x^2$. It also contains the term $4x$, which is a term of degree 1, but we label the polynomial by the term with the highest degree. What is the degree of the constant term 5? If we wrote this term as $5x^0$, it makes sense to say it is a monomial of degree 0.

DEGREE
The *degree* of a polynomial in one variable x, written in simplest form, is the exponent of the highest power of x that appears in the polynomial. If a polynomial does not contain any variables, then its degree is 0 if the polynomial is nonzero, and undefined if the polynomial is zero.

Many activities in this section have been about finding expressions for area. We turn to expressions that describe volume.

EXAMPLE 2

Suppose we draw a rectangular box with each side having length x. What is the volume of this box?



SOLUTION The volume of a rectangular box is given by the formula $V = LWH$ where L is the length, W is the width and H is the height. In this case, each side has the same length, x. So, the volume is calculated as $V = (x)(x)(x) = x^3$. This box is a cube and we say its volume is "x cubed." We can think of V as a function so that for each possible value of $x > 0, V(x) = x^3$, the volume of the cube. Notice that this expression is a monomial of degree 3.

EXAMPLE 3

A rectangular box has sides of length x, $x + 2$ and $x - 4$ units. Write an expression for the volume of this box; that is, find a formula for $V(x)$. For what values of x does this formula make sense?

SOLUTION We write the formula for $V(x)$ as the product of the lengths of its three sides: $V(x) = (x)(x + 2)(x - 4)$.

Observe that this formula only gives the volume of the box if $x > 4$; if $x \leq 4$, then the side of the box of length $x - 4$ does not exist. So the domain of the function $V(x)$ is the set $\{x | x > 4\}$, even though the value of the polynomial $(x)(x+2)(x-4)$ can be calculated for all numbers x.

In section 7.3, we will learn that this product can be expanded to the equivalent expression:

$$V(x) = x^3 - 2x^2 - 8x.$$

The expression for V is a trinomial and has degree 3 because it has a term with x^3. Graph the two functions from Examples 2 and 3 with a graphing calculator and compare. Watch out for the domain!

EXERCISES

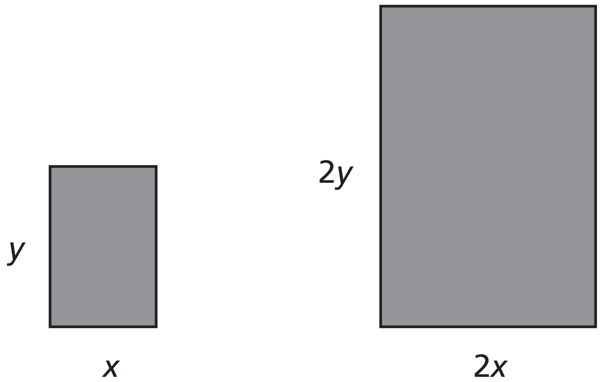
1. For each of the following polynomial functions, determine its degree and whether it is a monomial, binomial, trinomial, or a polynomial with more than 3 terms.
   a. $f(x) = 3x + 2$
   b. $f(x) = 4 - 2x$
   c. $f(x) = x^2 - 4x + 5$
   d. $f(x) = 3 + 2x - x^2$
   e. $f(x) = x^3 - 6x + 2$
   f. $f(x) = x^3 - 4x^2 + 2x - 3$
   g. $f(x) = x^3 + 3x^2 - 4x - 12$
   h. $f(x) = x^4 - 4x$
   i. $f(x) = x^4 - 5x^3 + 2x^2 - 3x + 1$
   j. $f(x) = x^5 - 2x^4 + 4x^3 - 2x^2 + 3x + 1$

2. Use a graphing calculator to graph each of the functions in Exercise 1. What do you notice about these graphs?

3. Suppose we define $f(x) = 2x^2 - 3x + 4$. Evaluate the following:
   a. $f(0)$
   b. $f(1)$
   c. $f(3)$
   d. $f(-2)$
   e. $f\left(\frac{1}{2}\right)$
4. For each of the following rectangles, draw a picture and label the area of each of the rectangles that make up the large rectangle. Use this information to obtain two equivalent expressions for the area of the large rectangle.
   a. An $(x + 2)$ by $(x + 1)$ rectangle.
   b. An $(x + 5)$ by $(x + 2)$ rectangle.
   c. An $(x + 2)$ by $(x + 2)$ rectangle.
5. Polynomials can involve more than one variable. For each of the following rectangles, draw a picture and label the area of each of the rectangles that make up the large rectangle. Use this information to obtain two equivalent expressions for the area of the large rectangle.
   a. An $(x + 2)$ by $(y + 3)$ rectangle.
   b. An $(2x + 3)$ by $(3y + 5)$ rectangle.
   c. An $(x + y)$ by $(x + y)$ rectangle.
   d. An $(x + y)$ by $(2x + 3y)$ rectangle.
   e. An $(x + 2)$ by $(x + 1)$ rectangle.
   f. An $(x + 5)$ by $(x + 2)$ rectangle.
   g. An $(x + 2)$ by $(x + 2)$ rectangle.
6. Greenfield Lawn Services offers a lawn-mowing service in which customers can have their lawns mowed for \$35 per hour plus a \$20 service charge. Suppose that on a given day, Greenfield mows L lawns, and each lawn takes H hours to mow. Which of the following expressions can be used to find the amount of money, in dollars, that Greenfield makes that day? (More than one expression may be correct.) For each expression, explain why the expression is correct or incorrect.
   a. $L(35H + 20)$
   b. $35H + 20L$
   c. $35HL + 20L$
   d. $35HL + 20$

7. Eric and Felicia are both working on an algebra problem. In one step, Eric replaces the expression (x^2+x) with x^3. In the same step, Felicia replaces the expression (x^2+x) with $3x$. Which student, if any, is correct? If a student performed an incorrect step, how might you convince him/her that the step is incorrect?
8. In this problem you will practice some skills that will come in handy in the following sections. For each of the following expressions, simplify the expression if possible. If the expression cannot be simplified, say so.
   a. $3+(-5)$
   b. $-3-5$
   c. $-3-(-5)$
   d. $3x-5x$
   e. $7x-(-2x)$
   f. $-x-6x$
   g. $5x-2y$
   h. $3x^2-5x^2$
   i. $-2x^2+5x$
   j. $3y+2y$
   k. $3x+2x+2y$
   l. $5x^2+4x-3x^2$
9. Can you find a polynomial function, in simplest form, that is a binomial of degree 6? How about a polynomial function, in simplest form, that has six terms and is of degree 2?
10. **Investigation:**
    Draw a rectangle with dimensions $(x+3)$ by $(x-2)$. Write two equivalent polynomial expressions for the area. Explain how the picture illustrates the areas of the smaller rectangles.

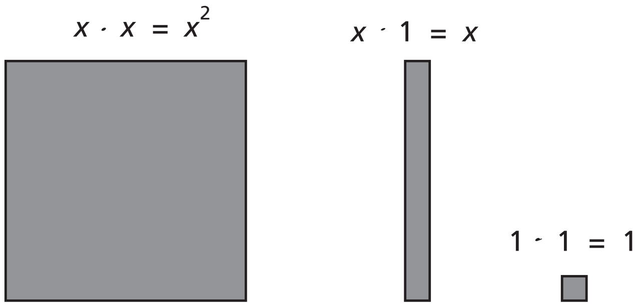
SECTION 7.2 POLYNOMIAL ADDITION AND SUBTRACTION



As you work with expressions involving polynomials, you will need to consider the sums and differences of these expressions. For example, the perimeter of a rectangle with sides of length x and y may be given by $x+x+y+y$ or by $2x+2y$. We say these expressions are equivalent; that is, $x+x+y+y=2x+2y$. The perimeter of a larger rectangle with sides twice as long as those of the first rectangle would be $2x+2x+2y+2y=4x+4y$. Let's explore the idea of polynomial addition and subtraction using algebra tiles to represent polynomial terms.

EXPLORATION 1

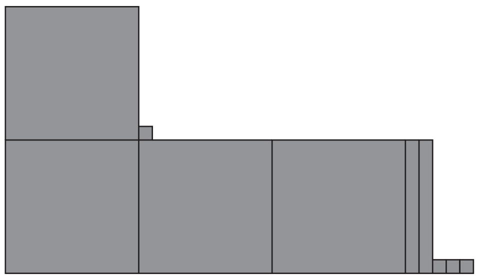
For this exploration, work with a partner. Your teacher will give you tiles that represent rectangles with areas x^2, x and 1, respectively.



1. Use appropriate numbers of the tiles to represent: $x^2 + 2x + 3$.
2. Use appropriate numbers of the tiles to represent: $2x^2 + 3x + 4$.
3. If we put all the tiles together, this represents the addition of the two polynomials: $(x^2 + 2x + 3) + (2x^2 + 3x + 4)$. What is the resulting polynomial?
4. Now, create a polynomial of degree 2, and have your partner create a different polynomial of degree 2. Add these polynomials together and determine the resulting polynomial. Write the process symbolically; that is, show how you would add the polynomials without using the algebra tiles.

EXPLORATION 2

Construct the figure below using your algebra tiles.



1. Find a polynomial expression for the area of the figure.
2. Find an expression for the perimeter of the figure.

EXPLORATION 3

The lengths of the sides of a triangle are as follows: Side b is 4 times as long as the shortest side a, and the third side c is 5 inches shorter than side b. Suppose the shortest side is x inches long.

1. Sketch the triangle. Label the length of the sides in terms of x.
2. Can $x = 2$? Explain. What are possible values for x?
3. Write an expression for the perimeter of the triangle. Simplify using the least number of terms possible.

PROBLEM 1

Suppose we have a rectangle with length 2 centimeters shorter than 3 times the width. Write an expression for the length and the perimeter of this rectangle using w as the width of the rectangle. Simplify each expression using the least number of terms possible.

In Exploration 3, you noticed that $x + 4x + 4x = 9x$. No matter what the value of x, when you add 4 more of that quantity, namely $4x$, to x, then you have a total of $5x$. And when you add $4x$ more to the $5x$, then you will have total of $9x$. We call x and $4x$ *like terms* because the variables are both x raised to the same power and terms differ only by their coefficients .

EXAMPLE 1

Add the following two polynomials: $5x^2 + 4x + 7$ and $2x^2 - 3x - 1$.

SOLUTION The trinomials each have the three terms: an x^2 term of degree 2, an x term of degree 1 and a constant term of degree 0.

When we add polynomials, we combine like terms; that is, terms that contain the same powers of the same variables. We have $5x^2$ and $2x^2$, that combine to make $7x^2$. We have $4x$ and $-3x$ that sum to $1x$ or x.

We have 7 and -1, which give a sum of 6 for the constant term. So we have

$$\begin{aligned}(5x^2 + 4x + 7) + (2x^2 - 3x - 1)\\ &= (5x^2 + 2x^2) + (4x - 3x) + (7 - 1)\\ &= 7x^2 + x + 6\end{aligned}$$

You can also align the polynomials vertically to add them. Our usual number addition algorithm starts from the right, but would it matter in this case? Why or why not?

$$\begin{array}{rrcrcr} & 5x^2 & + & 4x & + & 7 \\ + & 2x^2 & - & 3x & - & 1 \\ \hline & 7x^2 & + & x & + & 6 \end{array}$$

Just as we can add polynomials, we can subtract polynomials. For example, when we compute $4x - x$, we are taking one x away from $4x$. Because $4x = x+x+x+x$, we have $4x-x = x+x+x+x-x = 3x$. You can also see by the distributive property that $4x - x = (4 - 1)x = 3x$. In either case, just as we add the coefficients of like terms when adding polynomials, we subtract the coefficients of like terms when subtracting polynomials.

EXAMPLE 2

Subtract $2x^2 - 3x - 1$ from $5x^2 + 4x + 7$.

SOLUTION There are two ways to express the subtraction: horizontally or vertically. If we write the problem horizontally we get:

$$(5x^2 + 4x + 7) - (2x^2 - 3x - 1)$$

$$\begin{aligned}
&= (5x^2 + 4x + 7) + (-1)(2x^2 - 3x - 1) \\
&= (5x^2 + 4x + 7) + (-2x^2 + 3x + 1) \\
&= (5x^2 - 2x^2) + (4x + 3x) + (7 + 1) \\
&= 3x^2 + 7x + 8
\end{aligned}$$

Notice in the first step we write the problem as an addition problem. In the second step, we use the distributive property to multiply $(2x^2 - 3x - 1)$ by -1. This changes the sign of each term in the second polynomial. Then we add like terms, which also uses the distributive property.

We can also write the problem vertically:

$$\begin{array}{rrrrrrr}
 & 5x^2 & + & 4x & + & 7 & \\
- & (2x^2 & - & 3x & - & 1) & \\
\hline
 & 3x^2 & + & 7x & + & 8 &
\end{array}$$

It is important to subtract for each term. For example, the middle term is $4x - (-3x) = 7x$.

EXERCISES

1. Make your own algebra tiles. Rectangle A has length x and width y. Rectangle B has length twice as long as rectangle A and width 3 times as wide as rectangle A. Sketch and label a picture of the 2 rectangles cut out the rectangles to model computing the difference between the areas of rectangle A and rectangle B. Write an equation that expresses what is happening.
2. The sides of Triangle A have lengths x, y and z units. The sides of Triangle B have lengths $x - 4$, $y - 3$ and $z - 2$ units. Sketch and label a picture of the 2 triangles. Find the sum of the perimeters of Triangle A and Triangle B. Simplify your answer. Assume all lengths are positive.
3. Perform the indicated operation of addition:
   a. $(2x^2 + 3x + 5) + (4x^2 + 2x + 7)$

b. $(2x^2 - 3x + 5) + (4x^2 + 2x - 7)$

c. $(2x^2 + 3x - 5) + (4x^2 - 2x + 7)$

d. $(-2x^2 + 3x + 5) + (4x^2 - 2x - 7)$

e. $(x^2 + 3x + 5) + (4x^2 + 2x + 7)$

4. Perform the indicated operation of subtraction:

a. $(3x^2 + 5x + 8) - (x^2 + 2x + 3)$

b. $(3x^2 - 5x + 8) - (x^2 + 2x - 3)$

c. $(3x^2 - 5x - 8) - (x^2 - 2x + 3)$

d. $(4x^2 + 5) - (2x^2 + 2x + 3)$

e. $(3x^2 - 5x) - (x^2 + 3)$

f. $(3x^2 + 5x + 4) - (-2x^2 - 2x - 2)$

g. $(-3x^2 - 4x + 1) - (-3x^2 - 2x - 3)$

5. Mr. Uribe writes two polynomials on the blackboard in his classroom; he writes the polynomial $x^2 + 6x + 9$ on the left and another polynomial on the right. He asks his students to subtract the two polynomials in whatever order they choose. Some students subtract the polynomial on the left from the one on the right and get $3x^2 - 5x - 3$. The other students subtract the polynomial on the right from the one on the left. What do they get?

6. Tina fires two balls directly upward from different cannons (which are situated at ground level) at the same time. One cannon fires a ball at a speed of 40 meters per second, so that the ball's height (in meters) at a given time t (in seconds) is given by $f(t) = 40t - 5t^2$. The other cannon fires a ball at a speed of 45 meters per second, so that the ball's height (in meters) at a given time t (in seconds) is given by $g(t) = 45t - 5t^2$.

a. Compute $f(0)$ and $f(8)$. What do these values mean?

b. For values of t such that $0 \leq t \leq 8$, what is the vertical distance between the two balls? Write your answer as an expression in terms of t. Why is this formula only valid for $0 \leq t \leq 8$?

c. What is the vertical distance between the balls 3 seconds after Tina fires the cannons?

d. What do you think the $-5t^2$ in each formula represents?

7. Williamson Widgets can produce up to 1000 widgets each month and sell them to factories. The monthly cost of producing n widgets, in dollars, is given by the function $C(n) = 2500 + 40n - \frac{n^2}{100}$. If Williamson sells n widgets in a month, the amount of money it makes, in dollars, is given by the function $R(n) = 60n$.
   a. If Williamson produces and sells n widgets in a given month, how much profit, in dollars, do they make? Give your answer as an expression in terms of n.
   b. By plugging in values of n and keeping track of your results, find an estimate for the number of widgets Williamson needs to produce and sell in a given month in order to make at least $1000 of profit.

8. In this problem you will practice skills with numbers that will come in handy in the following sections. Fill in the following table with integers. Note that the first line is given to you. Try to do the problems in your head as quickly as you can.

a	b	$a+b$	ab
4	-3	1	-12
-4	-3		
4	3		
-4	3		
3	-2		
-2	3		
2	-3		
-2	-3		
2	3		
4	5		
-4	-5		
3	-3		
2	7		
2	-7		

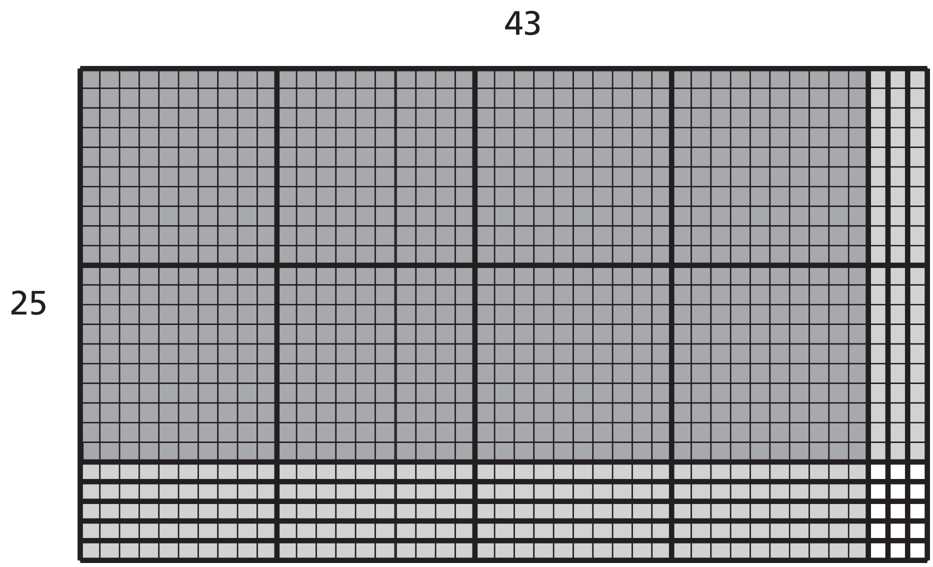
SECTION 7.3 POLYNOMIAL MULTIPLICATION

We worked with addition and subtraction of polynomials in section 7.2. In this section, we will learn how to multiply polynomials. In section 7.1, we computed the areas of rectangles, such as a $(x+2)$ by $(x+3)$ rectangle, by computing the area of smaller rectangles. We found that the total area was x^2+5x+6 by adding the areas of the 4 smaller rectangles. But we also know that the area of the large rectangle is its length times its width; that is, $(x+2)(x+3)$. So these two expressions must be equivalent. This means that:

$$(x+2)(x+3) = x^2+5x+6.$$

In this section, we will study the distributive property and see how it is related to the areas of rectangles. Once we get good at using the distributive property, we can multiply polynomials without drawing the rectangles each time.

Let's first review one way of modeling the multiplication of numbers using an area model. For example, consider multiplying the numbers 43 × 25.

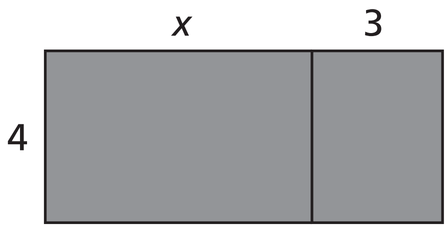


Just as in the numerical example above, where we viewed 43×25 as $(40 + 3)(20 + 5)$, explore the following problem.

EXAMPLE 1

Compute the area of a rectangle with dimensions $(x + 3)$ and 4.

SOLUTION We want to use the visual model from section 7.1. So we sketch and label a rectangle:



We find the areas of the two smaller rectangles: $(x)(4) = 4x$ and $(3)(4) = 12$. We then add them to get the total area, $4x + 12$. We also noticed in section 7.1 that the total area could also be expressed as the product of the length and the width of the large rectangle: $(x + 3)(4)$. This means that the two expressions of area are equivalent; that is, $(x + 3)(4) = 4x + 12$ for all values of $x > 0$.

Why does this make sense? Recall from Chapter 1 that the distributive property tells us how multiplication and addition interact. The distributive property says that if each of A, B and C is a number, then $(A + B)C = AC + BC$, and equivalently, $C(A + B) = CA + CB$. This is exactly the pattern that our area model of multiplication produces. Now we will explore how to use the distributive property to compute the area of the $(x + 2)$ by $(x + 3)$ rectangle we discussed at the beginning of this section.

PROBLEM 1

Use the distributive property to compute the area of a $(x+2)$ by $(x+3)$ rectangle.

EXPLORATION 1

Use the area model to represent the multiplication of the two binomials: $(4x+3)(2x+5)$. Then simplify to get the smallest possible number of terms.

Notice that Exploration 1 asked you to determine the product $(4x+3)(2x+5)$. We can use the properties of arithmetic to justify the answer as well. By the distributive property, $(4x+3)(2x+5) = (4x+3)2x+(4x+3)5$. By the commutative, associative and distributive properties, the right hand side of the equation is equal to

$$\begin{aligned}(4x)(2x) + 3(2x) + (4x)(5) + 3(5) \\ &= 4 \cdot 2 \cdot x \cdot x + 3 \cdot 2 \cdot x + 4 \cdot 5 \cdot x + 3 \cdot 5 \\ &= 8x^2 + 6x + 20x + 15 \\ &= 8x^2 + 26x + 15.\end{aligned}$$

This is precisely the sum of the partial products represented in the area model. The distributive property is an important property for simplifying expressions or manipulating them to get equivalent expressions.

EXAMPLE 2

Multiply the following polynomials:

1. $(x+3)(2x+5)$
2. $3x(4x-5)$
3. $6x^2(9x^2+7x-3)$
4. $(8x-2)(5x+1)$

SOLUTION

1. $(x+3)(2x+5) = (x+3)2x + (x+3)5 = 2x^2 + 6x + 5x + 15 = 2x^2 + 11x + 15$
2. Using the distributive property, we have $3x(4x-5) = (3x)(4x) - (3x)(5)$, which can be simplified to $12x^2 - 15x$. Notice that we use another form of the distributive property: A(B - C) = AB - AC.
3. Using the distributive property, we have

$$6x^2(9x^2) + 6x^2(7x) - 6x^2(3) = 54x^4 + 42x^3 - 18x^2$$

4. Using the distributive property, we have

$$\begin{aligned}(8x-2)(5x) + (8x-2)(1) &= (8x)(5x) - 2(5x) + 8x(1) - 2(1)\\ &= 40x^2 - 10x + 8x - 2\\ &= 40x^2 - 2x - 2.\end{aligned}$$

EXPLORATION 2

Rectangle A has length x and width y. Rectangle B is 2 units longer and 3 units wider than rectangle A. Sketch and label the rectangles. Find the difference between the areas of rectangle A and rectangle B. Simplify your answer as a polynomial with the least possible number of terms.

There are several products that occur often in mathematics. Familiarity with the following types of expressions is helpful for your work in algebra.

EXAMPLE 3

Find the products of the following expressions and simplify your answers.

1. $(x+y)(x-y)$
2. $(x+y)(x+y) = (x+y)^2$

3. $(x - y)(x - y) = (x - y)^2$

SOLUTION Using the distributive property:

1. $(x+y)(x-y) = x(x-y)+y(x-y) = xx-xy+yx-yy = x^2-y^2$.
2. $(x + y)(x + y) = x(x + y) + y(x + y) = xx + xy + yx + yy = x^2 + 2xy + y^2$.
3. $(x - y)(x - y) = x(x - y) - y(x - y) = xx - xy - yx + yy = x^2 - 2xy + y^2$.

We have been investigating products of binomials but of course polynomials may also be monomials, trinomials and expressions with even more terms. The distributive property and exponent properties can be useful for finding the products of these expressions.

PROBLEM 2

Find the product of the following polynomials and simplify your answers:

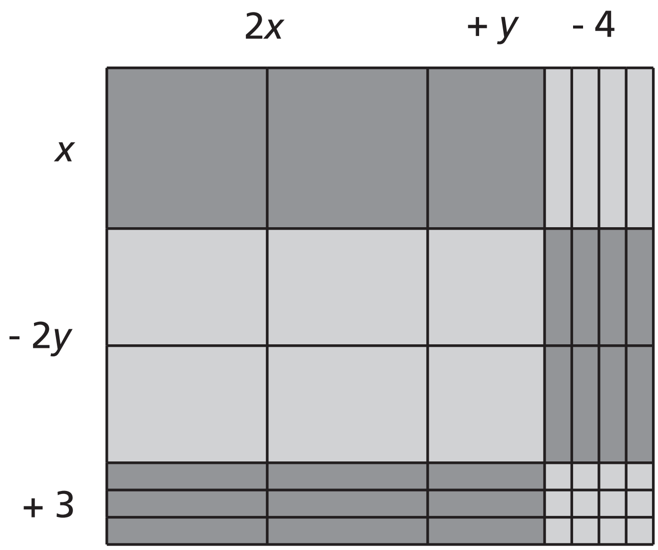
1. $5x^2y(3xy + 6x)$
2. $(3x^2 - 4x + 9)(x - 2)$
3. $(x - y + 2)(x + y - 3)$

As you can see, the distributive property is often when we perform polynomial multiplication. This method can become cumbersome when we multiply polynomials with more than two terms. However, there is a way to keep our work organized when we perform these operations.

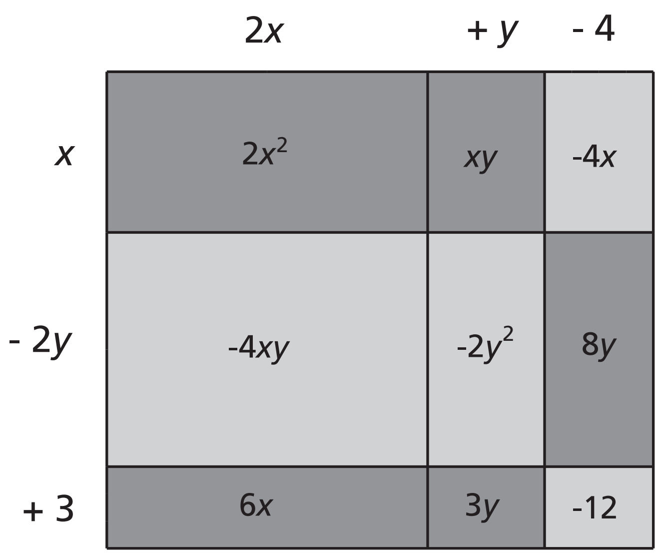
EXAMPLE 4

Multiply the expression $(2x + y - 4)(x - 2y + 3)$.

SOLUTION First we'll use the area model to perform this multiplication, using different-colored tiles to represent positive and negative terms in the product.



We can find products of certain pairs of terms in this expression in our picture:



We see that, in order to multiply $(2x + y - 4)(x - 2y + 3)$, we must multiply each term of $(2x + y - 4)$ by each term of $(x - 2y + 3)$. But this area picture also suggests a way for us to keep our work organized as we do this. We can organize our work in a table as follows:

	$2x$	y	-4
x	$2x^2$	xy	$-4x$
$-2y$	$-4xy$	$-2y^2$	$8y$
$+3$	$6x$	$3y$	-12

So the product $(2x+y-4)(x-2y+3)$ is equal to the sum of the 9 terms in this table. After combining like terms, we find that $(2x+y-4)(x-2y+3)$ is equal to $2x^2 - 3xy - 2y^2 + 2x + 11y - 12$. This method of organizing our work is sometimes called the window pane or *box method*.

EXERCISES

1. Compute the area and perimeter of a rectangle with a length of $(x+4)$ and width of $(x+1)$. Sketch a picture to illustrate each part of the computation.
2. Compute the area and perimeter of a rectangle with a length of $(2x + 5)$ and width of $(2x + 3)$. Sketch a picture to illustrate each part of the computation.
3. Compute the area and perimeter of a rectangle with a length of $(x+5)$ and width of $(x-1)$. Sketch a picture to illustrate each part of the computation.

4. Compute the areas of the rectangles with the following dimensions. Compare the answers:
   a. $(x+3)$ by $(x+2)$
   b. $(x-3)$ by $(x+2)$
   c. $(x+3)$ by $(x-2)$
   d. $(x-3)$ by $(x-2)$
5. Compute the following products and simplify your answers:
   a. $5ab(3a+2b)$
   b. $x(x-4)$
   c. $3x(2x+1)$
   d. $4y(3-2y)$
   e. $y(1-y)$
6. Compute the following products and simplify your answers:
   a. $(x+3)(x+5)$
   b. $(2x+5)(x+3)$
   c. $(x-5)(x+4)$
   d. $(x+5)(x-4)$
   e. $(x-5)(x-4)$
7. Compute the following products and simplify your answers:
   a. $(x+3)^2 = (x+3)(x+3)$
   b. $(x-3)^2$
   c. $(x-3)(x+3)$
   d. $(x-5)^2$
   e. $(x+5)^2$
   f. $(2x+3)^2$
8. Compute the following products and simplify your answers:
   a. $(x-3)(x+3)$
   b. $(a+5)(a-5)$
   c. $(z-4)(4+z)$

d. $(2z+3)(2z-3)$

e. $(8x+7)(7-8x)$

f. $(x-3)(4x+12)$

9. Compute the following products and simplify your answers:
   a. $(2x+5)(3x+2)$
   b. $(2x-5)(3x+2)$
   c. $(5-2x)(2-3x)$
   d. $2x(2x+5)(3x+2)$

10. Compute the following products and simplify your answers:
    a. $(x+3)(x^2-4x+2)$
    b. $(x-3)(x^2-4x+2)$
    c. $(x+3)(x^2+4x-2)$
    d. $(x-3)(x^2+4x-2)$
    e. $(x+2)(x^2-2x+4)$
    f. $(x-2)(x^2+2x+4)$
    g. $(x+2)(x^2+2x+4)$
    h. $(x-2)(x^2-2x-4)$
    i. $(x+1)(x^2-2x-4)$

11. Two cattle farmers, Bob and Ed, want to make rectangular pens for their cattle. Each farmer has $4x$ feet of fencing material. Bob decides to make his pen an x-foot-by-x-foot square, so that the area of his pen is x^2 square feet. Ed believes that he can make a rectangular pen with greater area, so he decides to build his pen so that the length of the pen is $x+k$ feet.
    a. Assuming that Ed uses all of his fencing material, what will be the width of his pen?
    b. What will be the area of Ed's pen?
    c. Which pen has the greater area? Does this depend on the values of x and k?

12. Lindsey thinks that $(x+2)^2 = x^2+4$. Debra thinks that $(x+2)^2 = x^2+4x+4$. Who do think is correct? Sketch and label an area model to justify your answer.

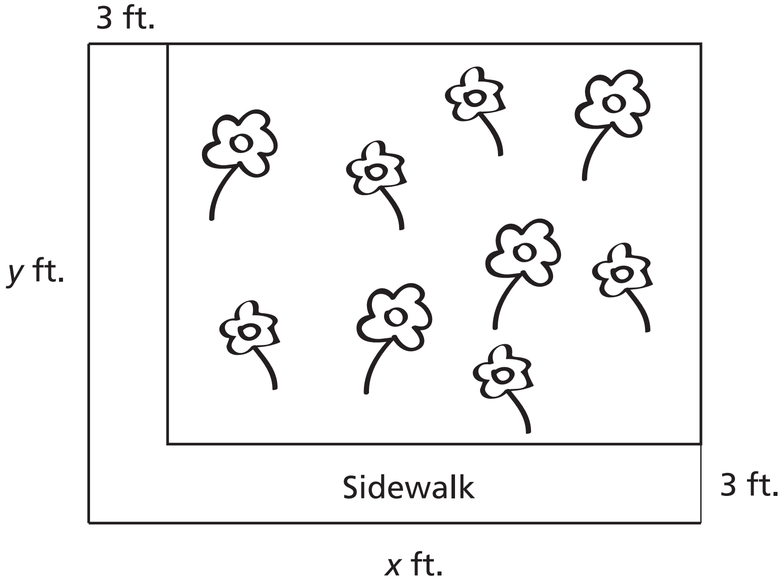
13. **Investigation:**
Often polynomial multiplication can help us understand how numbers behave.
    a. Find $11^2, 11^3, 11^4$. Perform these multiplications by hand.
    b. Find and simplify $(x+1)^2, (x+1)^3, (x+1)^4$.
    c. You should notice a pattern. What is it? Why do you think it works?
    d. Find 11^5 and $(x+1)^5$. The pattern stops working. Why do you think that is?

14. **Investigation:**
In each of (a) through (d), compute the given product.
    a. $(x-1)(x+1)$
    b. $(x-1)(x^2+x+1)$
    c. $(x-1)(x^3+x^2+x+1)$
    d. $(x-1)(x^4+x^3+x^2+x+1)$
    e. Explain the pattern that occurs in parts (a) through (d). Why does this pattern work?
    f. Use this pattern to predict the product $(x-1)(x^{54}+x^{53}+x^{52}+\cdots+x^2+x+1)$.

15. In this problem you will practice some skills with numbers that will come in handy in the following sections. Find in the following table with integers. Note the first line is given to you.

a	b	$a+b$	ab
4	−3	1	−12
−4		1	−20
		5	6
		1	−6
		−1	−6
		0	−9
		4	4
		−4	4

16. Rogelio has a rectangular garden of length x feet and width y feet. He wants to build a sidewalk 3 feet wide across 2 sides of the garden (see figure). Unfortunately, the garden is adjacent to his house, so he is forced to pave over part of the garden to make the sidewalk.



a. After Rogelio puts in the sidewalk, what is the area of the part of the garden that has not been paved over?
b. What is the area of the sidewalk?

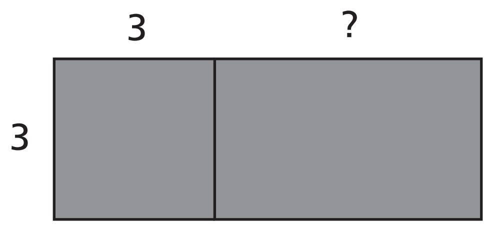
SECTION 7.4 COMMON FACTORS

In Section 7.3 we used the distributive property to multiply polynomials. One way to "see" the distributive property is to use an area model to represent the multiplication. So in Section 7.3, we were given expressions for the lengths of the sides of a rectangle, and we multiplied to find an expression for the area of the rectangle. In this section we will reverse the process. Given a polynomial expression, we will try to think of **factors** which, when multiplied, give the polynomial as the result. In the area model, this means we will be given an expression for the area of the rectangle and want to find expressions for the sides of the rectangles.

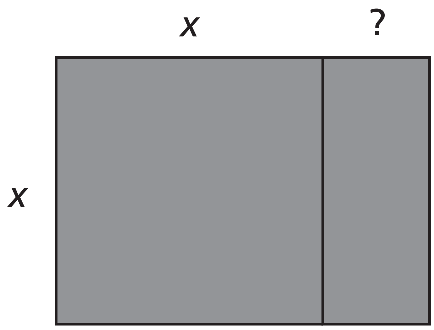
EXPLORATION 1

For each problem below, you are shown a rectangle and told its area. Find the missing lengths labeled with a "?". Write down the dimensions of the rectangle and multiply to verify that you get the given area.

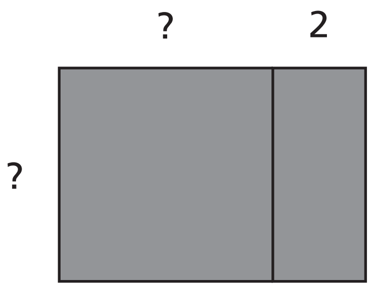
1. The area of the figure is 24 square units.



2. The area of the figure is $x^2 + 4x$ square units.



3. The area of the figure is $a^2 + 2a$ square units.



The process of starting with a polynomial and breaking it into factors is called *factoring* the polynomial. Factoring, then, is the reverse of multiplying. When we multiply, we write:

$$2(a + b) = 2a + 2b.$$

But if we switch sides and write

$$2a + 2b = 2(a + b),$$

then we have factored $2a + 2b$ as the product $2(a + b)$.

Often, however, factoring is harder than multiplying, because you have to imagine what could have been multiplied to get the expression you have. So the more experience you have with multiplying and familiarity you have with the patterns you find, the easier it is to find the factors. In the next exploration, you multiply in a group of problems, then factor in another set of problems. Try to use the patterns you see in the first set to solve the second set.

EXPLORATION 2

1. Use the distributive property to compute the each product:
   a. $x(x+4)$
   b. $3x(2x+5)$
   c. $y(1+y)$
   d. $3ab(a+2b)$
   e. $x(x-7)$
   f. $-x(x+2)$
2. Write each expression as a product of two factors:
   a. x^2+3x
   b. $6x^2+9x$
   c. $y-y^2$
   d. $15a^2b+10ab^2$
   e. x^2-6x
   f. $-x^2-3x$
   g. $x(x+3)-6(x+3)$

There are many techniques used to factor a polynomial. The method you used in Exploration 2 is called finding a *common factor*. This is very similar to finding common factors when you are working with numbers. In some problems, you can factor out more than one factor. In most cases it is preferable to factor out as much as you can, that is factor out the greatest common factor.

EXAMPLE 1

Factor each sum and pick out the greatest common factor. Use the distributive property to check your answer.

1. $x^2 + 10x$
2. $x^2 - 4x$
3. $2x^2 + 3x$
4. $15x^2y + 25xy^2$

SOLUTION

1. $x^2 + 10x = x(x + 10)$
2. $x^2 - 4x = x(x - 4)$
3. $2x^2 + 3x = x(2x + 3)$
4. $15x^2y + 25xy^2 = 5xy(3x + 5y)$

EXPLORATION 3

Sketch (or build using algebra tiles) a rectangle with area equal to $3x^2+6x$ square units. Find the lengths of the sides. Can you find another rectangle with the same area but with different sides? Explain.

EXAMPLE 2

Factor each sum. Factor out the greatest common factor.

1. $3x^2 + 12$
2. $6x^2 + 10x$
3. $a^2b^3 + a^2b$

SOLUTION

1. $3x^2 + 12 = 3(x^2 + 4)$. The common factor need not involve the variable. Here, 3 is the only common factor of the two terms.
2. $6x^2 + 10x = 2x(3x + 5)$. Here we need to find the greatest common factor of the coefficients 6 and 10, which is 2. x is a common factor as well.
3. $a^2b^3 + a^2b = a^2b(b^2 + 1)$. Remember what the exponents mean and that we want to find the highest common factor. So since $a^2b^3 + a^2b = a \cdot a \cdot b \cdot b \cdot b + a \cdot a \cdot b$, we can factor out $a \cdot a \cdot b = a^2b$, which is common to both terms.

PROBLEM 1

Factor each of the following expressions:

1. $x^3 + 3x^2 + x$
2. $50x - 25$
3. $x^2yz + xy^2z + z^2$
4. $4x^2 - 22xy$

In some cases we can repeatedly use the common factor technique to factor more complicated polynomials. Again, the essential idea is to use the distributive property to undo the mulitplication. To see how this works, carefully write out the steps used to find the product $(x+a)(x+b)$:

$$\begin{aligned}(x + a)(x + b) &= x(x + b) + a(x + b)\\ &= (x^2 + bx) + (ax + ab)\\ &= x^2 + bx + ax + ab\end{aligned}$$

Notice that we used the distributive property twice. Hence we can use the common factor technique twice to undo the multiplication.

EXAMPLE 3

Factor $x^2 + 3x + bx + 3b$

SOLUTION Notice that the first two terms have the common factor x: $x^2 + 3x = x(x + 3)$. The next two terms also have a common factor, but this time it is b: $bx + 3b = b(x + 3)$. So we get:

$$x^2 + 3x + bx + 3b = x(x + 3) + b(x + 3).$$

We see two terms each with $(x + 3)$ as a factor. So using the common factor method once more, we get:

$$\begin{aligned} x^2 + 3x + bx + 3b &= x(x + 3) + b(x + 3) \\ &= (x + b)(x + 3). \end{aligned}$$

EXAMPLE 4

Factor $x^2 + bx - 2x - 2b$.

SOLUTION Again we notice that the first two terms have the common factor x: $x^2 + bx = x(x + b)$. The next two terms also have a common factor. But we have a choice: we can factor out 2: $-2x-2b = 2(-x-b)$, or we can factor out -2: $-2x - 2b = -2(x + b)$. If we think about reversing the multiplication process shown above, we want to create terms that have a second common factor, so that we can factor once again. If we choose -2 we get:

$$x^2 + bx - 2x - 2b = x(x + b) - 2(x + b)$$

and we see two terms each with $(x+b)$ as a factor. So using the common factor method once more, we get:

$$\begin{aligned} x^2 + bx - 2x - 2b &= x(x + b) - 2(x + b) \\ &= (x - 2)(x + b). \end{aligned}$$

The method used in Examples 3 and 4 is called *factoring by grouping*. Though the idea is fairly simple, it takes practice to recognize when grouping will work and which common factors to use. In the following exploration, you multiply in the first set of problems, then factor in the second set of problems. Try to use the patterns you see in the first set to solve the second set.

EXPLORATION 4

1. Use the distributive property to compute the following products. Do not combine like terms even if it is possible.
   a. $(2x + 1)(x + 3)$
   b. $(x - 2)(x + 4)$
   c. $(x^2 + 2)(x + 3)$
   d. $(x^2 + 2)(x - 1)$
2. Factor each of the following polynomials using the grouping method.
   a. $2x^2 + 4x + 3x + 6$
   b. $3x^2 - 3x + 2x - 2$
   c. $x^3 + 2x^2 + 3x + 6$
   d. $x^3 - 3x^2 - x + 3$

Many polynomials can not be factored using grouping.

EXAMPLE 5

Try to factor $x^3 + x^2 + 2x + 4$ by grouping. Explain what goes wrong.

SOLUTION To factor by grouping, we first need to group two terms together and take out the common factor. If we group the first two terms, we notice the common factor x^2, so $x^3 + x^2 = x^2(x + 1)$. This leaves the last two terms with a common factor of 2, or $2x + 4 = 2(x + 2)$. Putting this together:

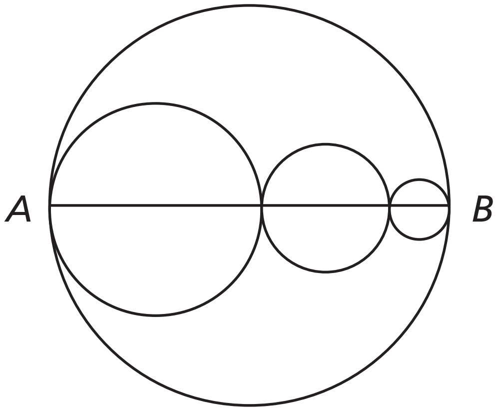
$$x^3 + x^2 + 2x + 4 = x^2(x + 1) + 2(x + 2)$$

But we are left with no common factor, since $x + 1 \neq x + 2$. There are other ways we could group, but they all lead to the same problem. It turns out that this polynomial cannot be factored, by grouping or by any other method. Polynomials that cannot be factored into polynomials of smaller degree are said to be *irreducible*.

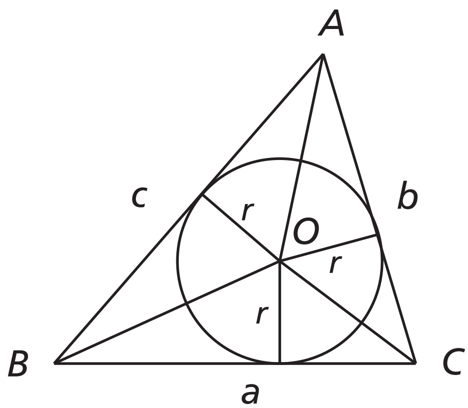
EXERCISES

1. Factor the following polynomials by finding the greatest common factor. If a polynomial does not contain any common factors, say so.
   a. $8x + 2$
   b. $-x^2 + 5x$
   c. $3c^2 + 7c + 3$
   d. $3x^2 - 3x + 6$
   e. $a^2b + 3a^2 + 4ab$
   f. $x^2yz - x^3y + xz - y^2z^2$
   g. $18k^3 - 6k^2$
   h. $14r^4s + 4r^2s^3 - 8r^2s^2$
2. Factor the following polynomials by grouping. If a polynomial cannot be factored by grouping, say so.
   a. $xy + 5x + 3y + 15$
   b. $xy + 8x - 7y - 21$
   c. $x^2 - 8x + 4x - 32$
   d. $r^3 - 3r^2 + 2r - 6$
   e. $x^3 - x^2 - x + 1$
   f. $x^4 + 4x^3 + x + 4$
   g. $20ab - 4a - 5b + 1$
   h. $p^4 + p^3 + 2p^2 + p + 1$ (*Hint:* Try separating the $2p^2$ into $p^2 + p^2$.)
3. Suppose we draw a line segment $\overline{AB}$, and then draw a circle with

$\overline{AB}$ as a diameter. We then draw 3 smaller circles inside the large circle, with each circle having part of $\overline{AB}$ as its diameter. (See the figure on the next page.) Which is greater: the circumference of the large circle, or the combined circumferences of the 3 smaller circles? (*Hint:* Let the diameters of the smaller circles be x, y and z, and write an expression for the sum of the circumferences of these circles in terms of x, y and z. Recall that the circumference of a circle of diameter d is πd.)



4. Suppose we inscribe a circle with center O inside a triangle ΔABC so that the circle just touches each side of the triangle (see figure). Then the point O is the same distance from all 3 sides of the triangle; this distance is r, the radius of the circle. Let a, b and c be the lengths of the sides of the triangle. Suppose we divide the triangle into 3 smaller triangles by drawing a line segment from each vertex to O.



a. In terms of a, b, c and r, what are the areas of the 3 smaller triangles?
b. In terms of a, b, c and r, what is the area of ΔABC?
c. In terms of P and r, where P is the perimeter of ΔABC, what is the area of ΔABC?

5. **Ingenuity:**
Find all pairs of positive integers such that the product of the 2 integers plus the sum of the 2 integers equals 44. (*Hint:* Write an equation representing this condition. Then add 1 to each side, and factor by grouping.)

6. **Investigation:**
In this problem we investigate a simple but important property that we will use in the next section.
a. Find the following products:
   i. $3 \cdot 0$
   ii. $0 \cdot -4$
   iii. $0x$
   iv. $a \cdot 0$
b. Suppose $p \neq 0$ and $q \neq 0$ are 2 nonzero numbers. Is it possible that $pq = 0$? Explain.
c. In the next section, we will use the property: if p and q are two numbers and $pq = 0$ then $p = 0$ or $q = 0$ (possibly both). Explain why this makes sense.
d. Suppose $pq = 6$, is it possible that $p = 6$? Is it true that either $p = 6$ or $q = 6$? Explain.

SECTION 7.5 FACTORING $x^2 + bx + c$

In Chapter 1, we saw how to solve linear equations such as:

$$2x + 3 = 15$$

or

$$6 - 3(x - 5) = 2x + 10.$$

In this section, we explore equations involving polynomials of degree greater than 1. Consider the following example:

EXPLORATION 1

A rectangle is 10 inches longer than it is wide. The area of the rectangle is 75 square inches. Suppose we want to find the dimensions of the rectangle.

1. Call the width (in inches) of this rectangle x. In terms of x, what is the length of the rectangle? In terms of x, what is the area of the rectangle?
2. What equation do we need to solve in order to find the dimensions of the rectangle?
3. How do you solve linear equations? Try to use these techniques from Chapter 1 to solve this equation. What happens?

The techniques we used to solve linear equations don't seem to work when we try to solve *quadratic equations* – that is, those involving quadratic polynomials. In order to solve quadratic equations, let's try a technique we developed in Section 7.4:

EXPLORATION 2

Recall Exercise 6 on page 394, where Tina shot two balls from different cannons at the same time. We found that one ball falls to the ground 8 seconds after being fired from the cannon. Suppose we want to determine how long it takes for the other ball to fall to the ground. This ball's height (in meters) at a given time t (in seconds) was given by the function $g(t) = 45t - 5t^2$. Since we want to determine when the ball hits the ground, we need to find a value of t for which the ball's height is zero; that is, $45t - 5t^2 = 0$.

1. Factor the expression $45t - 5t^2$.
2. In order for this expression to be equal to zero, what must be true about the two factors?
3. Find two values of t for which $45t - 5t^2 = 0$.

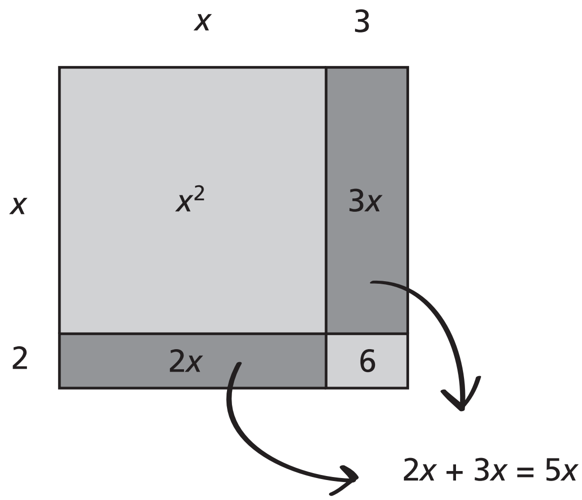
As we see, we can learn much about a quadratic equation by using the *zero product property* we investigated in the previous section: if $xy = 0$, then $x = 0$ or $y = 0$. Since this property deals with factors of a given expression, we need to be able to find the factors of a given polynomial. In this section, we will learn how to find factors of a quadratic polynomial of the form $x^2 + bx + c$, where b and c are real numbers.

EXPLORATION 3

Let's use algebra tiles to investigate factoring.

1. Use appropriate numbers of the tiles to represent: $x^2 + 5x + 6$.
2. Arrange the tiles into a rectangle. What do you have to do to the bars that represent $5x$?
3. What are the dimensions of the rectangle?
4. Write $x^2 + 5x + 6$ as a product of two factors. Use the distributive property to check your answer.

To make the rectangle in Exploration 3, you must split the middle term $5x$ into two pieces. To understand why this happens, we can think about multiplying $(x + 3)(x + 2)$.



$$\begin{aligned}(x + 3)(x + 2) &= x(x + 2) + 3(x + 2)\\ &= x^2 + 2x + 3x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

We first use the distributive property twice and **then** combine the like terms $2x$ and $3x$. To find factors we have to think of the process in reverse. We need to think of how to split the middle term. To do this, notice that the last term results from a product $2 \cdot 3$. Hence, when we factor, we are looking for two numbers whose sum is 5 and whose product is 6. Since $2 + 3 = 5$ and $2 \cdot 3 = 6$, we can factor $x^2 + 5x + 6$ into $(x + 2)(x + 3)$. Look at the figure above to see why this makes sense.

EXAMPLE 1

Factor $x^2 - 7x + 10$. Multiply to verify your answer.

SOLUTION The coefficient of the middle term is -7 and the constant term is 10. Thus, when we factor we are looking for two numbers whose sum is -7 and whose product is 10. We can find this pair of numbers by listing all possible factors of the constant term 10 (including negative numbers)and then computing their sums.

Factor pairs of 10	1,10	-1, -10	2, 5	-2, -5
Sum	11	-11	7	-7

The factors of 10 that add up to -7 are -2 and -5. Thus

$$x^2 - 7x + 10 = (x - 2)(x - 5).$$

We can use the distributive property to verify our answer.

$$\begin{aligned}(x-2)(x-5) &= x(x-5) - 2(x-5) \\ &= x^2 - 5x - 2x + 10 \\ &= x^2 - 7x + 10\end{aligned}$$

Do you see where we use the condition that $-2 + (-5) = -7$?

In Example 1 we listed all the possible factors. For many polynomials, this quickly becomes a tedious process. As you become more familiar with multiplying and factoring, you begin to remember patterns that allow you to find the factors without making the table.

EXPLORATION 4

1. Use the distributive property to compute the following products. Simplify by combining like terms where possible.
   a. $(x + 4)(x + 2)$
   b. $(x + 3)(x + 7)$
   c. $(x + 5)(x + 6)$
   d. $(x - 3)(x - 4)$
   e. $(x - 1)(x - 8)$

f. $(x - 5)(x - 5)$

2. What patterns do you notice about the signs of the factors and the signs of the coefficients?
3. Factor each of the following polynomials.
   a. $x^2 + 10x + 24$
   b. $x^2 - 10x + 24$
   c. $x^2 + 8x + 15$
   d. $x^2 - 11x + 18$
   e. $x^2 - 20x + 100$

PROBLEM 1

Factor each of the following polynomials. Multiply to verify your answers.

1. $x^2 + 5x + 6$
2. $x^2 - 7x + 6$
3. $x^2 - 9x + 20$
4. $x^2 + 2x + 1$

We know how to factor quadratic polynomials of the form $x^2 + bx + c$, where c is positive. What happens if c is negative?

EXPLORATION 5

1. Use the distributive property to compute the following products. Simplify by combining like terms where possible.
   a. $(x - 5)(x + 2)$
   b. $(x + 3)(x - 4)$
   c. $(x + 1)(x - 3)$
   d. $(x - 2)(x + 8)$
   e. $(x - 9)(x + 5)$
   f. $(x - 6)(x + 6)$
2. What patterns do you notice about the signs of the factors and the signs of the coefficients?

3. Factor each of the following polynomials:
   a. $x^2 - 3x - 4$
   b. $x^2 + x - 6$
   c. $x^2 + 4x - 21$
   d. $x^2 - 5x - 50$
   e. $x^2 - 16$

PROBLEM 2

Factor each of the following polynomials. Multiply to verify your answers.

1. $x^2 - 2x - 24$
2. $x^2 + 6x - 27$
3. $x^2 + 5x - 66$
4. $x^2 - 49$

Two special kinds of quadratics frequently appear in factoring problems. A quadratic of the form $(x-a)^2$ is called a *perfect square*. A quadratic of the form $x^2 - a^2$ is called a *difference of squares*. We encountered these special quadratics in Example 3 from Section 7.3, but they deserve special attention here. Once you learn to recognize the patterns, these types of quadratics are especially simple to factor.

EXPLORATION 6

1. Use the distributive property to compute the following products. Simplify by combining like terms where possible.
   a. $(x+1)^2$
   b. $(x-2)^2$
   c. $(x+3)^2$
   d. $(x+4)^2$
   e. $(x-5)^2$
2. What patterns do you see here? Use these patterns to compute the product $(x+6)^2$ without actually using the distributive property.

3. Use the distributive property to compute the following products. Simplify by combining like terms where possible.
   a. $(x + 1)(x - 1)$
   b. $(x + 2)(x - 2)$
   c. $(x + 3)(x - 3)$
   d. $(x + 4)(x - 4)$
   e. $(x + 5)(x - 5)$
4. What patterns do you see here? Use these patterns to compute the product $(x + 6)(x - 6)$.
5. Factor each of the following expressions:
   a. $x^2 - 100$
   b. $x^2 + 16x + 64$
   c. $x^2 - 2x + 1$
   d. $x^2 - 144$

PROBLEM 3

Label each of the following quadratic expressions as a perfect square, a difference of squares, or neither. Then factor it.

1. $x^2 - 9$
2. $x^2 + 5x + 4$
3. $x^2 - 10x + 25$

EXPLORATION 7

Let's use algebra tiles to further investigate factoring.

1. Use appropriate numbers of the tiles to represent: $x^2 + 3x + 4$.
2. Try to arrange the tiles into a rectangle. What goes wrong?
3. What do you think this says about factoring $x^2 + 3x + 4$?

Using the tiles above, it appears that $x^2 + 3x + 4$ can not be factored.

Let's see why our factoring methods above don't work for this polynomial.

EXAMPLE 2

Explain why $x^2 + 3x + 4$ can not be factored using only integer factors.

SOLUTION The coefficient of the middle term is 3 and the constant term is 4. Hence, we are looking for two integers whose sum is 3 and whose product is 4. Try to find this pair of numbers by listing all possible factors of 4 and then computing their sums.

Factor Pairs of 4	1,4	-1, -4	2,2	-2, -2
Sum	5	-5	4	-4

None of the factor pairs of 4 add up to 3. So we cannot factor x^2+3x+4 into linear polynomials with integer coefficients.

EXERCISES

1. Factor $x^2+8x+15$. Draw a rectangle with area $x^2+8x+15$. Explain how the rectangle is related to the factors.
2. Factor each of the expressions, if possible. If an expression cannot be factored, say so. Multiply to verify your answers.
   a. $x^2 + 4x + 3$
   b. $p^2 + 4p - 32$
   c. $x^2 + 2x + 2$
   d. $x^2 - 3x$.
   e. $x^2 + 6x - 16$.
   f. $x^2 - 2x - 120$.
   g. $x^2 - 7x - 7$
3. Label each of the following quadratic expressions as a perfect square, a difference of squares or neither. Then factor it.

a. $x^2 - 64$

b. $x^2 - 5x + 4$

c. $x^2 + 22x + 121$

d. $x^2 + 3x - 10$

e. $x^2 - 12x + 36$

f. $x^2 - 81$

4. Combining the methods of this section with factoring the common factor, one can factor more complicated expressions. For example $5x^2 + 20 + 20 = 5(x^2 + 4x + 4) = 5(x + 2)^2$. Use this idea to factor each of the following expressions.

a. $3x^2 - 24x - 60$

b. $-2x^2 - 2x + 4$

c. $4x^2 - 36$

d. $x^3 + x^2 - 6x$

5. Factor each of the following expressions:

a. $x^2 - 13x + 36$

b. $x^4 - 13x^2 + 36$

c. $x^2 - 5x + 4$

d. $x^4 - 5x^2 + 4$

e. $x^4 - 10x^2 + 9$

6. Sometimes if we factor out a fraction, we can use the methods of this section to factor the remaining expression. For example,

$$\frac{1}{2}x^2 + \frac{3}{2}x + 1 = \frac{1}{2}(x^2 + 3x + 2) = \frac{1}{2}(x + 1)(x + 2).$$

For each of the following, first factor out $\frac{1}{2}$ and then factor the remaining expression:

a. $\frac{1}{2}x^2 - 2x - 6$

b. $\frac{1}{2}x^2 - \frac{1}{2}x - 3$

c. $\frac{1}{2}x^2 - \frac{5}{2}x - 18$

7. If we write down the perfect squares, starting with 4, and then subtract 4 from each square, we get the following sequence:

$$0, 5, 12, 21, 32, 45, \ldots$$

How many numbers in this never-ending sequence are prime? How do you know?

8. **Investigation:**
In each of the following polynomials, fill in the blank so that the polynomial is a perfect square, and factor the resulting polynomial. *For example, if the given polynomial is $x^2 - \_\_\_ + 16$, then the blank should contain the term $8x$, because $x^2 - 8x + 16 = (x - 4)^2$.*
   a. $x^2 + \_\_\_ + 25$
   b. $x^2 - 14x + \_\_\_$
   c. $x^2 - \_\_\_ + 400$
   d. $x^2 + 36x + \_\_\_$
   e. $x^2 - 98x + \_\_\_$
   f. $x^2 + bx + \_\_\_$

SECTION 7.6 SOLVING $x^2 + bx + c = 0$

Now that we know how to factor quadratic polynomials, we can begin solving problems that involve quadratic equations.

EXAMPLE 1

A rectangle is 10 inches longer than it is wide. The area of the rectangle is 75 square inches. Find the dimensions of the rectangle.

SOLUTION As we discovered in Exploration 1, to solve this problem, we need to solve the equation $x(x + 10) = 75$, where x represents the width of the rectangle (in inches). We can rewrite this equation as $x^2 + 10x = 75$.

We can change a statement about a quadratic equation into statements about linear equations by using the zero product property, as we did in Exploration 2. But in order to do this, we must rearrange our equation so that it contains a zero. We can do this by subtracting 75 from each side to obtain $x^2 + 10x - 75 = 0$.

Now let's try factoring the polynomial $x^2 + 10x - 75$. We know that we are looking for numbers a and b such that $a + b = 10$ and $ab = -75$. Since $a+b$ is positive and ab is negative, one of a and b must be positive, and must be negative. We also need the positive term to "dominate" the negative term. We explore the possibilities by making a table:

Factor pairs of -75	$75, -1$	$25, -3$	$15, -5$
Sum	74	22	10

We find that $a = 15$ and $b = -5$ works, so we have $x^2 + 10x - 75 = (x + 15)(x - 5) = 0$.

Since $(x + 15)(x - 5) = 0$, by the zero product property we know that one of the factors $x + 15$ and $x - 5$ must be zero. If $x + 15 = 0$, then we have $x = -15$; if $x - 5 = 0$, then we have $x = 5$. Since x is the width of a rectangle, the solution $x = -15$ doesn't make any sense. Therefore, we must have $x = 5$. So the width of the rectangle is 5 inches, and the length of the rectangle is $5 + 10 = 15$ inches.

Let's try using this technique to solve another quadratic equation.

EXAMPLE 2

Solve the equation $x^2 = 16x - 28$.

SOLUTION Use the zero product property to solve this equation. Begin by rearranging the terms of this equation so that one side of the equation is equal to zero. We do this by subtracting $16x$ from each side, and then adding 28 to each side:

$$x^2 - 16x + 28 = 0.$$

Now we'll factor the left side of the equation. We are looking for numbers a and b such that $a + b = -16$ and $ab = 28$. We find that $a = -14$ and $b = -2$ works, so we have $x^2 - 16x + 28 = (x - 14)(x - 2) = 0$. Since the product of $(x - 14)$ and $(x - 2)$ is zero, we know that one of these factors must be zero. So we have:

$$\begin{array}{ccc} x - 14 = 0 & \text{or} & x - 2 = 0 \\ x = 14 & & x = 2. \end{array}$$

So this equation has two solutions: $x = 14$ and $x = 2$. We say that the solution set of this equation is $\{2, 14\}$.

PROBLEM 1

Solve each of the following equations. Substitute to check your answer.

1. $x^2 - 2x - 120 = 0$

2. $x^2 + 6x = 16$
3. $2x^2 + 6x - 20 = 0$

In Chapter 8 we will carefully examine the graphs of quadratic functions. But first, let's get a preview of coming attractions.

EXPLORATION 1

Let $f(x) = x^2 + 5x - 6$ for any number x.

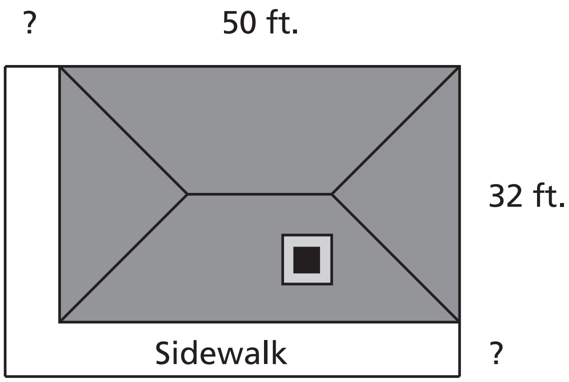
1. Use a graphing calculator to graph $f(x)$.
2. Where does the graph cross the x-axis? Where is $f(x) < 0$? Where is $f(x) > 0$?
3. Solve $x^2 + 5x - 6 = 0$. What do you notice? What do the solutions of $f(x) = 0$ tell you about the graph of $f(x)$?

PROBLEM 2

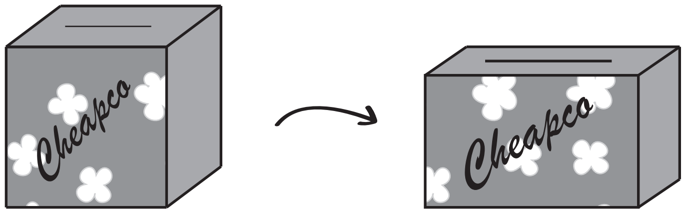
Without graphing, predict where $f(x) = x^2 - 15x + 56$ will cross the x-axis. Then use a calculator to graph the $f(x)$ to verify your answer.

EXERCISES

1. Solve each of the following equations. Check your solutions.
   a. $x^2 - 3x + 2 = 0$
   b. $x^2 + 14x + 48 = 0$
   c. $x^2 - 5x - 36 = 0$
   d. $x^2 = 100$
   e. $x^2 = 3x + 54$
   f. $x^2 + 6x = -9$
   g. $x^2 + 12x + 35 = 0$
   h. $x^2 = 11x - 30$
   i. $x^2 - 14x + 49 = 0$
   j. $x^2 - 225 = 0$
2. Margaret wants to build a sidewalk of constant width along two sides of her house. Her house is rectangular with a width of 32 feet and a length of 50 feet. (See the figure below.) If she has enough concrete to pour 255 square feet of sidewalk, how wide can she make the sidewalk?



3. Until recently, Cheapco Facial Tissue has been sold in boxes that are perfect cubes. After doing some market research, Cheapco has found that if they reduce the height and width of the box by 1 inch each, and increase the length by 3 inches, then the volume of the box will decrease by 1 cubic inch, but consumers will still perceive the new box as larger because the surface area is greater. What are the dimensions of the new box?



4. a. Solve $x^2 - 3x - 18 = 0$.
   b. Without graphing predict where $f(x) = x^2 - 3x - 18$ will cross the x-axis.
   c. Graph $f(x)$ to verify your answer.

5. a. Solve $x^2 + 5x = 14$.
   b. What do your answers say about the function $f(x) = x^2 + 5x$?

6. We can use our technique of factoring and using the zero product property to solve equations involving polynomials of degree greater than 2. Solve the following equations:
   a. $x^3 - 6x^2 + 9x = 0$
   b. $x^3 + 4x^2 - 4x - 16 = 0$
   c. $x^3 + x^2 + x + 1 = 0$

7. Two legs of a right triangle have length $2x - 1$ and $2x + 2$, where x is a real number, and the hypotenuse has length $3x$. What is the area of the triangle?

SECTION 7.7 SOLVING $ax^2 + bx + c = 0$

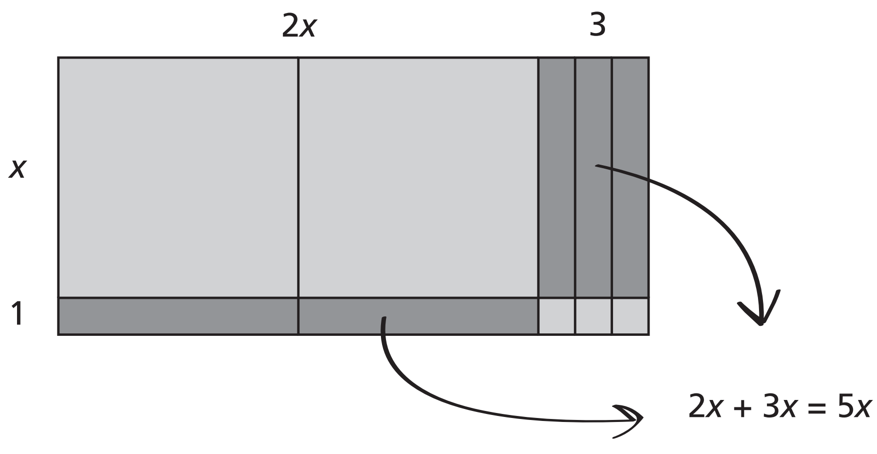
In this section we will discuss how to factor second degree polynomials: $ax^2 + bx + c$ where a, b and c are numbers. Once we know how to factor, we can use the zero product property to solve equations of the form $ax^2 + bx + c = 0$.

EXPLORATION 1

In this exploration we will use algebra tiles to investigate factoring.

1. Use appropriate numbers of the tiles to represent: $2x^2 + 5x + 3$.
2. Arrange the tiles into a rectangle.
3. What are the dimensions of the rectangle? Explain where each of the terms $2x^2$, $5x$ and 3 are in your model.
4. Write $2x^2 + 5x + 3$ as a product of two factors. Use the distributive property to check your answer.

To make the rectangle in Exploration 1, notice that the middle term $5x$ was split into two pieces: one that is $2x$ by 1 and one that is 3 by x. To understand why this happens algebraically, we can think about multiplying $(2x + 3)(x + 1)$.



$$\begin{aligned}(2x+3)(x+1) &= 2x(x+1) + 3(x+1) \\ &= 2x^2 + 2x + 3x + 3 \\ &= 2x^2 + 5x + 3\end{aligned}$$

We first use the distributive property twice and then combine the like terms $2x$ and $3x$. To find factors we have to think of the process in reverse. As in Section 7.5, we need to think of how to split the middle term. However, in this case we have to think about the role played by the coefficient of x^2. The steps for factoring a polynomial of the form $Ax^2 + Bx + C$ are then:

Step 1: First, check whether the terms of the polynomial have any common integer factors greater than 1. If so, factor these out first. We now have a polynomial of the form $k(ax^2 + bx + c)$, where a, b, and c have no common factors greater than 1.

Step 2: Find the product of the first and last coefficients: ac.

Step 3: Find the factor pairs of ac that add up to b. In other words find p and q so that $pq = ac$ and $p + q = b$.

Step 4: Write $ax^2 + bx + c = ax^2 + px + qx + c$.

Step 5: Factor this last expression by grouping.

EXAMPLE 1

Factor $4x^2 - 2x - 12$. Multiply to verify your answer.

SOLUTION

Step 1: We begin by noticing that the coefficients 4, -2, and -12 have a common factor of 2. So we factor this out to obtain $4x^2 - 2x - 12 = 2(2x^2 - x - 6)$. Comparing $2x^2 - x - 6$ to $ax^2 + bx + c$, we see that $a = 2$, $b = -1$ and $c = -6$.

Step 2: $ac = 2 \cdot (-6) = -12$.

Step 3: List all possible factor pairs (including negative numbers) of -12 and compute their sums.

Factor pairs of -12	1,-12	-1, 12	2, -6	-2, 6	3, -4	-3, 4
Sum	-11	11	-4	4	-1	1

The factor pair that adds up to $b = -1$ is 3 and -4.

Step 4: $2(2x^2 - x - 6) = 2(2x^2 + 3x - 4x - 6)$.

Step 5: By grouping we get:

$$\begin{aligned} 2(2x^2 + 3x - 4x - 6) &= 2(x(2x + 3) - 2(2x + 3)) \\ &= 2(x - 2)(2x + 3). \end{aligned}$$

We can use the distributive property to verify our answer:

$$\begin{aligned} 2(x - 2)(2x + 3) &= 2(x(2x + 3) - 2(2x + 3)) \\ &= 2(2x^2 + 3x - 4x - 6) \\ &= 4x^2 - 2x - 12. \end{aligned}$$

Notice that this is nothing more than Step 5 in reverse.

One interesting question to ask is: What would have happened if we had written the terms $3x$ and $-4x$ in a different order in Step 4? Would we still have been able to factor this polynomial? Let's see what happens if we do this:

$$\begin{aligned} 2(2x^2 - x - 6) &= 2(2x^2 - 4x + 3x - 6) \\ &= 2(2x(x - 2) + 3(x - 2)) \\ &= 2(2x + 3)(x - 2). \end{aligned}$$

So we end up with the same factorization. The only difference is that the factors appeared in a different order.

In the previous section, we saw some useful relationships between the signs of the coefficients of a polynomial and the signs of its factors. We can use these relationships to our advantage here as well.

EXAMPLE 2

Factor the polynomial $3x^2 + 10x - 8$.

SOLUTION

Step 1: Again, we begin by looking for common factors of the coefficients 3, 10, and -8. There are none.

Step 2: Here $a = 3$, $b = 10$, and $c = -8$. So $ac = -24$.

Step 3: We are looking for numbers p and q such that $p + q = 10$ and $pq = -24$. As we discovered in the previous section, in order for this to happen, we need p and q to have opposite signs, and we need the one that is positive to "dominate" the one that is negative. So we can use the following condensed table to search for suitable values of p and q:

Factor pairs of -24	$24, -1$	$12, -2$	$8, -3$	$6, -4$
Sum	23	10	5	2

So the pair $p = 12$, $q = -2$ works!

Step 4: This time we will use the *box method* from Example 4 on page 400 to determine the factors, but in reverse. As before:

$$3x^2 + 10x - 8 = 3x^2 + 12x - 2x - 8.$$

Place these four terms in the box below.

$3x^2$	$12x$
$-2x$	-8

Step 5: To factor the polynomial, we now look at the greatest common factor of the first column and write it on the top of the box. We then take the greatest common factor of the first row and write to the left of the box.

	x	
$3x$	$3x^2$	$12x$
	$-2x$	-8

Finally, to fill in the top of the second column we divide $12x$ by $3x$ and get 4. To fill in the second row we divide $-2x$ by x and get -2. Do you see why? This method mimics the area model.

	x	4
$3x$	$3x^2$	$12x$
-2	$-2x$	-8

So we have $3x^2 + 10x - 8 = (3x - 2)(x + 4)$.

PROBLEM 1

Factor each of the following polynomials. Multiply to verify your answer.

1. $2x^2 + 3x + 1$
2. $3x^2 + 10x + 8$
3. $3x^2 - 14x + 8$

Now that we know how to factor polynomials of the form $ax^2 + bx + c$, we can use this knowledge to solve quadratic equations of the form $ax^2 + bx + c = 0$. Consider the following example.

EXAMPLE 3

In triangle ΔABC, $AB = 3x + 7$, and the height of ΔABC, measured from C to the line AB is x. The area of ΔABC is 24. What is the value of x?

SOLUTION We know that the area of a triangle is given by the formula $\frac{1}{2}bh$, where b is the length of a base of the triangle, and h is the corresponding height. Since we are given that the area is 24, we have:

$$\frac{1}{2}(3x + 7)(x) = 24.$$

We can rewrite this equation without fractions by multiplying both sides by 2 to get:

$$(3x + 7)(x) = 48.$$

Multiplying the left side, we get

$$3x^2 + 7x = 48.$$

Now let's rearrange the terms of this equation so that one side is equal to zero:

$$3x^2 + 7x - 48 = 0.$$

Since $ac = 3 \cdot -48 = -144$, we need to find factors p and q of -144 such that $p + q = 7$. It turns out that $p = 16$ and $q = -9$ works, so we rewrite the term $7x$ and factor by grouping:

$$\begin{aligned} 3x^2 + 16x - 9x - 48 &= 0 \\ x(3x + 16) - 3(3x + 16) &= 0 \\ (x - 3)(3x + 16) &= 0 \end{aligned}$$

So we must have $x-3 = 0$ or $3x+16 = 0$. If $3x+16 = 0$, then $x = -\frac{16}{3}$. This doesn't make sense in this context, since this would mean that the height of our triangle is negative. So we must have $x - 3 = 0$, and therefore, $x = 3$.

As was the case for polynomials of the form x^2+bx+c, some polynomials of the form $ax^2 + bx + c$ cannot be factored.

EXAMPLE 4

Explain why $2x^2 + 7x + 2$ cannot be factored using only integer coefficients.

SOLUTION Comparing $2x^2 + 7x + 2$ to $ax^2 + bx + c$, we see that $a = 2$, $b = 7$ and $c = 2$. $ac = 2 \cdot 2 = 4$. We are looking for integers p and q such that $p + q = 7$ and $pq = 4$. We know that in order for this to happen, p and q must both be positive. Now we list all positive factor pairs of 4 and compute their sums.

Factor Pairs of 4	1,4	2,2
Sum	5	4

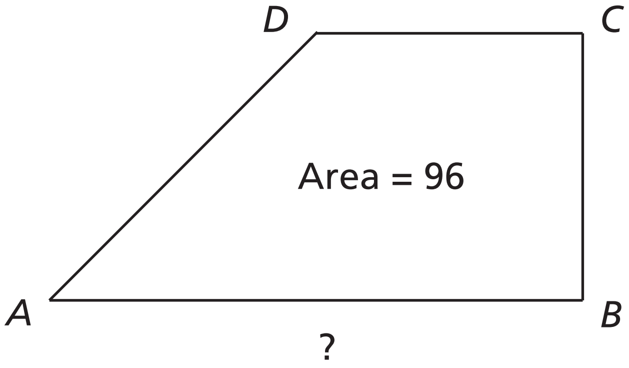
None of the factor pairs of 4 that add up to 7. So we can not factor $2x^2 + 7x + 2$ using only integer coefficients.

EXERCISES

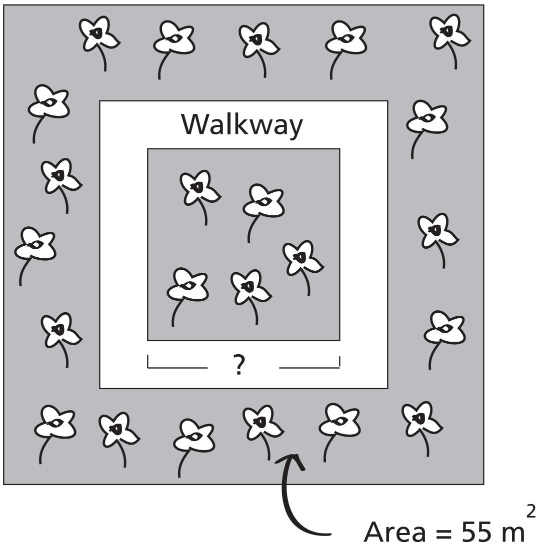
1. Factor each of the following polynomials as completely as possible. If a polynomial cannot be factored, say so.
   a. $2x^2 - 3x - 9$
   b. $4x^2 + 10x + 6$
   c. $3x^2 - 8x - 3$
   d. $6x^2 - 7x + 1$
   e. $5x^2 + 5x + 15$
   f. $10x^2 + 21x - 10$
   g. $4x^2 - 13x + 8$
   h. $3x^2 + 12x + 12$
   i. $7x^2 + 12x - 4$

j. $12x^2 + 22x + 8$

2. Solve the following quadratic equations by factoring:
   a. $2x^2 - 8x - 10 = 0$
   b. $3x^2 - 14x = 5$
   c. $2x^2 + x - 3 = 0$
   d. $5x^2 = 17x + 12$
   e. $6x^2 - 5x + 1 = 0$
   f. $2x^2 + x = 21$
   g. $4x^2 + 4x + 1 = 0$
   h. $100x^2 - 9 = 0$
   i. $15x + 27 = 2x^2$
   j. $22x^2 - 117x - 22 = 0$
3. a. Solve $2x^2 + x - 6 = 0$.
   b. Without graphing, predict where the graph of $f(x) = 2x^2 + x - 6$ crosses the x-axis.
   c. Graph $f(x)$ to verify your answer.
4. a. Solve $3x^2 + 13x = 10$.
   b. What does your answer say about the function $f(x) = 3x^2 + 13x$?
5. Terrence is standing at the top of a cliff that is 44 meters above the ground. He throws a ball directly upward at a speed of 12 meters per second so that when the ball falls, it will fall down the cliff and hit the ground. The height (in meters, measured from the ground) of the ball t seconds after Terrence throws it is given by the equation $h(t) = 44 + 12t - 5t^2$. How long does it take for the ball to hit the ground?
6. In trapezoid $ABCD$, side $\overline{AB}$ is perpendicular to side $\overline{BC}$, and side $\overline{BC}$ is perpendicular to side $\overline{CD}$. Side $\overline{AB}$ is twice as long as side $\overline{CD}$, and $\overline{BC}$ has the same length as $\overline{CD}$. The area of $ABCD$ is 96. What is the length of the side $\overline{AB}$?



7. Janalyn has a square flowerbed in her backyard. She wants to put a one-meter-wide walkway around this flowerbed, and then put in a new flowerbed surrounding the walkway. The width of the new flowerbed is equal to one half of the side length of the original flowerbed. Together, the original flowerbed, the walkway, and the new flowerbed form a square. If the area of the new flowerbed is 55 square meters, what is the side length of the original flowerbed?



8. Find all integers b such that the trinomial $4x^2 + bx + 9$ can be factored into two binomials with integer coefficients.
(Be sure to look for both positive and negative values of b.)

9. **Ingenuity:**
Airport A is 500 miles due north of airport B. At noon, a plane leaves airport A and flies due north at 350 miles per hour. At the same time, another plane leaves airport B and flies due west at 250 miles per hour. At what time will the two planes be exactly 1300 miles apart?

SECTION 7.8 CHAPTER REVIEW

Key Terms

algebraic expression
area model
associative property
binomial
box method
common factor
commutative property
degree of polynomial
distributive property
equivalent expression
factor
factoring by grouping
greatest common factor
irreductible
monomial
polynomial expression
quadratic equations
term
trinomial
variable
volume
zero product property

Most Common Forms

Perfect Square I:
$(x+y)^2 = x^2 + 2xy + y^2$

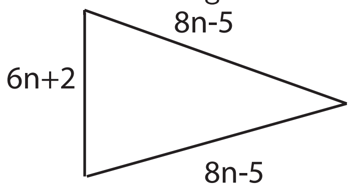
Perfect Square II:
$(x-y)^2 = x^2 - 2xy + y^2$

Difference of Squares:
$x^2 - y^2 = (x-y)(x+y)$

Practice Problems

1. Classify each of the following polynomials by its degree and number of terms.
   a. $5x^2 - 7$
   b. $2x^2 + 3xy + 6y^2$
   c. $-7x^3 - 4x^2 + 5x + 9$
   d. $18y^3 + 7z^3 - 15$
   e. $4y^3 + 9x$
2. For each of the following dimensions of a rectangle, draw a picture, label it and write a simplified expression that represents the area.
   a. $x + 5$ by $x + 3$
   b. $2x + 1$ by $x + 7$
   c. $n + 7$ by $3n + 1$

3. Simplify the following polynomial expressions.
   a. $(3x^2 + 2x + 5) + (2x^2 + 4x + 7)$
   b. $(-2n^2 + 8n + 6) + (3n^2 - 6n + 3)$
   c. $(5x^2 - 3x - 8) - (2x^2 - x + 3)$
   d. $(7n^2 + 5n - 3) - (6n^2 - n - 4)$
4. Mia is making a flag in the shape of an isosceles triangle. She wants to line the flag with a border. See figure below.



   a. Can n be equal to $\frac{1}{2}$? What are the possible values for n? Explain.
   b. Write a simplified expression that represents the length of the border.
   c. Mia has 102 cm of material to use to make the border. What should the value of n be if she wants to use all of it?
5. Compute the area and the perimeter of a $(3x+2)$ by $(x+1)$ rectangle. Use the area model to show your work.
6. Compute the following products and simplify your answers.
   a. $4y(3y + 5)$
   b. $(x - 3)(x + 2)$
   c. $(2n + 3)(2n - 3)$
   d. $(k - 5)^2$
   e. $(x + 2)(x^2 + 4x + 7)$
   f. $(3n - 1)(n^2 - 2n + 2)$
   g. $(y - 1)^3$
7. Factor out the greatest common factor from the following polynomials.
   a. $4n^2 + 2n - 6$
   b. $m^3n + 3m^2n^2 - 7m^2$
   c. $5y^2 + 7y - 8$

d. $14x^2z^3 - 8x^4z^2$

8. Factor the polynomials using grouping. Multiply to verify your answer.
   a. $n^2 + 2n + 5n + 10$
   b. $x^2 - 6x + 2x - 12$
   c. $mn + 2m - 9n - 18$
   d. $n^4 + 2n^2 - 3n^2 - 6$
9. Factor each of the expressions. Multiply to verify your answer.
   a. $n^2 + 7n + 12$
   b. $p^2 - 3p - 18$
   c. $r^2 - 9$
   d. $t^2 - 16t + 64$
10. Solve. Check your solutions.
   a. $y^2 - 6y + 5 = 0$
   b. $z^2 + 9z + 20 = 0$
   c. $a^2 + 2a = 15$
   d. $d^2 = 14d - 49$
   e. $f^2 + 11f - 21 = 0$
   f. $g^2 - 196 = 0$
11. Sketch and label a right triangle with legs of length $3n + 7$ and $2n - 4$.
   a. Can $n = 1$? Explain. What are the possible values of n?
   b. Write an expression for the area in simplified form.
   c. If the area of the triangle is 16 square units, what is the value of n? Substitute and multiply to verify your answer.
   d. Write an expression for the perimeter of the triangle.
12. Factor each of the following expressions. Multiply to verify your answer.
   a. $3x^2 - 7x - 6$
   b. $2x^2 + 8x - 42$
   c. $3x^2 + 21x + 18$
   d. $n^2 - 6n + 9$
13. Solve. Check your solutions.
   a. $3x^2 - 12x - 15 = 0$
   b. $z^2 + 26z + 48 = 0$

c. $w^2 + 12x + 20 = 0$
d. $f^2 - 11f = -28$
e. $2n^2 + 9n = 5$

QUADRATIC FUNCTIONS

8

SECTION 8.1 QUADRATIC FUNCTIONS

In Chapter 7, we explored properties of polynomial expressions. We will now explore second degree polynomial functions, which are usually referred to as *quadratic functions*. You noticed in Section 7.1 that all these second degree functions had similar graphs. The shape of these graphs is called a *parabola*.

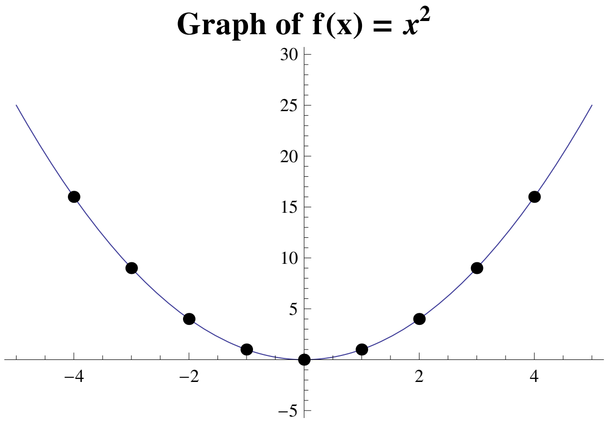
EXAMPLE 1

Make a table of at least 10 points on the graph of $f(x) = x^2$. Plot these points with graph paper, and connect them.

SOLUTION Let's make a table of points for the graph of $f(x) = x^2$. Substitute different values for x to find the corresponding values of $f(x)$.

x	$f(x) = x^2$
-4	
-3	
-2	
-1	
0	0
1	1
2	4
3	
4	

Next graph the points (x, y) where $y = f(x)$, and connect the points.



In the graph, did you notice one special point which is the lowest? This point is called the vertex. For the parabola $f(x)$ the vertex is like the bottom of the valley. For other parabolas the vertex is the highest point (or top of the mountain). In the next exploration, we will graph some more parabolas with vertices at the bottom.

EXPLORATION 1

1. Using a graphing calculator graph these quadratic functions on the same coordinate plane. *Set the window so that x goes from -4 to 4 and y from -5 to 20.*

a. $f(x) = x^2$
b. $p(x) = x^2 + 1$
c. $p(x) = x^2 + 2$
d. $p(x) = x^2 - 1$
e. $p(x) = x^2 - 2$

2. What do you notice about the shapes of the graphs?
3. These graphs are called parabolas. We say that all the parabolas graphed here "face upward". Why does that make sense?
4. When a parabola faces upward, the vertex is the lowest point, the point on the graph where $f(x)$ has the smallest value. Find the coordinates of the vertex of each of the graphs. What pattern do you notice?

The graphs you made in the last exploration look similar. For this reason, we can think of the quadratic functions as a family: while they look alike, they are not all identical. This family also has a parent. The parent of a family of functions is the simplest example in that family, in this case, $f(x) = x^2$. Notice that the vertex of $f(x)$ is $(0, 0)$. A nice way to think about any quadratic function is how it is related to the parent function.

Now consider what happens if we change the graph slightly.

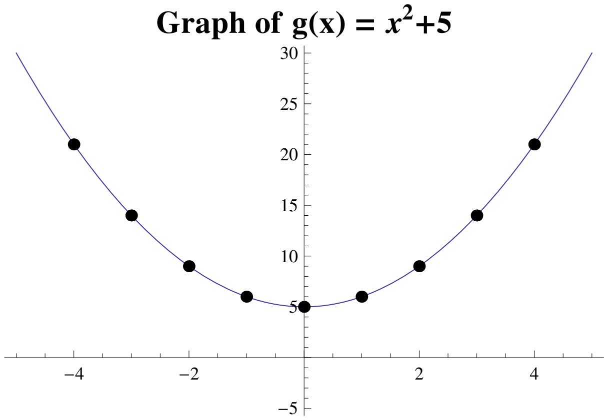
EXAMPLE 2

Make a table of at least 10 points of the graph of $g(x) = x^2 + 5$. Plot these points on the same graph as Example 1 and connect them. Locate and label the vertex. Compare the table and the graphs of $g(x)$ to that of $f(x)$ from Example 1.

SOLUTION Again, make a table.

x	$g(x) = x^2 + 5$
-4	
-3	
-2	
-1	
0	5
1	6
2	9
3	
4	

Next, graph these points, and connect the points to create the new graph. Observe that $g(x) = f(x) + 5$. Each value in the second column is 5 greater than it was for $f(x)$, and the graph $f(x)$ has been shifted up by 5 units to form the graph of $g(x)$. The vertex for $g(x)$ is $(0, 5)$.



Now let's shift the graph down.

PROBLEM 1

Write a function whose graph has the same shape as $f(x) = x^2$ but is shifted down by 5 units.

Another way of thinking of this is that when we replace our original function f with this new function, the graph now is shifted down by 5 units. The new function is $q(x) = x^2 - 5$.

In the same way, we can change the x-coordinate of the vertex:

EXPLORATION 2

1. Using a graphing calculator graph these quadratic functions on the same coordinate plane.
   a. $f(x) = x^2$
   b. $p(x) = (x + 3)^2$
   c. $p(x) = (x + 4)^2$
   d. $p(x) = (x - 7)^2$
   e. $p(x) = (x - 5)^2$
2. What do you notice about the shapes of the graphs?
3. Find the coordinates of the vertex of each of the graphs. What pattern do you notice?

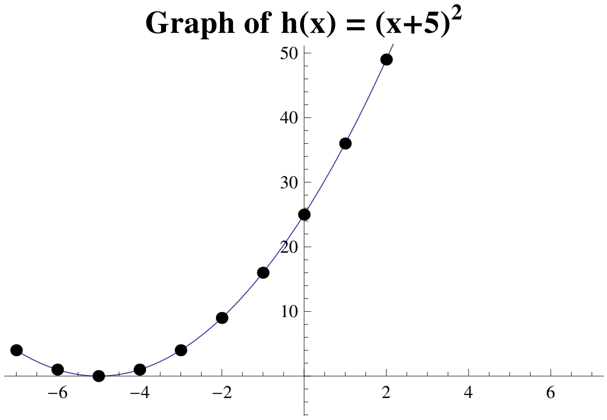
Starting with our parent function $f(x) = x^2$, a good way to think about Example 2 is that the second coordinates are determined by $y = f(x)+5$, so new function is $g(x) = f(x) + 5$. In Problem 1 the y coordinates are determined by $y = f(x) - 5$, so the new function is $q(x) = f(x) - 5$. In the same way, we can think of what is happening in Exploration 2 as a change in the first coordinate. In the example below we replace $f(x)$ by $f(x + 5)$. What will happen to the graph this time?

EXAMPLE 3

Make a table of at least 10 points of the graph of $h(x) = f(x + 5) = (x + 5)^2$. Plot these points on graph paper and connect them. Locate and label the vertex. Compare the table and the graphs of $h(x)$ to that of $f(x)$ from Example 1.

SOLUTION

x	$h(x) = (x+5)^2$
-9	
-8	
-7	
-6	
-5	0
-4	1
-3	4
-2	
-1	



Notice that we decided to start our table at $x = -9$ and stopped at $x = -1$. Can you explain why this makes sense? Looking at the graph and the table, we see that the vertex of $h(x)$ is the point $(-5, 0)$.

It is important to think about how the graph of $h(x) = (x+5)^2$ compares to the graph of our original function $f(x) = x^2$. We can think of $h(x)$ as being a translation of $f(x)$ where we replace x from $f(x)$ by $(x+5)$. But now when $(x+5) = 0, x = -5$. So replacing x by $x+5$ will shift the entire graph 5 units to the left! This may seem the opposite of what you would expect. Do you see why the graph shifts in this direction? Another way to think about it is that for the function $h(x) = (x+5)^2$, when $x = -5$, $h(-5) = 0$, so the vertex for the new graph will occur at $x = -5$.

Now let's practice shifting the graph of $f(x) = x^2$ left and right as well as up and down.

EXPLORATION 3

1. Using a graphing calculator, graph these quadratic functions on the same coordinate plane.
   a. $f(x) = x^2$
   b. $p(x) = f(x) - 3 = x^2 - 3$
   c. $p(x) = f(x - 1) = (x - 1)^2$
   d. $p(x) = f(x - 1) - 3 = (x - 1)^2 - 3$
   e. $p(x) = (x + 1)^2 + 3$
2. What do you notice about the shapes of the graphs?
3. Find the coordinates of the vertex of each of the graphs. What patterns do you notice?
4. Each of the graphs above has a line of symmetry. What kind of line is it: vertical, horizontal, neither? This line is called the *axis of symmetry*. Find the equation of the axis of symmetry for each of the graphs.

PROBLEM 2

Make a table of 6 points on the graph of $k(x) = (x+2)^2 + 5$. Plot these points on graph paper and connect them. Locate and label the vertex.

In this section, all the quadratic functions can be thought of as translations of the parent function. In other words, we slide the vertex to a new point in the plane and copy the graph of $f(x) = x^2$ starting from this new vertex. In many cases we can determine the location of the vertex easily from the function.

EXAMPLE 4

For the following quadratic functions, determine the coordinates of the vertex and the equation for the axis of symmetry.

1. $p(x) = (x-5)^2 + 3$
2. $p(x) = (x+2)^2 - 5$
3. $p(x) = (x-4)^2 - 2$
4. $p(x) = (x+7)^2 + \frac{1}{2}$
5. $p(x) = (x-a)^2 + b$

SOLUTION As before, let $f(x) = x^2$. Recall that its vertex is $(0, 0)$. To find the new vertex we need to think about how the coordinates of the graph of $f(x)$ are changed to form the new graph.

1. Notice that $(x-5)^2 + 3 = f(x-5) + 3$. So the first coordinate is shifted 5 units to the right and the second coordinate is shifted 3 units upward. So the new vertex is $(5, 3)$. The axis of symmetry is a vertical line through the vertex, so its equation is $x = 5$.
2. V:$(-2, -5)$, axis of symmetry: $x = -2$
3. V:$(4, -2)$, axis of symmetry: $x = 4$
4. V: $(-7, \frac{1}{2})$, axis of symmetry: $x = -7$
5. V: (a, b), axis of symmetry: $x = a$

EXERCISES

1. For each quadratic function below, make a table of points on the parabola and graph them. Locate and label the vertex.
   a. $p(x) = (x-3)^2 + 5$
   b. $p(x) = x^2 - 11$
   c. $p(x) = (x+3)^2 + 3$
   d. $p(x) = 2x^2$

e. $p(x) = 2(x+1)^2$

2. Graph the quadratic function $f(x) = x^2$ and the linear function $h(x) = 2x + 3$. Find all the points of intersection.

3. Write the formula for the quadratic function created by each of the following transformations of the parent function $f(x) = x^2$.
   a. Shift $f(x)$ to the right by 3 units.
   b. Shift $f(x)$ to the left by 6 units.
   c. Shift $f(x)$ up by 2 units.
   d. Shift $f(x)$ down by 5 units.
   e. Shift $f(x)$ to the right by 1 unit and up by 4 units.

4. Without graphing, determine the coordinates of the vertex and the equation of the axis of symmetry for each of the following quadratic functions.
   a. $p(x) = (x-2)^2 + 4$
   b. $p(x) = (x+3)^2 + 3$
   c. $p(x) = (x-10)^2 - 7$
   d. $p(x) = \left(x - \frac{1}{2}\right)^2 - 11$

5. Use vertical and horizontal shifts to graph each of the following functions. *Do not use a graphing calculator or a table of values.*
   a. $q(x) = x^2 + 1$
   b. $q(x) = (x-3)^2$
   c. $q(x) = (x+2)^2$
   d. $q(x) = x^2 - \frac{1}{2}$
   e. $q(x) = (x - \frac{1}{2})^2 - 2$
   f. $q(x) = (x+4)^2 + 4$

6. When we transform a function, we don't always start with the parent function $f(x) = x^2$. Let $q(x) = (x-1)^2 + 2$. Write the formula for the quadratic function created by each of the following transformations.
   a. Shift $q(x)$ to the right by 2 units.
   b. Shift $q(x)$ to the left by 5 units.
   c. Shift $q(x)$ up by 7 units.
   d. Shift $q(x)$ down by 5 units.
   e. Shift $q(x)$ to the right by 1 unit and up by 4 units.

7. Let $q(x) = x^2 - 9$.
   a. Graph $q(x)$. Where is the vertex of the parabola? Use the graph to determine the coordinates of the points where the graph of $q(x)$ crosses the x-axis.
   b. Imagine we drew the graph of $p(x) = (x - 100)^2 - 9$. Predict where the vertex of the graph will be. Predict where the graph will cross the x-axis.

8. **Investigation:**
   a. Graph $q(x) = (x + 3)^2 - 1$. Locate and label the vertex.
   b. How many times does the graph of $q(x)$ cross the x-axis? Use the graph or table to determine the coordinates of the points where the graph of $q(x)$ crosses the x-axis.
   c. Show that $q(x) = x^2 + 6x + 8$.
   d. Factor $x^2 + 6x + 8$.
   e. Graph the two linear functions $g(x) = x + 2$ and $h(x) = x + 4$. What do you notice about where the $q(x)$ intersects with $h(x)$? With $g(x)$? Why does this make sense?

9. **Ingenuity:**
   Suppose that the quadratic function $q(x) = x^2 + bx + 15$ (where b is a number) has its vertex on the line $x - y = 5$. What are the possible values of b?

SECTION 8.2 MORE QUADRATIC FUNCTIONS

In 8.1 we graphed quadratic functions that resulted from horizontal and vertical shifts of the function $f(x) = x^2$. In this section, we will see what happens when we replace x^2 by some multiple of x^2, for example $3x^2$ or $-x^2$. These transformations are called **scale** changes.

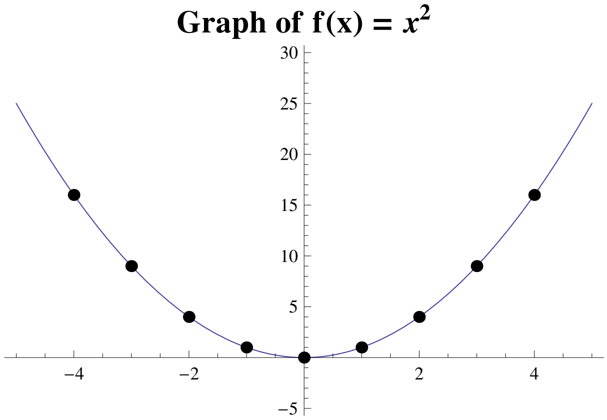
EXPLORATION 1

1. Using a graphing calculator, graph these quadratic functions on the same axes:
   a. $f(x) = x^2$
   b. $p(x) = -x^2$
   c. $p(x) = -(x+1)^2 + 3$
   d. $p(x) = -(x-1)^2 + 3$
   e. $p(x) = -(x-1)^2 - 3$
2. What do you notice about the shapes of the graphs?
3. We say that all the parabolas graphed here except for $f(x)$ "face downward". Why does that make sense?
4. When a parabola faces downward, the vertex is the highest point. That is the point on the graph where $f(x)$ has the greatest value. Find the coordinates of the vertex of each of the graphs. What pattern do you notice?

To better understand why the parabolas face downward, let's carefully plot $g(x) = -x^2$ by hand.

In Section 8.1 we made a table and graph of $f(x) = x^2$. For the sake of comparison, let's copy that below.

x	$f(x) = x^2$
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16



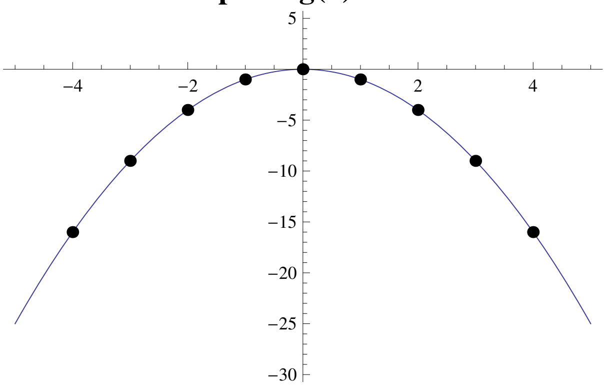
EXAMPLE 1

Make a table of points on the graph of $g(x) = -x^2$. Plot these points and connect them. Locate and label the vertex. Compare the table and the graph of $g(x)$ to that of $f(x)$.

SOLUTION Make a table of points.

x	$g(x) = -x^2$
-4	
-3	
-2	
-1	
0	0
1	-1
2	-4
3	
4	

Graph of g(x) = $-x^2$



For parabolas that face downward, the vertex is the highest point. So the vertex of $g(x)$ is $(0, 0)$. Comparing the tables, we see that second column for $g(x)$ is just -1 times the corresponding value in the table for $f(x) = x^2$. And the graph $g(x)$ is like the graph of $f(x)$ turned upside down.

PROBLEM 1

Without graphing, predict which of the parabolas determined by the following functions will face upward and which will face downward.

1. $p(x) = (x+2)^2$
2. $p(x) = -(x-3)^2 + 1$
3. $p(x) = -(x+2)^2 - 5$
4. $p(x) = (x-3)^2 - 2$

Now let's see what happens if we multiply by a number other than -1.

EXPLORATION 2

1. Using a graphing calculator graph these quadratic functions on the same coordinate plane.
   a. $f(x) = x^2$
   b. $p(x) = 2x^2$
   c. $p(x) = 3x^2$
   d. $p(x) = \frac{1}{2}x^2$
   e. $p(x) = \frac{1}{3}x^2$
2. What relationship do you notice between the coefficient of x^2 and the shape of the graph?
3. Comparing the graphs to $f(x) = x^2$, which graphs would you call "narrower" than $f(x)$? Which graphs are "wider"?

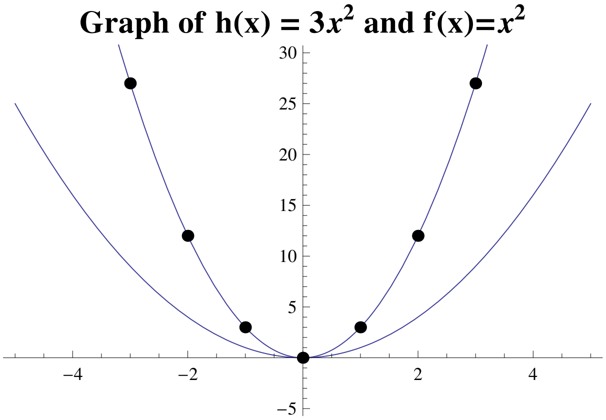
To better understand how changing the scale works, let's graph $h(x) = 3x^2$ by hand.

EXAMPLE 2

Make a table of points of the graph of $h(x) = 3x^2$. Plot these points and connect them. Locate and label the vertex. Compare the table and the graphs of $h(x)$ to that of $f(x) = x^2$ shown above.

SOLUTION Make a table of points.

x	$h(x) = 3x^2$
-4	
-3	
-2	
-1	
0	0
1	3
2	12
3	
4	



Multiplying by 3 did not change the vertex. So the vertex is $(0, 0)$. Comparing the tables, we see that second column for $h(x)$ is just 3 times the corresponding value in the table for $f(x) = x^2$. This makes the value of $h(x)$ increase faster as x moves away from 0 than the value of $f(x)$ did. On the graph above we have graphed both $h(x)$ and $f(x)$. Can you tell which one is which? The graph of $h(x)$ is narrower than that of $f(x)$.

Let's combine the horizontal and vertical shifts with the change in scale.

EXPLORATION 3

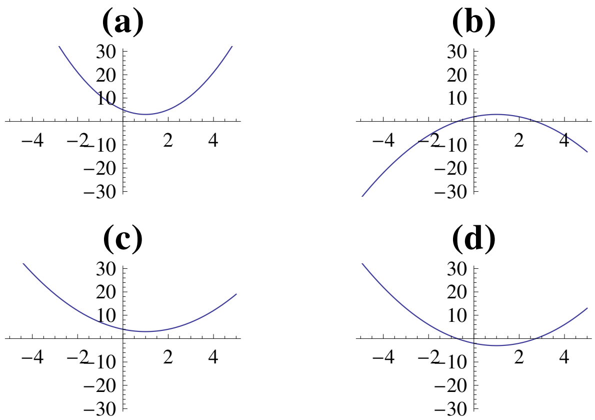
Let's explore how to find the graph of $p(x) = 2(x+2)^2 - 3$ by transforming $f(x)$ step by step. For each step use the graphing calculator to find the graph.

1. Graph $f(x) = x^2$. What is its vertex?
2. Graph $p(x) = (x+2)^2$. What is its vertex? How does the graph relate to $f(x)$?
3. Graph $q(x) = 2(x+2)^2$. What is its vertex? How does the graph relate to $p(x)$?
4. Graph $r(x) = 2(x+2)^2 - 3$. What is its vertex? How does the graph relate to $q(x)$?

PROBLEM 2

The graphs of the following quadratic functions are shown on the next page. Match the function to its graph.

1. $p(x) = (x-1)^2 - 3$
2. $q(x) = (x-1)^2 + 3$
3. $r(x) = 2(x-1)^2 + 3$
4. $s(x) = -(x-1)^2 + 3$



So far in this chapter, we have worked with a particular form of quadratic functions that makes it easy to determine the vertex.

VERTEX FORM OF A PARABOLA
We say that $f(x) = a(x-h)^2 + k$ is the *vertex form* of the quadratic function f. The *vertex* of the parabola is (h, k). The coefficient a is called the *scale factor*. The *axis of symmetry* is a vertical line through the vertex given by the equation $x = h$. For example, for $f(x) = 3(x-1)^2 + 5$, the vertex is $(1, 5)$ and the axis of symmetry is $x = 1$

If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. If $|a| > 1$, the parabola will be narrower than the parent function $f(x) = x^2$. If $|a| < 1$, the parabola will be wider than $f(x)$.

EXAMPLE 3

Consider the quadratic function $q(x) = -2(x-2)^2 + 3$. Determine a, h and k from the definition of the vertex form. Find the coordinates of the vertex and the equation for the axis of symmetry. Does the parabola open upward or downward? Is it wider or narrower than $f(x) = x^2$.

SOLUTION Comparing $-2(x-2)^2 + 3$ to $a(x-h)^2 + k$, we see that $a = -2$, $h = 2$ and $k = 3$. So the vertex is $(2, 3)$ and the axis of symmetry is $x = 2$. Since $a = -2$ is negative, the parabola will open downward. Since $|a| > 1$, the parabola will be narrower than $f(x) = x^2$.

PROBLEM 3

For each of the following quadratic functions, find the vertex and the axis of symmetry. Determine if the parabola opens upward or downward. Also, determine whether the parabola is narrower, wider or the same shape as the parabola $f(x) = x^2$.

1. $p(x) = 2(x+2)^2 - 3$
2. $p(x) = -(x-1)^2 + 2$
3. $p(x) = -\frac{1}{2}(x+2)^2 + 4$

EXAMPLE 4

A toy cannon shoots small rubber cannonballs. When the cannon is on the ground, the height, $h(x)$, in feet of the ball at time x seconds is described by the quadratic function $h(x) = -2(x-3)^2 + 19$.

1. What is the domain of h? Sketch a graph of $h(x)$.
2. What is the vertex of the parabola $h(x)$? What is the maximum height attained by the cannonball?
3. What $h(0)$ equal? What does this number represent?
4. Suppose we put the cannon on top of a table that is three feet tall. Write a quadratic function describing the height of the cannonball. How does the vertex and the maximum height change?
5. Cody is taking measurements as he shoots the cannon. He places the cannon on the ground. He fires the cannon and uses his stopwatch to measure the time, t, and a special sensor to measure the height

of the cannon ball, $q(t)$. The problem is that Cody's reflexes are a little slow. So there is a 1 second delay between the time he fires the cannon and the time he clicks the stop watch. So all his times are 1 second off. Write a quadratic function describing the height of the cannon ball $q(t)$. How is this different from $h(x)$?

SOLUTION

1. x is the time, so $x \geq 0$.
2. $h(0) = 1$, which is the height of the barrel of cannon off the ground.
3. $(3, 19)$. The cannonball reaches its maximum height of 19 feet after 3 seconds.
4. We have moved the cannon up three feet, so the new quadratic function is $p(x) = h(x) + 3 = -2(x - 3)^2 + 22$. The vertex is now $(3, 22)$. The cannonball reaches a maximum height of 22 feet after 3 seconds.
5. The time Cody measures is always 1 second less than the actual time since the cannon was fired. We can write this as $t = x - 1$ or $x = t + 1$. Since Cody put the cannon on the ground, the height as a function of t is $q(t) = h(t+1) = -2(t+1-3)^2 + 19 = -2(t-2)^2 + 19$. The vertex is now $(2, 19)$. The cannonball reaches its maximum height when Cody's stopwatch reads $t = 2$ seconds.

EXERCISES

1. Graph each quadratic function. Locate and label the vertex.
   a. $p(x) = -2x^2$
   b. $p(x) = -(x - 1)^2 + 3$
   c. $p(x) = 3x^2$
   d. $p(x) = 2(x + 3)^2 - 2$
2. Graph the quadratic function $f(x) = 2x^2$ and the linear function $h(x) = 5x + 3$. Find all the points of intersection. Use factoring to solve the quadratic equation $2x^2 - 5x - 3 = 0$. What do you notice?

3. Write the formula for the quadratic function created by each of the following transformations of $q(x) = -x^2$.
   a. Shift $q(x)$ to the right by 3 units.
   b. Shift $q(x)$ to the left by 6 units.
   c. Shift $q(x)$ up by 2 units.
   d. Shift $q(x)$ down by 5 units.
   e. Shift $q(x)$ to the right by 1 unit and up by 4 units.
4. For each of the following quadratic functions determine the coordinates of the vertex and the equation of the axis of symmetry. Does the parabola open upward or downward? Will the parabola be wider or narrower than $f(x) = x^2$?
   a. $p(x) = -(x-2)^2 + 4$
   b. $p(x) = 2(x+3)^2 + 5$
   c. $p(x) = \frac{1}{2}(x-4)^2 - 6$
   d. $p(x) = -2\left(x - \frac{1}{2}\right)^2 + 5$
5. Let $q(x) = 2(x-1)^2 - 2$. Write the formula for the quadratic function created by each of the following transformations.
   a. Shift $q(x)$ to the right by 2 units.
   b. Shift $q(x)$ to the left by 5 units.
   c. Shift $q(x)$ up by 7 units.
   d. Shift $q(x)$ down by 5 units.
   e. Shift $q(x)$ to the right by 1 unit and up by 4 units.
   f. Leave the vertex the same, but make the parabola open downward.
6. You throw a ball into the air. The height $h(t)$ in meters of the ball at time t in seconds can be described by the equation $h(t) = -5(t-2)^2 + 25$ where $t = 0$ is the time at which you throw the ball.
   a. What is the domain of h? Graph $h(t)$.
   b. What is the vertex?
   c. What is the maximum height of the ball?

d. At what time does the ball reach its maximum height?

7. The vertex of a quadratic function is $(1, 3)$ and the graph of the function passes through the point $(0, 5)$. Find the formula for the function.

8. The vertex of a quadratic function is $(1, 3)$ and the graph of the function passes through the point $(3, 1)$. Find the formula for the function.

9. **Investigation:**

   a. Make a table of points on the graph of $q(x) = 2(x + 1)^2 - 8$. Plot these points on graph paper, and connect them. Locate and label the vertex.

   b. How many times does the graph of $q(x)$ cross the x-axis? Use the graph or table to determine the coordinates of the points where the graph of $q(x)$ crosses the x-axis.

   c. Show that $q(x) = 2x^2 + 4x - 6$.

   d. Factor $2x^2 + 4x - 6$.

   e. Graph the two linear functions $g(x) = x + 3$ and $h(x) = x - 1$. What do you notice?

SECTION 8.3 x-INTERCEPTS OF QUADRATIC FUNCTIONS

Imagine you throw a ball into the air and the height $h(x)$ of the ball (in meters) at time $x > 0$, in seconds, can be described by the quadratic function $h(x) = -4(x-2)^2 + 24$, where $x = 0$ is the time at which you throw the ball. At what time will the ball hit the ground? In other words, what value of x makes $h(x) = 0$? If we graph $h(x)$, this will be the x-coordinate of the point where the graph crosses the x-axis. We will call this x value the x-intercept.

EXPLORATION 1

1. Using a graphing calculator, graph each of the following quadratic functions:
   a. $p(x) = (x-2)^2$
   b. $p(x) = (x+3)^2$
   c. $p(x) = -(x-5)^2$
2. For each function, how many times does the parabola touch or cross the x-axis?
3. For each function, find the vertex.
4. For each function, what is the x-intercept? What is the relationship between the x-intercept and the vertex?

Recall from Chapter 3 all linear functions, except for $y = 0$, have exactly one x-intercept. All of the parabolas in Exploration 1 also had exactly one x-intercept. Do you think this is true for all quadratic functions?

EXPLORATION 2

1. Using a graphing calculator, graph each of the following:
   a. $p(x) = (x-2)^2 - 4$
   b. $p(x) = (x+3)^2 - 9$
   c. $p(x) = -(x-5)^2 + 1$

d. $p(x) = (x-4)^2 + 1$

2. For each function, how many times does the parabola touch or cross the x-axis?
3. For each function, find the vertex.
4. For each function, find the x-intercepts.
5. Is there any relationship between the number of x-intercepts and the coordinates of the vertex?

In Exploration 2 we saw a relationship between the number of x-intercepts and the location of the vertex. A parabola can have either 0, 1 or 2 x-intercepts. If the vertex lies on the x-axis, there is only one intercept. If the parabola faces upward and the y-coordinate of the vertex is negative, there are 2 intercepts. If the parabola faces upward and the y-coordinate of the vertex is positive, there are no intercepts. Similarly, if the parabola faces downward and the y-coordinate of the vertex is positive, there are 2 intercepts. If the parabola faces downward and the y-coordinate of the vertex is negative, there are no intercepts.

PROBLEM 1

For each the following, predict how many x-intercepts the parabola will have.

1. $p(x) = (x-5)^2$
2. $p(x) = (x-5)^2 + 2$
3. $p(x) = -(x-5)^2 + 2$

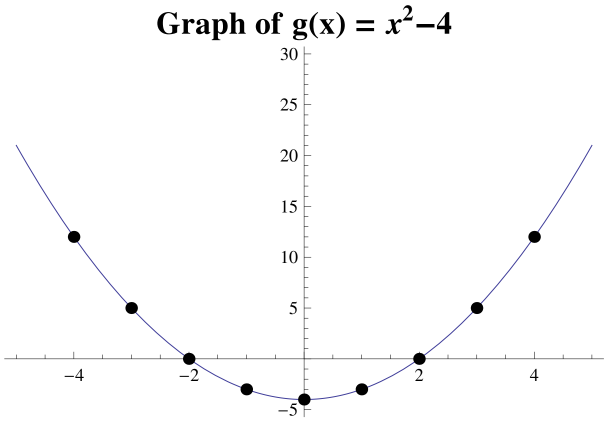
EXAMPLE 1

Make a graph of $g(x) = x^2 - 4$. Locate and label the x-intercepts. Solve for the exact values of the x-intercepts.

SOLUTION Make a table.

x	$g(x) = x^2 - 4$
-4	
-3	
-2	
-1	
0	-4
1	-3
2	
3	
4	

Next, graph these points, and connect the points to create the graph.



Observe that the x-intercepts are $x = 2$ and $x = -2$. Why does this parabola have two x-intercepts? First of all, the vertex, $(0, -4)$, lies below the x-axis and the parabola faces upward. So as the x value moves away from 0, $g(x)$ increases. This occurs both for positive values of x and negative values of x, since $(-2)^2 = 2^2 = 4$, $g(-2) = g(2) = 4 - 4 = 0$.

How could we use algebra to find the x-intercepts without making the table or reading from the graph?

To find the x-intercept we need to find the value of x for which $g(x) = 0$. In other words, we want to solve the equation:

$$x^2 - 4 = 0.$$

Two different methods can be used to solve this equation: we can use the factoring technique we discovered in Chapter 7, or we can use the square root.

Factor: We recognize that the left hand side is the difference of squares, so:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0. \end{aligned}$$

This means that either $x - 2 = 0$ or $x + 2 = 0$. By solving each simple linear equation we get the x-intercepts we found in the graph, $x = 2$ and $x = -2$.

Square Root: Since x^2 is a perfect square, we can rearrange the equation and take the square root to find the solution.

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \end{aligned}$$

So we are looking for values of x such that $x^2 = 4$. From the graph (and our factoring method) it is clear that there are two values that make this true, since $2^2 = (-2)^2 = 4$. Hence:

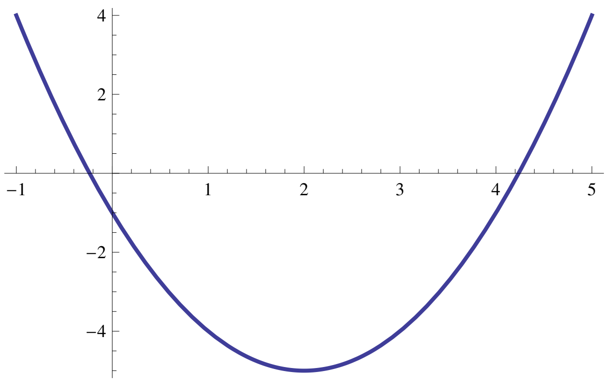
$$x = \pm\sqrt{4} = \pm 2.$$

The x-intercepts of a quadratic function are a very important subject in mathematics. Perhaps for this reason, mathematicians have coined 3 different names for them. Since we can use the square root method to find them, sometimes they are referred to as *roots* of the function. Since the x-intercepts are the inputs that make the function equal to 0, they are also sometimes called the *zeroes* of the function. While this can be confusing, it is important to remember that roots, zeroes, and x-intercepts all refer to the same thing.

For the parabolas in Explorations 1 and 2, it was easy to read the exact value of the x-intercept from the graph. This is not always the case. Often we will need to use algebraic methods to find the exact values of the x-intercepts.

EXAMPLE 2

The graph of the quadratic function $p(x) = (x-2)^2 - 5$ is shown below.



First use the graph to estimate the x-intercepts. Solve for the exact values of the x-intercepts algebraically.

SOLUTION The parabola crosses the x-axis at two points. The left x-intercept appears to be slightly less than $-.2$. The x-intercept on the right is a bit greater than 4. To find the x-intercepts exactly, we need to find the value of x for which $p(x) = 0$. In other words we want to solve the equation:

$$(x-2)^2 - 5 = 0.$$

Since the left side does not fit our pattern of difference of squares, let's use the square root method to solve the equation.

Square Root: Since $(x-2)^2$ is a perfect square, we can rearrange the equation and take the square root to find the solution.

$$\begin{aligned}(x-2)^2-5&=0\\(x-2)^2&=5\\\sqrt{(x-2)^2}&=\sqrt{5}\\x-2&=\pm\sqrt{5}\end{aligned}$$

This leads to two equations $x-2=\sqrt{5}$ and $x-2=-\sqrt{5}$. Solving each linear equation for x gives two solutions $x=2+\sqrt{5}$ or $x=2-\sqrt{5}$. We can use the calculator to find approximate values, $x\approx-.236$ or $x\approx 4.236$.

PROBLEM 2

Graph the quadratic function $q(x)=-(x-3)^2+7$. Use the graph to estimate the x-intercepts. Then use algebra to find the exact value of the x-intercepts.

Let's return to the problem about throwing the ball described in the beginning of the section.

EXAMPLE 3

Imagine you throw a ball into the air and the height $h(x)$, in meters, of the ball at time $x\geq 0$, in seconds, can be described by the quadratic $h(x)=-4(x-2)^2+24$ where $x=0$ is the time at which you throw the ball.

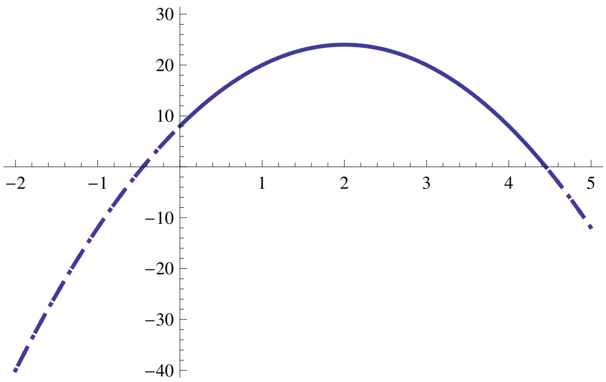
1. In this context, what is the domain of h?
2. What is the range of h?
3. What is the initial height of the ball, when it is thrown?
4. Graph the function.

5. What is the maximum height? At what time does the ball reach its maximum height?
6. Use the graph to estimate the time the ball hits the ground.
7. Find the exact time the ball hits the ground algebraically.

SOLUTION

1. When a function is used to model a real life situation, we have to think about the context to determine the domain. Since x represents time, the domain is the set of numbers greater than or equal to 0.
2. $h(x)$ represents the height of the ball. So the range is the set of numbers greater than or equal to 0.
3. The initial time is $x = 0$, so the initial height is:

$$h(0) = -4(0-2)^2 + 24 = -4 \cdot 4 + 24 = -16 + 24 = 8 \text{ meters.}$$



4. We have graphed $h(x)$ for all values of x, but we have used a dashed line to represent the portions of the graph that are outside of our domain and range.
5. Since the parabola faces downward, the maximum value occurs at the vertex $(2, 24)$. The maximum height is 24 meters, and the maximum occurs at 2 seconds.
6. The left x-intercept in the graph occurs near $x = -.4$ that is outside of the domain. The right intercept occurs near $x = 4.4$. So the ball takes nearly 4.4 seconds to hit the ground.

7. We use the square root method to find the exact value of the x-intercept:

$$\begin{aligned} -4(x-2)^2+24 &= 0 \\ -4(x-2)^2 &= -24 \\ (x-2)^2 &= 6 \\ \sqrt{(x-2)^2} &= \pm\sqrt{6} \\ x-2 &= \pm\sqrt{6}. \end{aligned}$$

This leads to two equations, $x-2=\sqrt{6}$ and $x-2=-\sqrt{6}$. Solving each linear equation for x gives two solutions: $x=2+\sqrt{6}$ and $x=2-\sqrt{6}$. Only $x=2+\sqrt{6}$ is in our domain. Use the calculator to find the approximate value, $x=4.449$.

EXERCISES

1. For each quadratic function, predict the number of x-intercepts.
   a. $p(x)=(x-3)^2+5$
   b. $p(x)=x^2-11$
   c. $p(x)=(x+3)^2+3$
   d. $p(x)=2x^2$
   e. $p(x)=2(x+1)^2$
   f. $p(x)=-2x^2$
   g. $p(x)=-(x-1)^2+3$
   h. $p(x)=3x^2$
   i. $p(x)=-2(x+3)^2-2$
2. For each of the following quadratic functions, find the vertex and the x-intercepts. What pattern do you notice?
   a. $q(x)=(x-1)^2$
   b. $q(x)=(x-1)^2-1$
   c. $q(x)=(x-1)^2-9$

d. $q(x) = (x-1)^2 - 16$

e. $q(x) = (x-1)^2 - 25$

3. Find the x intercepts for each of the following functions, if they exist. If there are no x-intercepts, write "no x-intercept".

 a. $h(x) = 10x^2$

 b. $h(x) = 2(x-3)^2 + 5$

 c. $h(x) = -(x-1)^2 + 16$

 d. $h(x) = 3x + 5$

 e. $h(x) = -2x - 2$

 f. $h(x) = 4$

4. Find the x-intercepts for each of the following quadratic functions.

 a. $q(x) = -2(x+3)^2 + 8$
 b. $q(x) = 3(x+2)^2 - 12$
 c. $q(x) = 2(x-1)^2 - 10$
 d. $q(x) = -(x-2)^2 + 7$

5. The vertex of a quadratic function is $(-2, 3)$. One x-intercept of the function is -1. What is the other x-intercept?

6. The vertex of a quadratic function is $(2, -5)$. One x-intercept of the function is 4. What is the other x-intercept?

7. The vertex of a quadratic function is $(3, 9)$. One x-intercept of the function is 0. What is the formula for the quadratic function?

8. Different quadratic functions can have the same x-intercepts. Find 3 different quadratic functions that have x-intercepts -2 and 2.

9. You throw a ball into the air. The height $h(t)$ of the ball (in meters) at time t (in seconds) is described by the equation $h(t) = -5(t-2)^2 + 25$ where $t = 0$ is the time at which you throw the ball.

 a. What is the domain of h? Graph $h(t)$.
 b. What is the initial height of the ball when it is thrown?
 c. What is the vertex of the parabola? What is the maximum height reached by the ball?
 d. Use the graph to estimate the time the ball will hit the ground.
 e. At what time does the ball hit the ground?

10. **Ingenuity:**
A quadratic function has a y-intercept at -10 and x-intercepts -2 and 5. What is the formula for the quadratic function?

SECTION 8.4 WRITING IN VERTEX FORM

So far, we have explored how to find the vertex and x-intercepts for any quadratic function in the vertex form $f(x) = a(x-h)^2 + k$. It turns out that all quadratic functions can be written in this form. Let's see how to do this.

EXAMPLE 1

Factor $x^2 - 6x + 9$. Write the quadratic function $q(x) = x^2 - 6x + 9$ in vertex form. Find the vertex and x-intercepts.

SOLUTION First, recognize that $x^2 - 6x + 9$ is a perfect square. So the factored form is $x^2 - 6x + 9 = (x-3)^2$. So:

$$q(x) = x^2 - 6x + 9 = (x-3)^2$$

is in vertex form. The vertex is $(3, 0)$. The parabola only has one x-intercept, $x = 3$.

PROBLEM 1

Write the the quadratic function $q(x) = x^2 + 8x + 16$ in vertex form. Find the vertex and x-intercepts.

Unfortunately, not all quadratic functions are this easy to convert to vertex form.

EXPLORATION 1

1. Using a graphing calculator, graph these quadratic functions:
   a. $p(x) = x^2 - 2x$
   b. $p(x) = x^2 + 2x$
   c. $p(x) = x^2 + 4x$

d. $p(x) = x^2 - 4x$
e. $p(x) = x^2 + 8x$

2. Each of the quadratic functions has the form $p(x) = x^2 + bx$ for different values of b. What relationship do you notice between b and the x-intercepts of the graph?
3. What relationship do you notice between b and the vertex of the graph?
4. Without graphing, can you predict the coordinates of the vertex for $p(x) = x^2 - 8x$?

Based on Exploration 1, notice that if the quadratic function has the form $f(x) = x^2 + bx$, then the x-intercepts are $x = 0$ and $x = -b$, and the vertex is $\left(-\frac{b}{2}, -\left(\frac{b}{2}\right)^2\right)$. In the next example, we give an algebraic reason why this is true.

EXAMPLE 2

Find the x-intercepts of the quadratic function $q(x) = x^2 + 6x$. Write $q(x)$ in vertex form and find the coordinates of the vertex.

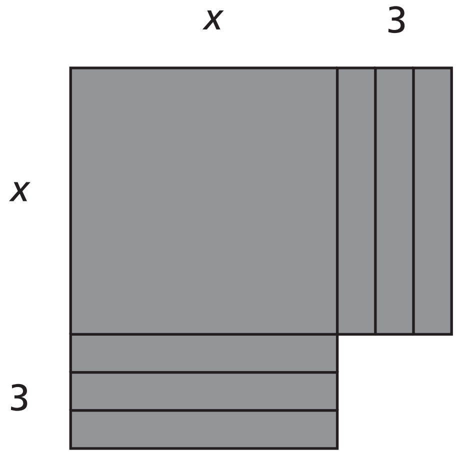
SOLUTION To find the x-intercepts, we are looking for the values of x which make $q(x) = 0$. This gives a quadratic equation, which we can solve by factoring.

$$x^2 + 6x = 0$$
$$x(x + 6) = 0$$

This leads to 2 simple linear equations: $x = 0$ and $x + 6 = 0$. Solving for x gives us $x = 0$ and $x = -6$. The same method would work for any quadratic function of the form $f(x) = x^2 + bx$. Do you see why $x = 0$ will always be one of the intercepts?

Finding the vertex algebraically is more difficult. We know how to convert quadratic functions that are perfect squares into vertex form.

Unfortunately, our quadratic function is not in the form of a perfect square. The following figure shows the algebra tile representation of $q(x)$. In the figure we try to form a square:



What goes wrong? We are missing the 9 unit squares we need to complete the square. Where did 9 come from? To make a square, we arranged one half of the 6 x tiles vertically, and the other half horizontally. These create the 3 by 3 missing square. What to do? You might simply add 9 to our function, but $x^2 + 6x + 9 = q(x) + 9$ is a different quadratic expression. What we can do is add add $0 = 9 - 9$ to our function without changing it:

$$q(x) = x^2 + 6x + 9 - 9 = (x + 3)^2 - 9$$

This is in vertex form. Hence the vertex is $(-3, -9)$.

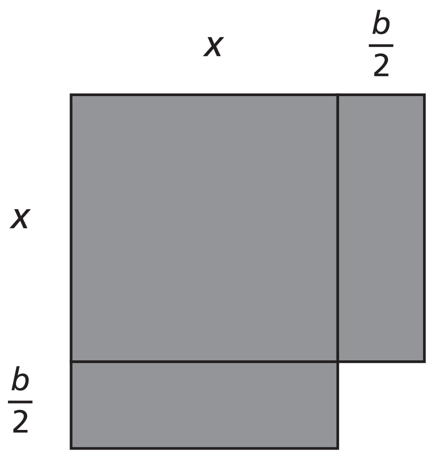
The method used to convert to vertex form in Example 2 is called *completing the square*. Thinking about this method carefully, we can see how this will work for any quadratic function of the form $f(x) = x^2 + bx$. Generalizing the figure from Example 2 we see that

$$f(x) = x^2 + bx$$

$$= x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2.$$

So the vertex will be $\left(-\frac{b}{2}, -\left(\frac{b}{2}\right)^2\right)$. Notice that the answer involves 2 parts, each of size $\frac{b}{2}x$. Looking at the figure, we see why this happens: to form the square, we needed to split the bx term into 2 parts.



We now look at quadratic functions of the form $f(x) = x^2 + bx + c$.

EXPLORATION 2

1. Using a graphing calculator graph these quadratic functions
   a. $p(x) = x^2 - 6x + 9$
   b. $q(x) = x^2 - 6x + 10$
   c. $q(x) = x^2 - 6x + 11$
   d. $q(x) = x^2 - 6x + 8$
   e. $q(x) = x^2 - 6x + 7$
2. Show that $p(x)$ is a perfect square. What is its vertex?
3. Each of the quadratic functions has the form $q(x) = x^2 - 6x + c$

for different values of c. What relationship do you notice between c and the vertex of the graph? How do they compare with the graph of $p(x)$?

4. Without graphing, can you predict what the coordinates of the vertex of $p(x) = x^2 - 6x + 12$ will be?
5. Write $q(x) = (x^2 - 6x + 9) + 1$ in vertex form.
6. Write $q(x) = x^2 - 6x + 8$ in vertex form.

PROBLEM 2

1. Write $q(x) = (x^2 + 10x + 25) + 2$ in vertex form. What is its vertex?
2. Write $q(x) = x^2 + 10x + 20$ in vertex form. What is its vertex?

In Exploration 2, you found that one way to convert a quadratic to vertex form is to compare the function to a perfect square. In the next example, we use completing the square to find the correct perfect square to compare with.

EXAMPLE 3

Write $q(x) = x^2 - 8x + 15$ in vertex form. Find the coordinates of the vertex.

SOLUTION Since the vertex form involves a perfect square, we want to use the completing the square method. If we think about the first two terms, we know that:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x - 4)^2 - 16.$$

However, $q(x)$ has a constant term. If we include this we get

$$\begin{aligned} q(x) &= (x^2 - 8x) + 15 \\ &= (x^2 - 8x + 16 - 16) + 15 \end{aligned}$$

$$= (x^2 - 8x + 16) - 16 + 15$$
$$= (x - 4)^2 - 1$$

So, the vertex is $(4, -1)$.

In Example 3 we completed the square to find the vertex form. From Example 3, we can see how this will work for any quadratic function of the form $f(x) = x^2 + bx + c$. We must complete the square and carefully keep track of the constant term.

$$\begin{aligned} q(x) &= x^2 + bx + c \\ &= \left(x^2 + bx\right) + c \\ &= \left(x^2 + 2\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right) + c \\ &= \left(x^2 + 2\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2\right) - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

So the vertex is $\left(-\frac{b}{2}, -\left(\frac{b}{2}\right)^2 + c\right)$ and the axis of symmetry is $x = -\frac{b}{2}$.

Now that we know how to use completing the square to convert to vertex form, let's find a method for determining the x-intercepts of a quadratic function.

EXAMPLE 4

Write $q(x) = x^2 + 6x + 5$ in vertex form. Use the vertex form to find the coordinates of the vertex and the x-intercepts.

SOLUTION We can either follow the steps of Example 3 or use the general version described above. Comparing $q(x)$ with $x^2 + bx + c$, we see that $b = 6$ and $c = 5$. Since $\frac{b}{2} = 3$, the vertex form is:

$$q(x) = (x + 3)^2 - 3^2 + 5 = (x + 3)^2 - 4.$$

Thus, the vertex is $(-3, -4)$. Once $q(x)$ is in vertex form, we can use the square root method to find the x-intercepts:

$$\begin{aligned} (x + 3)^2 - 4 &= 0 \\ (x + 3)^2 &= 4 \\ (x + 3) &= \pm 2. \end{aligned}$$

This gives 2 simple linear equations: $x + 3 = -2$ and $x + 3 = 2$. Solving the equations for x, we get $x = -5$ and $x = -1$.

If we write the intercepts in the form $x = -3 - \sqrt{4}$ and $x = -3 + \sqrt{4}$, we see an interesting relationship between the coordinates of the vertex and the x-intercepts.

We finish the section, by finding the vertex form of the general quadratic function.

EXAMPLE 5

Suppose a, b and c are numbers with $a \neq 0$. Write the quadratic function $q(x) = ax^2 + bx + c$ in vertex form.

SOLUTION For all the quadratic functions considered so far in this section $a = 1$. However, since $a \neq 0$, we can rewrite $q(x)$ to see how we can use the results we have developed so far:

$$q(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a})$$

From the general result above, we can write $h(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ in the vertex form:

$$h(x) = (x - \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}.$$

We now substitute into $q(x)$.

$$\begin{aligned} q(x) &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a\left((x - \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ &= a(x - \frac{b}{2a})^2 - a\left(\frac{b}{2a}\right)^2 + a\frac{c}{a} \end{aligned}$$

Yuck! The resulting form is quite complicated. However, we do notice one nice and important result. The x-coordinate of the vertex is $-\frac{b}{2a}$. So, for any quadratic function written in the form $q(x) = ax^2 + bx + c$, the vertex is $(-\frac{b}{2a}, q(-\frac{b}{2a}))$.

PROBLEM 3

Find the vertex for:

1. $q(x) = x^2 - 6x + 9$
2. $p(x) = x^2 - 6x + 5$
3. $s(x) = 2x^2 - 12x + 10$
4. $t(x) = -3x^2 + 18x - 15$

EXERCISES

1. Write each of the following in vertex form. Find the vertex.
   a. $q(x) = x^2 + 4x + 4$
   b. $q(x) = x^2 - 6x + 9$

c. $q(x) = x^2 - 9$

d. $q(x) = x^2 - 10x + 25$

e. $q(x) = x^2 - x + \frac{1}{4}$

f. $q(x) = -x^2 + 4x - 4 = -(x^2 - 4x + 4)$

2. Write each of the following in vertex form. Find the vertex.
   a. $q(x) = x^2 + 6x$
   b. $q(x) = x^2 - 10x$
   c. $q(x) = x^2 + 10x$
   d. $q(x) = x^2 + 3x$

3. a. Graph the following quadratic functions.
      i. $q(x) = x^2 + 2x + 1$
      ii. $q(x) = x^2 - 2x + 1$
      iii. $q(x) = x^2 + 4x + 1$
      iv. $q(x) = x^2 + 6x + 1$

   b. All the functions above have the form $x^2 + bx + 1$. Describe the effect of b on the graph of the function.

   c. Without graphing, predict what the graph of $q(x) = x^2+10x+1$ will look like.

4. Write each of the following in vertex form. Find the vertex.
   a. $q(x) = x^2 + 4x + 10$
   b. $q(x) = x^2 - 6x + 12$
   c. $q(x) = x^2 + 12x - 24$
   d. $q(x) = x^2 - 5x + 2$

5. For each of the following, find the x-intercepts if they exist. If there are no intercepts, write "no intercepts".
   a. $q(x) = x^2 + 4x - 5$
   b. $q(x) = x^2 + 6x + 12$
   c. $q(x) = x^2 - 8x + 25$
   d. $q(x) = x^2 + 2x - 10$

e. $q(x) = x^2 - 10x + 4$

f. $q(x) = x^2 + 14x + 49$

6. Use factoring to solve the following quadratic equations.
   a. $x^2 - x - 12 = 0$
   b. $x^2 = 3x + 10$
   c. $x^2 - 8x = -15$
   d. $x^2 + 8x + 12 = 0$

7. Convert each of the following quadratic equations into vertex form. Find the x-intercepts.
   a. $q(x) = x^2 - x - 12$
   b. $q(x) = x^2 - 3x - 10$
   c. $q(x) = x^2 - 8x + 15$
   d. $q(x) = x^2 + 8x + 12$

8. Compare the x-intercepts you found in Exercise 7 to the solutions you found in Exercise 6. What do you notice? Explain why this happened.

9. Let $q(x) = x^2 - 2x - 3$.
   a. Write $q(x)$ in vertex form.
   b. Graph $q(x)$.
   c. Use the graph to predict how many solutions each of the following quadratic equations will have.
      i. $x^2 - 2x - 3 = 0$
      ii. $x^2 - 2x - 3 = 1$
      iii. $x^2 - 2x - 3 = -1$
      iv. $x^2 - 2x - 5 = 0$
   d. Find the vertex of:
      i. $q(x) = 3x^2 + 12x - 8$
      ii. $p(x) = 2x^2 + 24x - 10$
      iii. $r(x) = -x^2 + 4x + 5$
      iv. $s(x) = \frac{1}{2}x^2 + 4x - 2$

SECTION 8.5 THE QUADRATIC FORMULA

In the last section, we used the vertex form of a quadratic function to find the x-intercepts of the function. We now have 2 methods for finding the x-intercepts of a quadratic function: we can factor the quadratic, or we can write it in vertex form. In this section, we will develop a formula that we can use to find the x-intercepts of any quadratic polynomial, or show that the polynomial has no x-intercepts.

EXPLORATION 1

1. The table below is a list of quadratic functions in vertex form. For each quadratic function $q(x)$, find the vertex and solve the equation $q(x) = 0$ to find the x-intercepts. Then fill in the table.

$q(x)$	vertex	x-intercepts
$x^2 - 9$		
$(x-2)^2 - 9$		
$(x-2)^2 - 16$		
$(x-5)^2 - 9$		

In Exploration 1 you may have used the square root method to solve the quadratic equation and find the x-intercepts. In the next example, we look at the general form of this solution.

EXAMPLE 1

Let r and s be any two numbers and sketch the quadratic function $q(x) = (x-r)^2 - s^2$. Find the vertex. Solve the equation $q(x) = 0$ to find the x-intercepts.

SOLUTION Since we don't know the values of r and s, we can only give a very general sketch of $q(x)$. Since $q(x)$ is written in vertex form, we can see that the vertex is $(r, -s^2)$. To find the x-intercepts, we solve the equation $q(x) = 0$ using the square root method:

$$\begin{aligned} (x-r)^2 - s^2 &= 0 \\ (x-r)^2 &= s^2 \\ \sqrt{(x-r)^2} &= \pm\sqrt{s^2} \\ x - r &= \pm s. \end{aligned}$$

Solving the two linear equations for x, we find that the two x-intercepts are $x = r + s$ and $x = r - s$. Notice the relationship between the vertex and the x-intercepts.

In Example 1 the quadratic function was given in vertex form. In Section 8.4 we used "completing the square" to convert quadratic functions into vertex form. In the next exploration we use completing the square to explore the relationship between the coefficients of a quadratic function and the x-intercepts.

EXPLORATION 2

1. The table below shows a list of quadratic functions in standard form: $x^2 + bx + c$. For each quadratic function $q(x)$, use completing the square to rewrite the function in vertex form. Then find the vertex and solve the equation $q(x) = 0$ to find the x-intercepts. Fill in the table.

$q(x)$	vertex form	vertex	x-intercepts
$x^2 - 6x + 8$			
$x^2 - 6x + 5$			
$x^2 + 6x - 16$			
$x^2 + 8x + 15$			

2. What relationship do you notice between the coordinates of the vertex and the x-intercepts?
3. Each of the quadratic functions was given in standard form $x^2 + bx + c$. What relationship do you notice between the values of b and c and the x-intercepts?

In Exploration 1 and 2, you used "completing the square" to rewrite the function in vertex form, and then the square root method to solve the quadratic equation and find the x-intercepts. In the next two examples, we use this idea to find a general formula for solving quadratic equations.

EXAMPLE 2

Use completing the square to solve the quadratic equation $2x^2 + 16x - 40 = 0$. Check your answer.

SOLUTION In Section 8.4, we learned how to complete the square if the coefficient of the x^2 term was 1. We begin by dividing both sides of the equation by 2.

$$\begin{aligned} 2x^2 + 16x - 40 &= 0 \\ x^2 + 8x - 20 &= 0 \end{aligned}$$

Now that the coefficient of the x^2 term is one, we will rewrite the left hand side by completing the square.

$$
\begin{aligned}
x^2 + 8x - 20 &= 0 \\
x^2 + 8x + 16 - 16 - 20 &= 0 \\
(x+4)^2 - 36 &= 0
\end{aligned}
$$

Now use the square root method to solve the equation:

$$
\begin{aligned}
(x+4)^2 - 36 &= 0 \\
(x+4)^2 &= 36 \\
x + 4 &= \pm 6.
\end{aligned}
$$

That leads to two simple linear equations: $x + 4 = -6$ and $x + 4 = 6$. Solving for x, we get two solutions $x = -10$ and $x = 2$. We will often write the two solutions in the form $x = -4 \pm 6$.

We can then check our answer by substituting into our original equation. We see that

$$2(-10)^2 + 16(-10) - 40 = 200 - 160 - 40 = 0$$

and

$$2(2)^2 + 16(2) - 40 = 8 + 32 - 40 = 0.$$

PROBLEM 1

Use completing the square to solve the quadratic equation $3x^2+9x-12 = 0$. Check your answer.

If we think about the steps in Example 2, we can see how to find a formula to solve the general quadratic equation.

EXAMPLE 3

Use "completing the square" to solve the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are numbers and $a \neq 0$.

SOLUTION Notice $a \neq 0$. If a were equal to zero, then this would not be a quadratic equation because there would be no x^2 term. Now we will follow the steps from Example 2. To make completing the square easier, instead of dividing by a, we will multiply both sides by $4a$:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 4a^2x^2 + 4abx + 4ac &= 0. \end{aligned}$$

Now that the coefficient of the x^2 term is $4a^2$ (which is a perfect square). Let's rewrite the left side using "completing the square."

$$\begin{aligned} 4a^2x^2 + 4abx + 4ac &= 0 \\ 4a^2x^2 + 4abx + b^2 - b^2 + 4ac &= 0 \\ (2ax + b)^2 - b^2 + 4ac &= 0 \end{aligned}$$

Now use the square root method to solve the equation:

$$\begin{aligned} (2ax + b)^2 - b^2 + 4ac = 0 \\ (2ax + b)^2 = b^2 - 4ac \\ 2ax + b = \pm\sqrt{b^2 - 4ac}. \end{aligned}$$

This leads to the solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this example, we have discovered one of the most famous formulas in algebra.

THEOREM 8.5: QUADRATIC FORMULA

The solutions of the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are numbers and $a \neq 0$ are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4

Use the quadratic formula to solve $2x^2 - 3x = -1$. Check your answer.

SOLUTION To use the quadratic formula, we must first rewrite the equation with 0 on one side:

$$\begin{aligned} 2x^2 - 3x &= -1 \\ 2x^2 - 3x + 1 &= 0. \end{aligned}$$

Now we compare with the standard form $ax^2 + bx + c = 0$, and see that $a = 2$, $b = -3$ and $c = 1$. Substituting into the quadratic formula, we get:

$$x = \frac{-(-3) \pm \sqrt{9 - 4(2)1}}{2(2)} = \frac{3 \pm 1}{4}.$$

So the solutions are $x = \frac{4}{4} = 1$ and $x = \frac{2}{4} = \frac{1}{2}$. We can verify these solutions by substituting into our original equation:

$$2(1^2) - 3(1) = 2 - 3 = -1$$

and

$$2\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) = \frac{2}{4} - \frac{3}{2} = \frac{2}{4} - \frac{6}{4} = -\frac{4}{4} = -1.$$

PROBLEM 2

Solve each of the following quadratic equations in 2 ways. First, use factoring. Second, use the quadratic formula. Verify that you get the same answer.

1. $x^2 + x - 6 = 0$
2. $2x^2 + 5x - 3 = 0$
3. $3x^2 + 10x = 8$

In Exploration 2 and Problem 1 from Section 8.3 we saw that quadratic functions can have 0, 1 or 2 x-intercepts depending on the coefficients. Since solving the quadratic equation $ax^2 + bx + c = 0$ is the same thing as finding the x-intercepts of the quadratic function $q(x) = ax^2 + bx + c$, we see that quadratic equations can have 0, 1 or 2 solutions. Now let's explore how to tell how many different solutions a quadratic equation will have.

EXPLORATION 3

In the quadratic formula, the expression $b^2 - 4ac$ is called the *discriminant*.

1. The table below shows a list of quadratic functions in standard form. Graph each quadratic function $q(x)$ using a graphing calculator. Determine the number of x-intercepts. Compute the discriminant. Then fill in the table.

$q(x)$	number of x-intercepts	$b^2 - 4ac$
$x^2 + 2x + 1$		
$-4x^2 - 12x - 9$		
$x^2 - 4$		
$-x^2 - 4x + 5$		
$x^2 + 2x + 5$		
$-x^2 - 2x - 10$		

2. What relationship do you notice between the number of x-intercepts and the discriminant?
3. For the quadratic functions with only one x-intercept, what happens when you use the quadratic formula?
4. For the quadratic function with no x-intercepts, what happens when you use the quadratic formula?

In Exploration 3, we saw that the number of x-intercepts can be determined by the sign of the discriminant. If $b^2 - 4ac > 0$, then the graph of $q(x) = ax^2 + bx + c$ will have 2 x-intercepts. If $b^2 - 4ac < 0$, then the graph will have no x-intercepts. When $b^2 - 4ac = 0$, the graph will have only one x-intercept.

PROBLEM 3

1. Use the discriminant to predict the number of x-intercepts for each of the following quadratic functions:
   a. $q(x) = 4x^2 - 4x - 1$
   b. $q(x) = 3x^2 + 2x - \frac{1}{2}$
2. Use the discriminant to predict the number of solutions for each of the following quadratic equations:

a. $-x^2 + 2x - 7 = 0$
b. $x^2 - 3x = 15$

As you see, there is a close relationship between factoring quadratic expressions and using the quadratic formula to solve quadratic equations. Let's dif deeper into this relationship.

EXPLORATION 4

1. Factor each of the following:
   a. $x^2 + x - 6$
   b. $x^2 + 6x + 8$
   c. $x^2 - 7x + 10$
   d. $x^2 - 2x - 3$
2. Use the quadratic equation to solve:
   a. $x^2 + x - 6 = 0$
   b. $x^2 + 6x + 8 = 0$
   c. $x^2 - 7x + 10 = 0$
   d. $x^2 - 2x - 3 = 0$
3. What relationship do you notice between the solutions of the quadratic equations and the factors of the quadratic expressions?

In Exploration 4, we discovered that if we can use factoring to solve a quadratic equation, the quadratic formula will gives the same solution. In fact, relationship between the solutions of the quadratic equations and the factors is: if we know that the quadratic equation $x^2 + bx + c = 0$ has two solutions $x = r$ and $x = s$, then we can factor the left hand side as $x^2 + bx + c = (x - r)(x - s)$.

PROBLEM 4

Use the quadratic formula to factor each of the following quadratic expressions. Multiply to check your answer.

1. $x^2 - 3x + 2$
2. $x^2 + x - 20$
3. $x^2 + \frac{3}{2}x - 1$

The relationship between the factors of a quadratic expression and the solutions to the quadratic equation suggests another form of expressing a quadratic function.

We say that $q(x) = a(x - r)(x - s)$ is written in *factored form*. Notice that $q(x) = 0$ if $x - r = 0$ or $x - s = 0$. Hence, from the factored form it is easy to read off the x-intercepts.

EXAMPLE 5

Write $q(x) = 2x^2 + 8x - 10$ in factored form. What are its x-intercepts?

SOLUTION We factor $q(x)$ by finding the common factor 2 and then factoring the resulting expression:

$$q(x) = 2x^2 + 8x - 10 = 2(x^2 + 4x - 5) = 2(x - 1)(x + 5).$$

Now that we have written $q(x)$ in factored form, we can see that the x-intercepts will be 1 and -5.

PROBLEM 5

Write each function in factored form and find its x-intercepts.

1. $q(x) = x^2 + 7x + 12$
2. $q(x) = 3x^2 - 9x + 6$

Often we want to find a quadratic function that satisfies given conditions. Depending on the information given, it might be easier to start with the vertex form or factored form.

EXAMPLE 6

Find the formula for the quadratic function given by each condition.

1. The x-intercepts are 4 and -4. The y-intercept is 8.
2. The vertex is $(1, 3)$. The y-intercept is 7.

SOLUTION

1. We are given the x-intercepts $r = 4$ and $s = -4$, so it will be easiest to use the factored form $q(x) = a(x - r)(x - s) = a(x - 4)(x + 4)$. To find a, we use the second condition given. Since y-intercept is 8, we know that:

$$8 = q(0) = a(0 - 4)(0 + 4) = -16a.$$

   Solving this equation gives $a = -\frac{1}{2}$. So the function is $q(x) = -\frac{1}{2}(x - 4)(x + 4)$.
2. In this case we are given the vertex. So it makes sense to use the vertex form $q(x) = a(x - h)^2 + k = a(x - 1)^2 + 3$. Again we use the y-intercept to determine a:

$$7 = q(0) = a(0 - 1)^2 + 3 = a + 3.$$

   Solving this equation gives $a = 4$. So the function is $q(x) = 4(x - 1)^2 + 3$.

If factoring and the quadratic formula give the same answer, why do we need the quadratic formula? First of all, the quadratic formula gives a systematic (if a little complicated) way to find the factors. But more importantly, the quadratic formula works even in cases that are not easily factored. In Chapter 7 we discussed many different ways of factoring polynomials, but found out that not all quadratics could be factored using only integers. For example, we can't factor the polynomial $x^2 + 2x - 4$ using only integer coefficients. If we look at all the pairs of integers whose product is -4, we get the following:

Factor Pairs of -4	$-1, 4$	$1, -4$	$2, -2$
Sum	3	-3	0

None of the factor pairs add up to 2, so we can't factor the polynomial $x^2 + 2x - 4$ using the techniques from Chapter 7. In this next example, we see that we can use the quadratic formula to "factor" $x^2 + 2x - 4$, but the coefficients will involve square roots.

EXAMPLE 7

Use the quadratic formula to factor $x^2 + 2x - 4$. Check your answer.

SOLUTION If we compare x^2+2x-4 to the standard form ax^2+bx+c, we see that $a = 1$, $b = 2$ and $c = -4$. According to the quadratic formula, the solutions of the equation $x^2 + 2x - 4 = 0$ are:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{20}}{2} \\ &= \frac{-2 \pm \sqrt{4 \cdot 5}}{2} \\ &= \frac{-2 \pm 2\sqrt{5}}{2} \\ &= -1 \pm \sqrt{5}. \end{aligned}$$

This means that $\left(x - \left(-1 + \sqrt{5}\right)\right)$ and $\left(x - \left(-1 - \sqrt{5}\right)\right)$ should be factors of $x^2 + 2x - 4$. Let's check if this is true.

$$\begin{aligned}\left(x - \left(-1 - \sqrt{5}\right)\right)\left(x - \left(-1 + \sqrt{5}\right)\right) &= \\ &= x^2 - \left(-1 - \sqrt{5}\right)x - \left(-1 + \sqrt{5}\right)x + \left(-1 - \sqrt{5}\right)\left(-1 + \sqrt{5}\right) \\ &= x^2 + 2x + (1 + \sqrt{5} - \sqrt{5} - (\sqrt{5})^2) \\ &= x^2 + 2x - 4\end{aligned}$$

EXERCISES

1. Use the quadratic formula to solve:
   a. $x^2 - 4x + 2 = 0$
   b. $x^2 + 4x + 2 = 0$
   c. $x^2 - 4x - 1 = 0$
   d. $2x^2 = 1 - 2x$
   e. $9x^2 - 6x = -1$
   f. $2x^2 = 5x - 3$
2. Use the quadratic formula to factor each of the following expressions. Check your answer by multiplying.
   a. $x^2 + 2x - 3$
   b. $x^2 + 3x - 4$
   c. $x^2 - 5$
   d. $x^2 - \frac{8}{3}x - 1$
3. Find the x-intercepts of the following functions if they exist. If there are no x-intercepts write "no x-intercepts".
   a. $q(x) = x^2 + 6x + 9$
   b. $q(x) = 2x^2 - x + 2$
   c. $q(x) = -3x^2 + 3x - 2$
   d. $q(x) = -4x^2 - 6x + 2$
   e. $h(x) = 3 - 4x$
   f. $h(x) = -2$
4. Let $q(x) = 2x^2 - 3x + 3$ and $h(x) = 2x + 5$.
   a. Graph $q(x)$ and $h(x)$ on the same coordinate plane. Estimate

the points of intersection.

b. Use the quadratic formula to find the solutions of the quadratic equation $q(x) = h(x)$.

c. What is the relationship between the solutions of $q(x) = h(x)$ and the points of intersection?

5. **Investigation:**
Use "completing the square" to put each of the following quadratic functions into vertex form:
a. $q(x) = x^2 + 6x + 10$
b. $q(x) = 2x^2 + 12x + 20$
c. $q(x) = 2x^2 + 12x + 16$
d. $q(x) = ax^2 + 2abx + c$
e. $q(x) = ax^2 + bx + c$

6. **Ingenuity:**
Let $q(x) = 3x^2 + 8x - 35$. Use algebra, to find the vertex and the x-intercepts.

SECTION 8.6 CHAPTER REVIEW

Key Terms

axis of symmetry	roots
completing the square	scale factor
discriminant	translation
parabola	vertex
parent function	x-intercepts
quadratic functions	zeroes

Standard Form vs. Vertex Form

	Standard Form	Vertex Form
function:	$f(x) = ax^2 + bx + c$	$f(x) = a(x-h)^2 + k$
vertex:	$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$	(h, k)
axis of symmetry:	$x = -\frac{b}{2a}$	$x = h$
y-intercept:	$f(0) = c$	$f(0) = ah^2 + k$

Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ with $a \neq 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Practice Problems

1. Let $f(x) = (x+3)^2 - 5$. Write the formula for the quadratic function created by each of the following transformations.
   a. Shift $f(x)$ up by 7 units.
   b. Shift $f(x)$ right by 4 units.

c. Shift $f(x)$ left by 6 units.
d. Shift $f(x)$ right by 2 units and down by 4 units.

2. For each quadratic function below, build a table of at least 8 points on the graph of the function. Plot these points with graph paper and connect them. Locate and label the vertex. For parts b) through c) compare the graph of the function that for the parent function $f(x) = x^2$.
   a. $f(x) = x^2$
   b. $g(x) = x^2 + 4$
   c. $h(x) = (x - 2)^2 + 4$
   d. $j(x) = 3(x - 2)^2 + 4$
3. For each of the following quadratic functions, determine the coordinates of the vertex, if the parabola opens upward or downward and if the parabola will be wider or narrower than the parabola $f(x) = x^2$. Explain your answer. Do not use a calculator.
   a. $g(x) = 5x^2$
   b. $g(x) = -\frac{1}{3}(x + 5)^2 - 4$
   c. $g(x) = -(x - 3)^2 + 5$
   d. $g(x) = \frac{1}{2}(x - 2)^2$
4. Troy and Jose have a water balloon launcher and are up to no good. They are going to launch their ballon from the fourth story window. Ms. Cheevas intervenes and says they can't launch their water balloon unless they can determine the formula for the quadratic function that models the height of the balloon as time passes. Ms. Cheevas tells the boys that 2 seconds after releasing the balloon, the balloon will be at its highest point of 100 feet. After 4 seconds, the balloon will fall to a height of 36 feet. Help Troy and Jose find the formula so they can launch their balloon. Sketch a graph of the height of the balloon versus time.
5. Find the x-intercepts for each of the following quadratic functions:
   a. $f(x) = (x + 3)(x - 2)$
   b. $f(x) = x^2 - 64$
   c. $f(x) = x^2 + 5x - 24$
   d. $f(x) = 3(x + 2)^2 - 12$
   e. $f(x) = 4(x - 3)^2 - 8$

6. Troy and Jose have figured how to determine the formula for the quadratic functions that model the height of the water balloon as time passes. However, this time they will be launching from a third story window. They describe the height $h(t)$ in feet of the water balloon at t in seconds after release as

$$h(t) = -16(t+1)(t-4)$$

   a. What is the initial height of the water balloon?
   b. How many seconds will it take for the water balloon to hit the ground?

7. For each of the following find the x-intercepts if they exist. If there are no intercepts, write "no intercepts."
   a. $q(x) = x^2 + x - 30$
   b. $q(x) = x^2 - 12x + 20$
   c. $q(x) = x^2 - 6x + 15$
   d. $q(x) = x^2 + 10x + 10$

8. Write each of the following in vertex form. Find the vertex.
   a. $q(x) = x^2 - 16x + 64$
   b. $q(x) = x^2 + 8x$
   c. $q(x) = x^2 - 2x - 8$
   d. $q(x) = x^2 + 3x + 2$

9. Use the quadratic formula to solve.
   a. $2x^2 - 7x - 4 = 0$
   b. $x^2 + 6x - 1 = 0$
   c. $2x^2 - 1 = -3x$

10. For each of the following find the x-intercepts if they exist. If there are no intercepts, write "no intercepts."
   a. $q(x) = 2x^2 + 3x + 1$
   b. $q(x) = x^2 + 5x + 5$
   c. $q(x) = 3x^2 + 6x + 5$

MODELING 9

SECTION 9.1 Modeling Linear Data

In this section, we consider collecting data in pairs. Each data point represents the measurement of two variables on the same individual. For example, we could consider the height and weight of every one in your classroom. We call this collection of data in 2 variables a bivariate data set:

BIVARIATE DATA SET
Suppose n is a positive integer. A *bivariate data* set is a collection of pairs: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

To look at both variables at the same time we plot the pairs on a coordinate plane. In statistics, this is called a *scatter plot*.

Fitting a Line

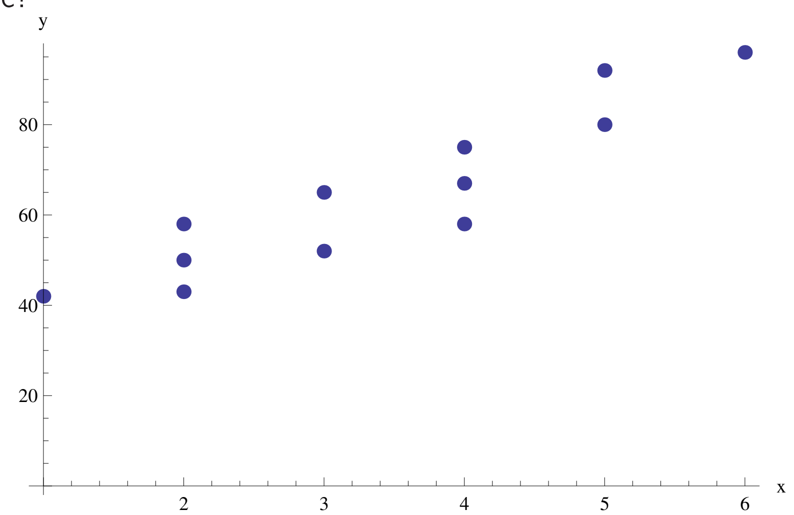
In many situations, we can see a pattern or trend in the scatter plot. We want to find a formula for a function that describes this pattern. This function can then be used to summarize the data, to predict the value of y for new values of x, and in some cases to help explain the underlying relationship between the two variables. Unlike what you have seen in previous chapters, though, the function may not fit the pattern perfectly. So we must try to find a function that fits the data as well as possible.

EXPLORATION 1

Ms. Sorto asks her students how many hours they studied for the final algebra test, x. She then records their grade on the test, y. The results are listed in the table below.

x	2	5	4	3	5	4	3	6	2	4	1	2
y	50	80	75	52	92	67	65	96	43	58	42	58

1. Below is a scatter plot of the data. Does the data fall exactly on a line?



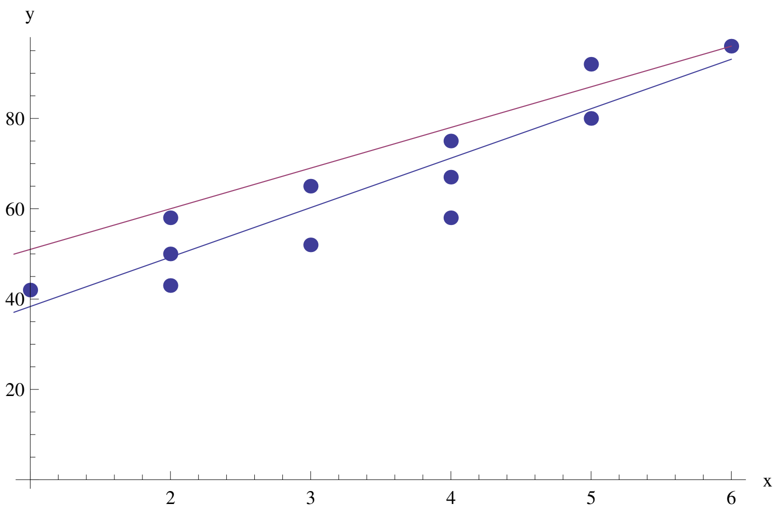
2. Draw a line on the graph which "fits" the graph. Explain how you chose your line.
3. Determine the equation of the line you drew. Be careful: note the scale on the plot and that the origin $(0, 0)$ is not displayed.
4. Write the equation of your line in slope-intercept form. Interpret

what the slope means in this context. What does the y-intercept mean?

5. Katie studied 200 minutes for the exam, but was sick on the day the exam was given. Use the your line to predict what score Katie would have gotten if she had taken the exam.

EXPLORATION 2

The graph below shows data from Exploration 1 and the graphs of two lines. Which line "fits" the data better? Explain. Is there something you could compute for each line to measure how well it fits?



How well a line fits the data is not well defined. Depending on the situation, this could be measured in different ways. However, in most situations we measure the "distance" between the line and the data using the squared distance between the y observation and the point on the line with the same x value.

EXPLORATION 3

The Least Square Mathematica Demo shows the data from Exploration 1, a movable line with slope m and intercept b, and some squares.

1. You can use the sliders to change the slope and intercept of the line. For instance, make the slope $m = 14$ and the y-intercept $b = 15$. Compute the y coordinate of the point on the line where $x = 5$. Find the point $(5, 80)$ on the scatter plot and compute the area of the square that touches this point.

2. The computer automatically computes the area for each square and shows the sum of the areas. This is called the *sum of squared error*. Change the y-intercept to $b = 16$. What happens to the sum of squared error? Explain why the smaller the sum of squared error is, the better the line fits.
3. Use the sliders to find the values of m and b that make the sum of squared error as small as possible. This is called the *best fit* or *least squares line*.

The Least Squares Line

LEAST SQUARES LINE

For a given bivariate data set: $(x_1, y_1), \ldots, (x_n, y_n)$, the *least squares line* is determined by choosing the slope m and y-intercept b, which makes the sum of squares:

$$(y_1 - (mx_1 + b))^2 + \cdots + (y_n - (mx_n + b))^2$$

as small as possible.

As you can imagine, the calculation to find the least squares line using the sliders as in Exploration 3 takes a long time. Fortunately, a formula can be found for the slope and intercept of the least squares and that formula can be easily computed by almost all scientific calculators. Each type of calculator has it own way to enter the data. Here we will give instructions for the TI-84 graphing calculator. If you have another calculator, the process will be similar, but the buttons may be very different.

EXPLORATION 4

1. Enter the data from Exploration 1 into the calculator. Use the following steps:
    - Press *STAT* key. Choose *ClrList*.
    - Enter in *ClrList L1, L2*

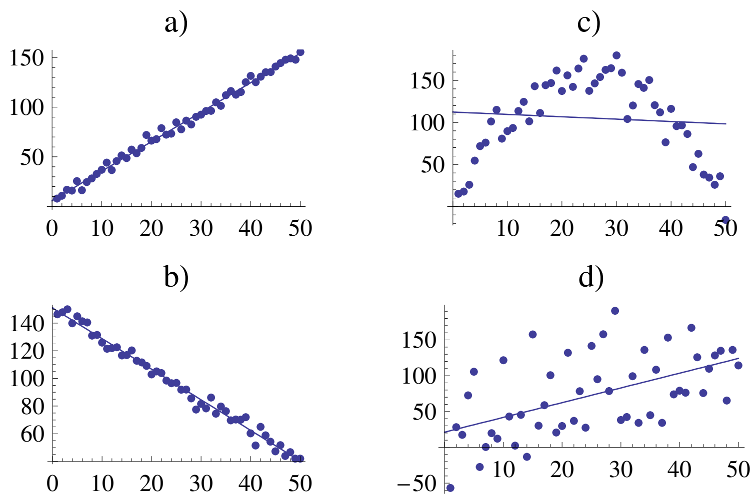
- Press *STAT* key. Choose *Edit*.
- Place the x data in *L1* and the y data in *L2*.
- Make sure you have one number per cell, and that L1 and L2 have the same number of cells filled.

2. Make a scatter plot of the data. Use the following steps:
   - Press *STAT PLOT* (this means Press *2ND Y=*). Choose *PlotsOff* to turn off any existing plots.
   - Press *STAT PLOT*. Choose *1: Plot 1*. Move blinking cursor to *On* and hit *ENTER* to turn plot on. Then move blinking cursor over the left most graph in the first row under *Type*, and hit enter to select the scatter plot type. Then make sure *Xlist* is set to *L1* and *YList* is set to *L2*.
   - Press *ZOOM*. Choose *9: ZoomStat*.
   - Press *TRACE* to see the data values.
3. Use your calculator to find and graph least squares line. Use the following steps:
   - Press STAT. Choose *4:LinReg(ax +b)* from the CALC menu.
   - Enter in *LinReg(ax+b) L1, L2*.
   - Write down the value for a and b. Watch out! The calculator uses the letter a for the slope instead of m.
   - Press Y=. Press CLEAR to remove any existing functions.
   - Move the blinking cursor to the right of $Y_1 =$.
   - Press *VARS*. Choose 5: Statistics. Choose *1: RegEQ* from the *EQ* menu. This should put the equation of the line in the Y= screen. It should match the slope and y-intercept you wrote down.
   - Press *GRAPH* to see the line and scatter plot together.
4. How close is the line computed by the calculator to the one you found in Exploration 3?

We use the least squares line for 2 main reasons. First, an equation for a line takes up a lot less space than a table or scatter plot. So the least squares line gives a brief summary of the data. Second, the line can be used to predict the y value for a new data point. For this reason, we often call y the response variable and x the predictor variable. We need to be careful, however. For any bivariate data set, we can use the calculator to find the least squares line, but not all data will look like a line. So the line could be a very bad summary of the data, and might make for very bad predictions.

PROBLEM 1

The figure below shows the scatter plots and least squares lines for four different bivariate data sets. For which of the data sets is the least square line a good summary of the data? Explain.



When the scatter plot of the data does not look like a line, as in problem 1c), you should not use the line to make good predictions of the y value. The *correlation coefficient*, r, can be used to determine how well the data fits a line. The formula for r is complicated, but fortunately we can have the calculator compute it and it is fairly easy to interpret. Here are some important properties of correlation coefficient:

- The value of r does not depend on the units of the variables. For example, if we converted the study time from minutes into seconds in the data set above, the correlation between studying and grade would not change.

- If $r > 0$ then the slope of the best fit line will be positive. So as x increases, y tends to increase.
- If $r < 0$ then the slope of the best fit line will be negative. So as x increases, y tends to decrease.
- If $r = 0$ then slope of the best fit line will be 0.
- If r is close to zero then the data is not close to a line. In this case, we say there is no correlation between x and y.
- The closer $|r|$ is to 1 then better the line fits the data. If $|r| = 1$ then the data points lie exactly on a line. If $|r|$ is close to 1, we say x and y are strongly correlated.

PROBLEM 2

Look at the graphs in Problem 1. Use the properties of r to match the scatter plot to the value of r: .07, -.96, .65, .99.

EXAMPLE 1

A student wanted to know the relationship between duration in seconds of a song and the amount of disk space in megabytes needed for the song. Using the YoTunes application, she gathered data for ten randomly selected songs shown in the table below.

x	141	144	177	200	248	248	301	295	299
y	4.5	5.3	6.1	6.9	8.3	8.6	9.7	9.8	9.9

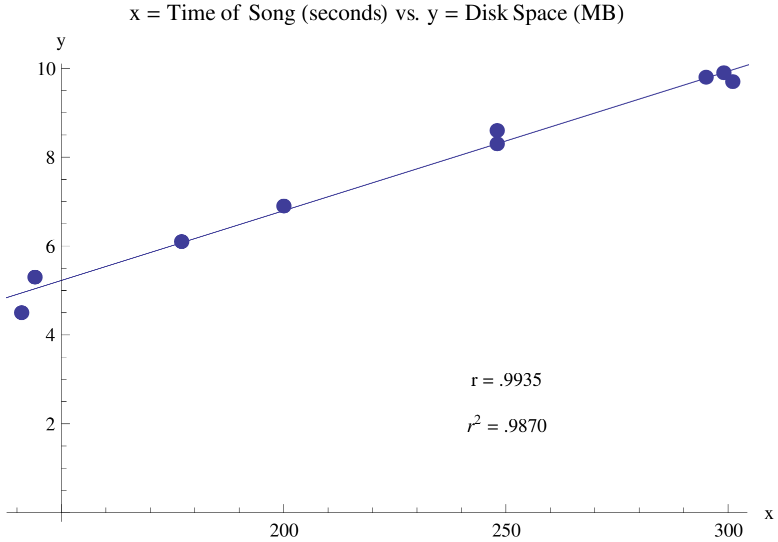
1. Use your calculator to make a scatter plot of the data, find the equation of the least squares line, and correlation coefficient.
2. Interpret the meaning of the slope and y-intercept.

3. What does the correlation coefficient tell you?
4. The song *Somebody that I used to know* by Gotye is 4 minutes and 4 seconds long. Predict the amount disk space needed to store this song.
5. John has 100 MB of disk space available, how many minutes of music do you predict he can store there?

SOLUTION

1. We can use the method described in Exploration 4 to enter the data and produce the scatter plot. Use your calculator to compute r. You only need to do the following steps once, and then it will report r every time you compute the least squares line.
   - Press CATALOG. Find *DiagnosticsOn*. Press *ENTER*. Press *ENTER* again.

   Now you can find the equation of the line as we did before:
   - Press *STAT*. Choose *4:LinReg(ax +b)* from the *CALC* menu.
   - Enter in *LinReg(ax+b) L1, L2*. This will give the least squares line again, but also report r and r^2 on the bottom.



2. The equation of the line is $y = .031x + .517$ and the correlation is $r = .9935$. The slope means that each extra second of music adds .031 MB of disk space. In this case, we see that the y-intercept does

not seem to make sense. It seems to say that a song which lasts 0 seconds still requires .517 MB of disk space. Why should this take any space at all? Often when the data that is observed is far from the origin, the calculated y-intercept can not be interpreted. Note that the shortest song in the data set is a 141 seconds long, so we really don't know what happens for very short songs.

3. Since the correlation coefficient is so close to 1, we see that the data fits the line very well.
4. The song by Gotye is $4 \cdot 60 + 4 = 244$ seconds long. So we predict that it will take $.031 \cdot 244 + .517 = 8.18$ MB of disk space.
5. To find out how much music fits on 100 MB, solve $100 = .031x + .517$ for x. Then $x = \frac{100-.517}{.031} = 3169$ seconds or 52 minutes and 49 seconds.

Even if the data does look like a line, you need to be careful about using the line to predict y values for a x value that is far away from the other data points.

EXPLORATION 5

Luis' parents gave him a puppy for his birthday. Each month Luis weighed his puppy in pounds. The results for the first 12 months are tabled below.

x: month	1	2	3	4	5	6
y: weight (lbs)	7.5	10.0	12.9	14.7	18.5	18. 6

x: month	7	8	9	10	11	12
y: weight (lbs)	21.0	22.8	23.8	23.9	27.0	25.9

1. Enter the data into the calculator, make a scatter plot and find the

equation of the least squares line. How well does the line fit the data?

2. Use the least squares line to the predict the weight of the dog in month 13 and in month 24.

Luis kept weighing the dog each month for another year. Here are the results:

x: month	13	14	15	16	17	18
y: weight (lbs)	29.0	29.6	30.0	30.3	30.5	31.1

x: month	19	20	21	22	23	24
y: weight (lbs)	31.3	31.4	31.5	31.7	32	32.4

3. How good was your prediction for month 13? How good was the prediction for month 24?
4. Add the data from month 13 to month 24 to the data set in your calculator. Make a scatter plot and find the equation of the least squares line for all 24 months of the data. What do you notice?
5. Luis wants to know how much his dog will weigh 10 years from now, or in month 144. Do you think you can use the least squares line from part 1 to predict the weight? How about the line from part 4? Explain.

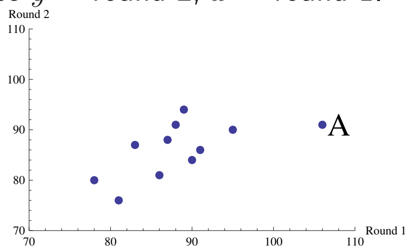
EXERCISES

1. A study of fast food measured the fat content (in grams) and the calories of 7 different types of hamburgers. Enter the data from the table below into your calculator.

x: fat(g)	19	31	34	35	39	39	43
y: Calories	920	1500	1310	860	1180	940	1260

a. Make a scatter plot of the data.
b. Find the equation of the least squares line. Graph it on the scatter plot. How well does the line fit the data?
c. A new hamburger has 36 grams of fat. How many calories do you predict the hamburger will have? How confident are you in this prediction?
d. A new restaurant chain, Burger Mean, offers a new low-fat hamburger with only 5 grams of fat. Do you think you could use this data and the least squares line to predict the number of calories? Why or why not?

2. The scatter plot below represents the round 1 and round 2 scores of 11 golfers. Suppose y = round 2, x = round 1.



a. Which of the following is most likely the formula for the least squares line? Explain.

$y = 46 + .45x$ $\qquad y = 46 + 4x$

$y = -46 - .45x$ $\qquad y = -46 - 4x$

b. Plot the least squares line you chose on the scatter plot.
c. If you remove point A from the data set, what will be the effect on the least squares line? Choose one option and explain:

Slope increases Slope Decreases Slope Unchanged

3. The ranges inhabited by the Indian gharial crocodile and the Australian saltwater crocodile overlap in Bangladesh. Suppose a large crocodile skeleton is found there, and we want to know which kind of crocodile it is. Wildlife scientists have measured lengths of the heads (x) and the complete bodies (y) of several crocs (in centimeters) of each species, and found the least squares line. Here are the results.

Australian Crocodile: $y = 7.7x - 20.2$, $r^2 = .98$
Indian Crocodile: $y = 7.4x - 69.4$, $r^2 = .97$

a. Based on the r^2 value, do the least squares lines seem to fit the data well?
b. What is the meaning of the slope in this context?
c. What is the meaning of the y-intercept in this context? Do you think you could use these lines to describe crocodiles whose length of head was close to 0? Explain.
d. Make a graph of both least squares line on the same coordinate plane with $0 < x < 100$. In what ways are the lines similar? In what ways are they different?
e. The crocodile skeleton found had a head length of 62 cm and a body length of 380 cm. Plot this point on the graph. If you had to guess, which kind of crocodile was this? Explain.

4. **Investigation:**
In this exercise we investigate what is the effect of changing a point on the least squares line. We start with a simple bivariate data set:

x	0	5	5	5	10
y	0	4	5	6	10

a. Enter the data into the calculator. Make a scatter plot and find the least squares line.
b. Change the point $(x_4, y_4) = (5, 6)$ to $(5, 10)$. Find the new least squares line. What do you notice?
c. Change the point (x_4, y_4) back to $(5, 6)$, but now change (x_5, y_5) from $(10, 10)$ to $(10, 14)$. Find the new least squares line. What do you notice?
d. Which changed the least squares line more?

5. Ten ninth graders with acne used a common home remedy: cooked oatmeal on their face. The table below shows their "acne score", as well as the number of minutes the oatmeal was applied.

x: minutes	10	20	20	20	20	0	40	25	15	15
y: acne score	7	5	6	5	4	9	1	4	5	6

a. Enter the data into the calculator. Make a scatter plot, find the least squares line and plot it on the scatter plot.

b. Suppose another ninth grader's acne was evaluated and we knew that he or she had used the oatmeal for 30 minutes. What would be your guess for this person's acne score? Use the least square line.

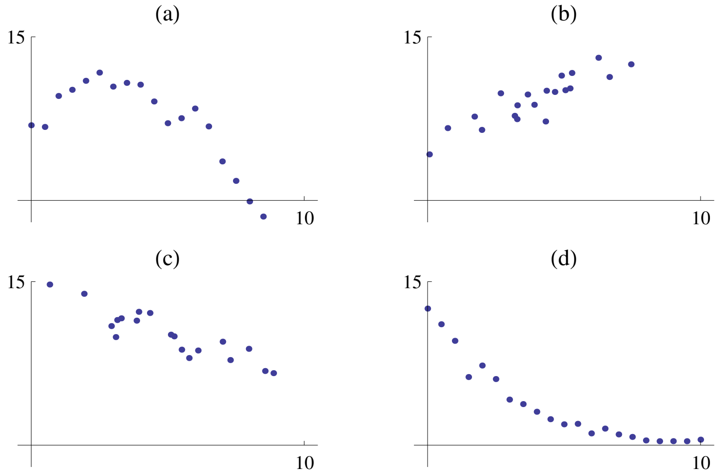
c. What is the y-intercept of the least squares line? What is its meaning in this case?

SECTION 9.2 Modeling Nonlinear Data

In Section 9.1, you used linear functions to describe the pattern in a scatter plot of data. Not all data exhibits a linear pattern, however. So let's explore fitting some nonlinear functions.

EXPLORATION 1

1. Think about the graphs of linear, quadratic and exponential functions. In general what do their graphs look like, how could you tell them apart?
2. The figure below shows the scatter plots for 4 different bivariate data sets. For each scatter plot, decide which type of function would fit the data the best. Explain.



EXAMPLE 1

Researchers studying how a car's fuel efficiency (y) varies with its speed (x) drove a compact car 200 miles at various speeds on a test track. The results are shown below:

i	1	2	3	4	5
x_i: Speed (mph)	35	40	45	50	55
y_i: Fuel Eff. (mpg)	25.9	27.7	28.5	29.5	29.2

i	6	7	8	9
x_i: Speed (mph)	60	65	70	75
y_i: Fuel Eff. (mpg)	27.4	26.4	24.2	22.8

1. Make a scatter plot and find and plot the least squares line. How well does the line fit the data?
2. In the scatter plot you see that shape of the data looks like an upside down "U". Find the least squares line when you only use the data points $i = 5, 6, 7, 8, 9$. Plot this new line on the scatter plot. Describe the fit of the line to the data.
3. Use your calculator to fit a quadratic function to the entire data set. Plot this on the scatter plot. Compare the fit to the 2 lines you have already found.
4. Use the quadratic function to find the speed where you expect to get the greatest fuel efficiency.

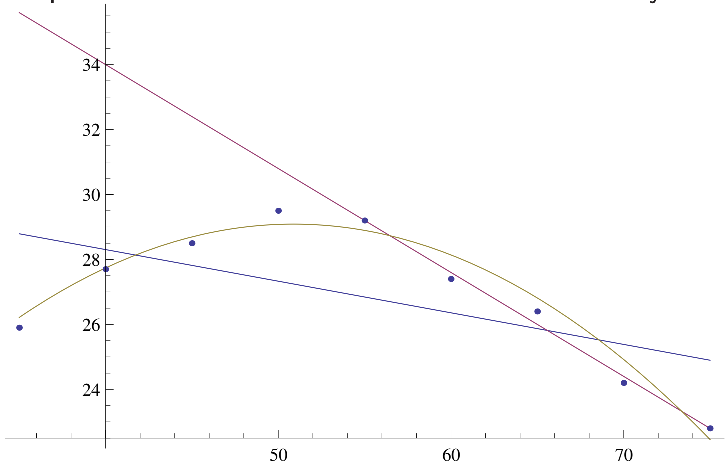
SOLUTION

1. The least squares line is $y = -.1x + 46.8$. Looking at the scatter plot, you can see that the line does not fit well. This explains why the correlation is $r = -.59$ is not particularly close to -1.
2. Now, redo the calculation of the least squares line, but only use the last 5 data points. This can be done by entering these points into new lists in your calculator. The resulting line is $y = -.32x + 46.8$.

The graph below shows that this fits one portion of the data fairly well, but it is way off for the low speeds.

3. Make sure the speed data is entered in the calculator in *L1* and the fuel efficiency data in *L2*. Use the calculator to find the quadratic function which best fits the data using the following steps:
   - Press *STAT*. Choose *5: QuadReg* from the *CALC* menu.
   - Enter in *QuadReg L1, L2*.

   This gives the quadratic function $q(x) = -0.42 + 1.16x - 0.011x^2$. Now you can graph $q(x)$ and compare to the graphs above. We see that the quadratic function fits the overall data set fairly well.



4. Notice that $q(x)$ is a downward facing parabola. So, the vertex tells us about the maximum possible fuel efficiency. In Chapter 8, we saw that the x-coordinate of the vertex for the general parabola $p(x) = ax^2 + bx + c$ is $-\frac{b}{2a}$. In this case, the vertex occurs when the speed $= -\frac{1.16}{2\cdot(-.011)} = 52.72$ mph. Substituting this value into $q(x)$, you find that $-.011 \cdot 52.72^2 + 1.16 \cdot 52.72 - .42 = 30.16$ mpg, or the maximum fuel efficiency.

EXPLORATION 2

The technology used to produce computer hard drives has improved over the years. For this reason, the price per gigabyte of storage space has fallen rapidly. The table below shows the price per gigabyte and the year sold for 10 hard drives.

year	2000	2001	2002	2003	2004
x	0	1	2	3	4
y	19.75	7.3	4.31	2.58	1.94

year	2005	2006	2007	2008	2009
x	5	6	7	8	9
y	0.75	0.5	0.4	0.27	0.07

1. Let $x = year - 2000$. So that $x = 0, 1, 2, \ldots, 9$. Let $y =$ price per gigabyte. Make a scatter plot of the data. Does the graph look linear, quadratic or exponential? Explain.
2. Use your calculator to fit a line to the data. Plot the line on the scatter plot. How good is the fit?
3. Use your calculator to fit a quadratic function to the data. Plot the function on the scatter plot. Which fit is better: the line or the quadratic function?
4. Consider fitting an exponential function of the form $f(x) = ab^x$. Explain why $0 < b < 1$ for this data.
   - Press *STAT*. Choose *0: ExpReg* from the *CALC* menu.
   - Enter in *ExpReg L1, L2*.
5. What does the value of a represent in this context?
6. What does the value of b represent in this context?
7. You have now fit 3 different functions to the data. Which do you think fits the data best?
8. 2009 is the last year in the data set. Use all 3 functions you fit to predict the price per gigabyte this year. What do you notice? If possible, find the current price per gigabyte, compare it to your predictions.

PROBLEM 1

The web site boxofficemojo.com reports how much movies earn at the box office. The table below shows the weekend gross box office for the top grossing film (in millions of dollars) for 1990 through 2011.

1. Make a scatter plot of the data.
2. Consider fitting an exponential function of the form $f(x) = ab^x$. Explain why $b > 1$ for this data.
3. Use you calculator to fit an exponential model. Use $x = year - 1990$. Graph the fitted model on the scatter plot of the data.
4. Use the model to predict the highest weekend gross box office for this year. Based on how well the model fits the existing data, how confident are you in the prediction?

Year	Film	Gross
2011	Harry Potter and the Deathly Hallows 2	169
2010	Toy Story 3	110
2009	Avatar	77
2008	Dark Knight	158
2007	Spider Man 3	151
2006	Pirates of the Caribbean: Dead Man's Chest	135
2005	Star Wars III	108
2004	Shrek 2	108
2003	Return of the King	73
2002	Spider Man	115
2001	Harry Potter and the Sorcerer's Stone	90
2000	The Grinch	55
1999	Star Wars I	65
1998	Saving Private Ryan	31
1997	Titanic	29
1996	Independence Day	50
1995	Toy Story	30
1994	Forrest Gump	24
1993	Jurassic Park	47
1992	Aladdin	19
1991	Terminator 2	32
1990	Home Alone	17

EXERCISES

1. The table below lists the total estimated numbers of AIDS cases, by year of diagnosis from 1999 to 2003 in the U.S.

Year	AIDS Cases*
1999	41356
2000	41267
2001	40833
2002	41829
2003	43171

* Source: US Dept. of Health and Human Services, Centers for Disease Control and Prevention, HIV/AIDS Surveillance, 2003.

   a. Make a scatter plot of the data.
   b. Researchers examined the data and fit a quadratic model. Look at the shape of the data in a scatter plot, explain why a quadratic model might fit the data.
   c. Fit a quadratic model.
   d. Use the fitted model to predict the number of AIDS cases in 2005.

2. Methylphenidate hydrochloride is a commonly used drug to treat attention deficit disorder in children ages 6 to 12. If child takes a 10 mg tablet, the drug takes a while to build up in the blood stream, reaches a peak concentration and then starts to decrease and the effect wears off. In order to study the right dose to recommend, manufacturers model this process by collecting data from lab experiments. The table below represents the concentration in the blood measured in micrograms per liter $\frac{\mu g}{L}$ for time in hours after taking a tablet from one experiment.

x: time is hours	0	1	2	3	4
y: drug concentration $\frac{\mu g}{L}$	0	8	11	7	2

   a. Make a scatter plot of the data.
   b. Look at the shape of the data in a scatter plot, explain why a quadratic model might fit the data.

c. Fit a quadratic model, $q(x)$. Plot $q(x)$ on the scatter plot.
d. Find the vertex of $q(x)$. When is the concentration of the drug largest? How many micrograms per liter is this?
e. As the drug wears offs the concentration decreases. Use $q(x)$ to predict the time when the concentration will reach 0 again. Explain why $q(x)$ can not be used to model the drug concentration after this time.

3. Ms. Eusebi likes to drink hot coffee. Every morning she brews a nice hot cup of coffee. However, she gets distracted by her students and by the time she takes her first sip, the coffee is no longer hot. To help her out, her class decided to measure the temperature of the coffee over time. The results of their experiment are shown in the table below. $x =$ time in minutes, and $y =$ temperature of coffee in degrees Fahrenheit.

x	0	1	2	3	4	5	6	7	8
y	104.7	102.7	98.8	96.8	95.1	93.3	91.6	89.7	88.4

x	9	10	11	12	13	14	15
y	86.5	85.	83.5	82.2	80.7	79.4	77.9

a. Make a scatter plot of the data.
b. Based on the shape of the data, explain why an exponential model might fit well. Will the base b be less than 1 or greater than 1? Explain.
c. Fit an exponential model. Plot the fitted model.
d. Ms. Eusebi likes her coffee to be at least 95$^\circ F$. Use the graph to estimate what is the longest time she can wait?

SECTION 9.3 CHAPTER REVIEW

Key Terms

bivariate	correlation coefficient
least squares line	sum of square error
exponential model	quadratic model

Practice Problems

1. The table below shows the median height for girls according to their age:

x_i: Age (yr)	2	3	4	5	6
y_i: Median Height (ft)	35.1	38.7	41.3	44.1	46.5

x_i: Age (yr)	7	8	9	10
y_i: Median Height (ft)	48.6	51.7	53.7	56.1

a. Make a scatter plot and find and plot the least squares line. How well does the line fit the data?
b. The typical age for eighth graders is 14 years old. Use the least squares line to predict the median height for 14 year old girls. Do you think this prediction will be correct? Explain.

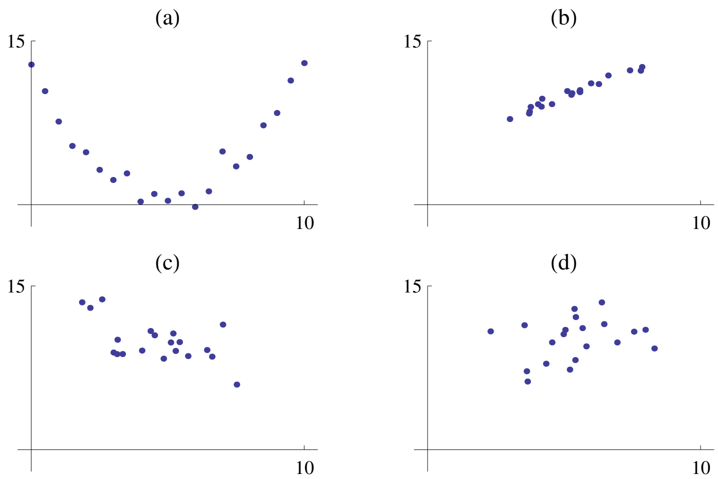
2. Darryl wants to buy a used truck. He goes on the internet to find recommended prices for Ford F-150's. He records x = the age in years and y = the recommended price in \$. Below is table of the results.

x: age	7	6	5	4	3	2	1
y: price	8930	10291	15402	17212	17563	19229	21451

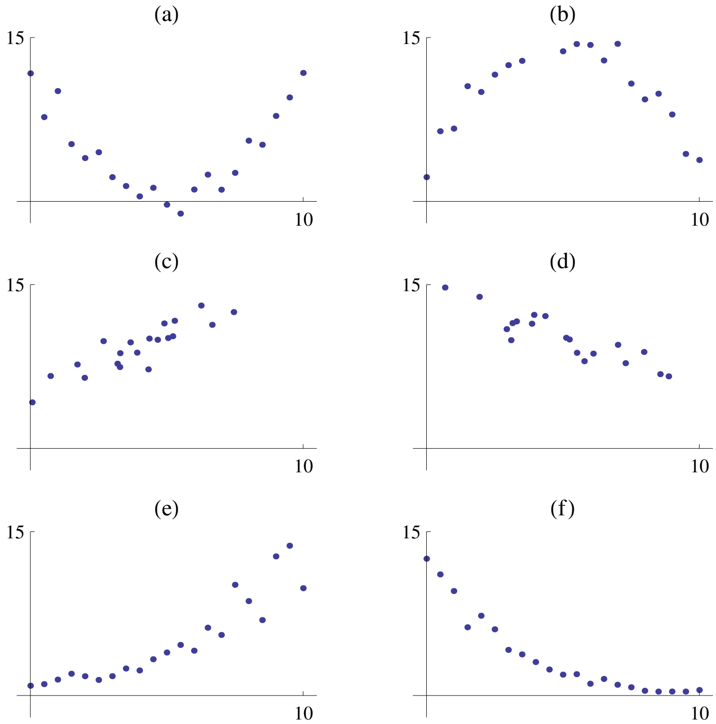
a. The least squares line for this data is $y = -2060x + 23954$. What does the slope mean in this context?
b. What does the intercept mean?
c. Darryl goes to the car lot and finds an 8-year old Ford 150 on sale for \$5000. According to the data, does this seem to be a

good deal? Explain. What other factors should Darryl check into before buying the car?

3. Consider the graphs below. Use the properties of the correlation coefficient to match the scatter plot to the value of r: $-.58$, $.06$, $.27$, $.98$.



4. Consider the graphs below.



Match the each graph to one of the following fitted functions:

i. $f(x) = 1.1x + 5.1$

ii. $f(x) = 12.9\left(\frac{1}{1.4}\right)^x$

iii. $f(x) = -.9x + 14.8$

iv. $f(x) = 1.2 \cdot 1.3^x$

v. $f(x) = -.5x^2 + 4.7x + 3.2$

vi. $f(x) = .5x^2 - 4.6x + 11.8$

5. The table below shows stopping distances in feet for a car tested 3 times at each of 5 speeds. We hope to create a model that predicts y = the stopping distance in feet from y = the speed in miles per hour.

x: Speed in mph	y: Stopping Distance in ft
20	63, 60, 61
30	114, 120, 104
40	152, 170, 164
50	230, 205, 238
60	317, 321, 270

a. Make a scatter plot of the data.

b. Fit a linear, quadratic and exponential model. Which model fits the best? Explain.

c. The fastest speed limit in the United States is 80 mph. Predict the stopping distance at that speed using the best fitting model.

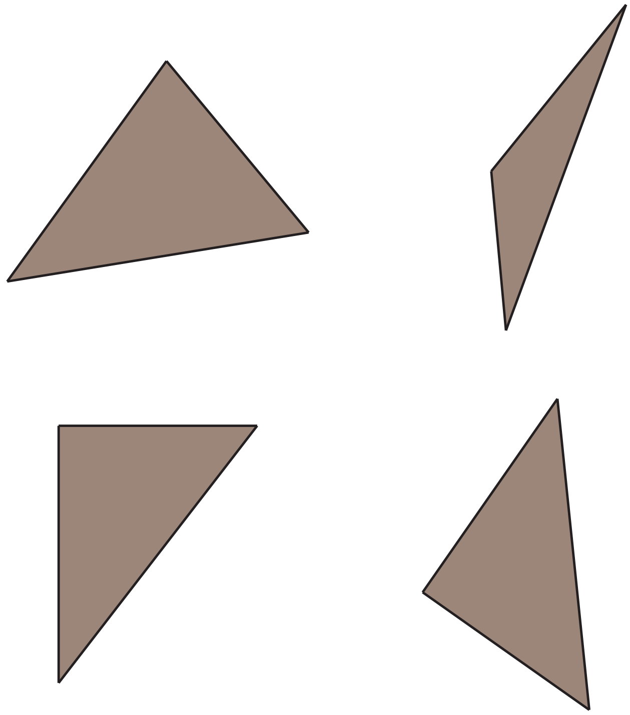
GEOMETRY 10

SECTION 10.1 THE PYTHAGOREAN THEOREM

In this section we will discuss one of the most important properties of right triangles. In the first exploration, we review one way to categorize triangles by their largest angle.

EXPLORATION 1

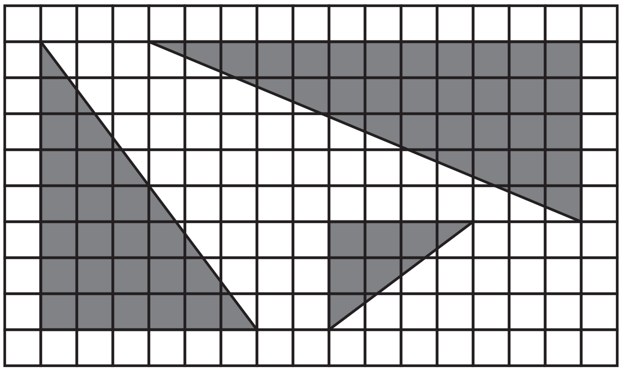
Look at the triangles in the figure below. For each triangle, circle the largest angle and then label the triangle as either acute, obtuse or right.



This section will focus on one property of the right triangles called the Pythagorean Theorem. First, however, let's review more terminology associated with right triangles.

EXPLORATION 2

In *right triangles*, the longest side is called the *hypotenuse* and the other 2 sides are called *legs*. We often label one leg the *base* and one the *height*. Look at the right triangles in the figure below. Using a piece of paper, create a ruler with equally spaced marks matching the grid given below. Use this ruler to measure the lengths of the sides.



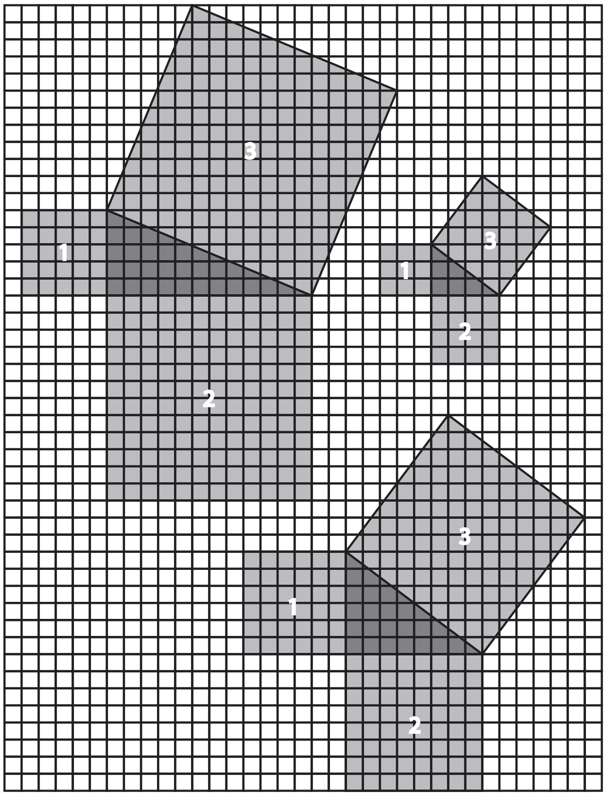
For each triangle:

1. Label the base, the height and the hypotenuse. Is there more than one way to do this? Explain.
2. Find the lengths of the base and the height. Then find the area.
3. To find the perimeter we need to know the lengths of all 3 sides. Use a ruler to find the length of the hypotenuse. Then find the perimeter.

In the next exploration we look for another way to find the lengths of the sides of a triangle without using a ruler.

EXPLORATION 3

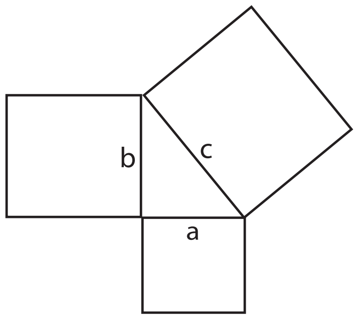
In the following diagram, squares are centered on each side of the triangles. Using a piece of paper, create a ruler with equally-spaced marks matching the grid given below. Use this ruler to measure the lengths of the sides. Fill out the table below to record the lengths of the sides and the areas of the attached squares. Look for a pattern involving the lengths of the sides of right triangles.



Base	Height	Hypotenuse	Area of Square 1	Area of Square 2	Area of Square 3

EXAMPLE 1

Looking at the triangle below, what is the area of each square?



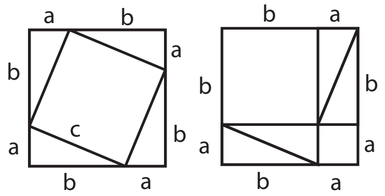
SOLUTION The area of the square with side length a is $a \cdot a = a^2$. The areas of the other 2 squares are b^2 and c^2.

The area of the square attached to the hypotenuse is equal to sum of the areas of the other 2 squares. This is usually written $a^2 + b^2 = c^2$, where c is the length of the hypotenuse, and a and b are the lengths of the legs. Check that this formula works for each of the right triangles in the exploration.

You discovered the pattern by looking at examples and computing areas. But how do you know that this is always true for any right triangle with legs of length a and b and hypotenuse of length c? There are many different proofs of this ancient theorem. Let's explore one of the most beautiful picture proofs.

EXPLORATION 4

In each picture, there are four copies of the original right triangle, with side lengths a, b, c.



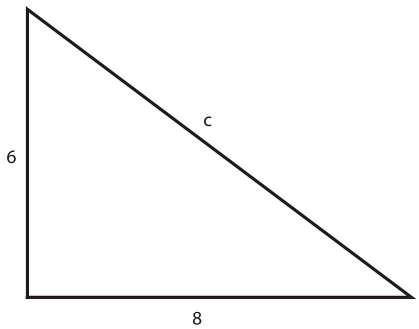
Each of the diagrams represents a square with side length $a + b$. On the left, there are 4 triangles and one square with side length c. On the right, there are the same 4 triangles and 2 squares, with side lengths a and b.

1. Use colors to match the corresponding triangles in the 2 diagrams.
2. Discuss in a group, why the diagrams prove that the area of the large square is the sum of the areas of the 2 smaller squares?

The result is the **Pythagorean Theorem**.

THEOREM 10.6: PYTHAGOREAN THEOREM
If a and b are the *lengths* of the legs of a *right triangle* and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$

EXAMPLE 2



Find c, the length of the hypotenuse.

SOLUTION By the Pythagorean Theorem,

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100$$

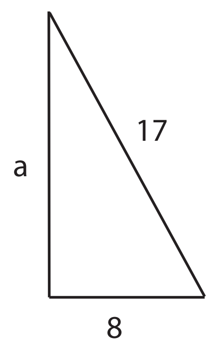
If $c^2 = 100$, what is the value of c? Explain why c must be 10.

In some cases, we know the hypotenuse and we use the Pythagorean Theorem to find the length of the legs.

EXAMPLE 3

A right triangle has hypotenuse 17 inches long, and one leg 8 inches long. Sketch and label the triangle. Find the length of the other leg. What are the area and perimeter of the triangle?

SOLUTION When a figure of the triangle is not provided, it is always a good idea to make a sketch.



From the information given, we know the length of the hypotenuse and one leg. We label the missing length a. By the Pythagorean Theorem:

$$a^2 + 8^2 = 17^2.$$

We must compute the squares, $8^2 = 64$, and $17^2 = 289$.

So we have $a^2 + 64 = 289$. From this we see that $a^2 = 289 - 64 = 225$. Since $15^2 = 225$, we know that $a = 15$.

PROBLEM 1

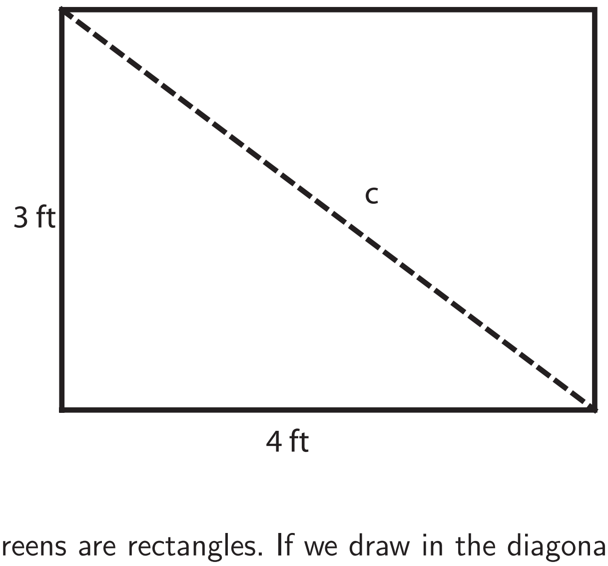
For each of the following, the lengths of 2 sides of a right triangle are given. Find the third side.

1. The lengths of the legs of the right triangle are 3 and 4.
2. The length of the hypotenuse is 13 and one of the legs is 12.

EXAMPLE 4

The "size" of a television screen is determined by the length of the diagonal of the screen. Suppose your television screen is 3 feet by 4 feet, what is the "size" of the screen?

SOLUTION



Television screens are rectangles. If we draw in the diagonal we see that this forms 2 right triangles whose hypotenuse is the diagonal. So we can use the Pythagorean Theorem to find the size of the television screen:

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25.$$

So the "size" of the screen is 5 feet or 60 inches.

The lengths of the sides of the right triangle in Example 4 were 3, 4 and 5. Can you construct a triangle with these lengths that is not a right triangle? The answer is no. If a triangle has 3 sides of length a, b and c and $a^2 + b^2 = c^2$, then the triangle is a right triangle. This is called the *Converse of the Pythagorean Theorem*.

PROBLEM 2

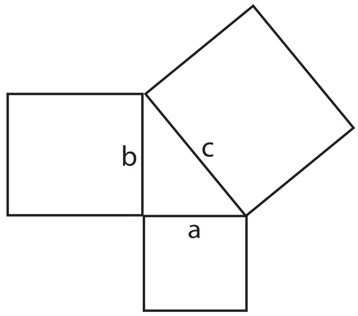
Use the Converse of the Pythagorean Theorem to determine if the following triangles are right triangles.

1. A triangle with sides of length 9, 12, 15.
2. A triangle with sides of length 3, 4, 6.
3. A triangle with sides of length 1.5, 2, 2.5.

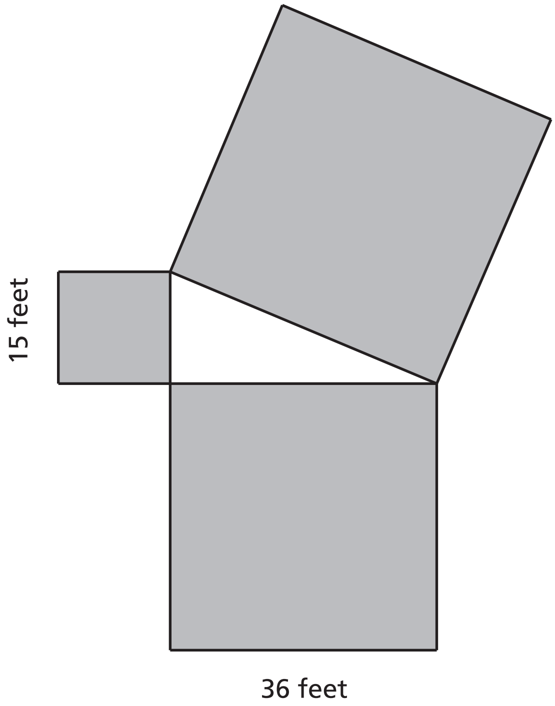
EXERCISES

1. A right triangle has a hypotenuse 10 inches long, and one leg 6 inches long. Sketch and label the triangle. Find the length of the other leg. What are the area and perimeter of the triangle?
2. A right triangle has a hypotenuse 17 inches long, and one leg 15 inches long. Sketch and label the triangle. Find the length of the other leg. What are the area and perimeter of the triangle?
3. A right triangle has a hypotenuse 13 inches long, and one leg 5 inches long. Sketch and label the triangle. Find the length of the other leg. What are the area and perimeter of the triangle?
4. Using a ruler draw a right triangle with legs of length 5 cm and 12 cm. Measure the length of the hypotenuse, then calculate the exact length of the hypotenuse.

5. Draw a grid on each of the squares in the figure below. You can use any grid unit you like. Use the grid to estimate the area of each of the three squares. Do your area estimates agree with the Pythagorean Theorem. Explain.

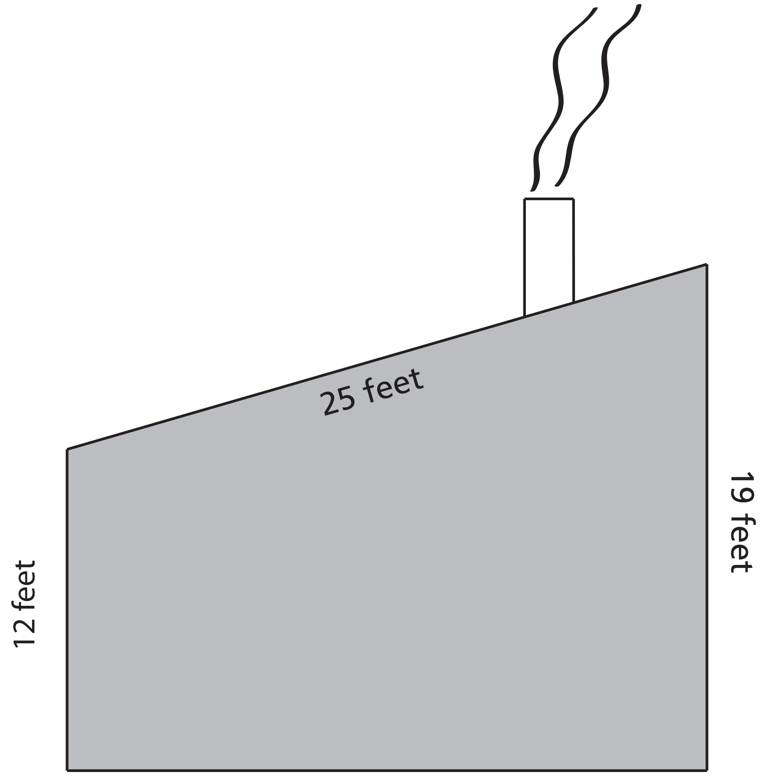


6. Use the Converse of the Pythagorean Theorem to determine if the following triangles are right triangles:
   a. A triangle with sides of length 6, 12, 15.
   b. A triangle with sides of length 10, 6, 8.
   c. A triangle with sides of length $\frac{9}{2}$, 6, $\frac{15}{2}$.
7. Television sets are sold in many different sizes. For "standard" sets the ratio between the width and the height of the screen is 4 : 3. The "size" of a television screens is determined by the length of the diagonal of the screen. Suppose Michael wants to buy a standard screen that is 12 inches wide. How tall will the screen be? What size screen is it?
8. Jessica wants to build 3 square barns around a triangular yard. The yard is a right triangle, with the right angle formed by sides of length 15 feet and 36 feet (see the diagram below). What will the area of the largest barn be?



9. **Investigation:**
In this investigation you will make cut outs that match the diagrams in Exploration 4. Measure the dimensions carefully and then cut out the triangles and squares of each diagram. If done correctly you should be able to arrange the pieces of one diagram on the pieces of the other. To check that it works for different values of a, b and c. Choose a value $a > 0$. Then make a version of the diagrams where $b = a$ and one where $b = 2a$. Be prepared to explain how the cut outs relate to the Pythagorean Theorem.

10. Matthew wants to paint the side of his house. To do this, he needs to know the area. He knows that the front of his house is 12 feet tall, and the back of his house is 19 feet tall. The length of his roof, from the front of the house to the back of the house, is 25 feet. (See the picture below.) What is the area of the side of the house?



11. **Investigation:**
A *Pythagorean triple* is an ordered triple (a, b, c) of positive integers such that $a \leq b \leq c$ and a, b, and c are the lengths of the 3 sides of a right triangle. That is, (a, b, c) is a Pythagorean triple if and only if $a < b < c$ and $a^2 + b^2 = c^2$. In this section, you encountered several Pythagorean triples, including $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$ and $(8, 15, 17)$.
   a. Find the value of c for each of the following Pythagorean triples: $(9, 40, c)$ and $(11, 60, c)$.
   b. What do you notice about the first number in each ordered triple? What do you notice about the second and third numbers in each ordered triple? (Hint: Try adding the second and third numbers together.)
   c. Find values of b and c for each of the following Pythagorean triples: $(13, b, c)$ and $(15, b, c)$. Can you find more than one Pythagorean triple of the form $(15, b, c)$?

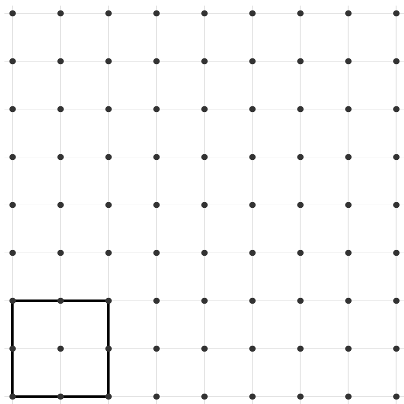
SECTION 10.2 SQUARE ROOTS

In Chapter 8 you used square roots to solve some quadratic equations. In this chapter, you will investigate the square root function in more depth.

First let's explore the meaning of $\sqrt{2}$.

EXPLORATION 1

The figure below represents a square with area equal to 4 square units on a geoboard. Can you see how to determine the area? What is the length of a side of the square?



1. Using a geoboard or drawing on graph paper, create a square with area equal to 9 square units. What is the length of the side?
2. Create a square with each of the following areas: 16, 25, 36. Make a table with the area of the squares and the lengths of the sides of the different squares. What do you notice about the numbers?
3. One can create more squares than it first appears. We have already created squares with area equal to $4, 9, 16, 25, \ldots$. Can you create squares with smaller areas, like 2 or 5? What do you think the length of the side of this square is?
4. If you haven't already, create a square with area 2. What name

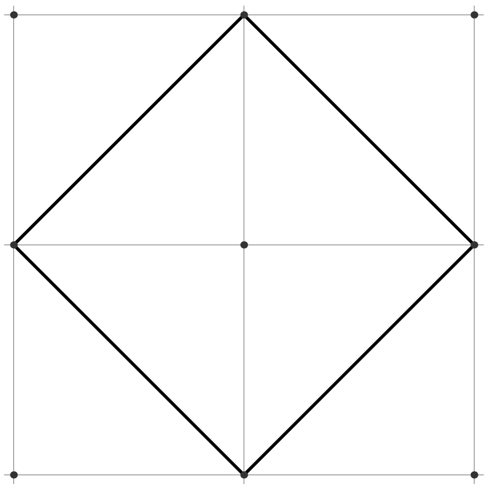
would you give to the length of the side?

5. On the geoboard, create squares with as many different areas as you can. For each square you find, label the length of the side.

In the next exploration, we look a little deeper at the meaning of the square root.

EXPLORATION 2

The figure below represents a square on a geoboard (or part of a geoboard).

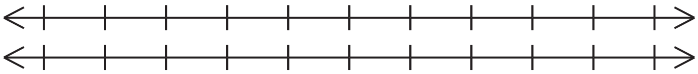


1. Explain why the area of the square is 2 square units.
2. Explain why the length of a side of the square is $\sqrt{2}$ units.
3. If we let $c = \sqrt{2}$ then $c^2 = 2$, but what is the value of c exactly? Explain why c must be bigger than 1. Explain why c must be smaller than 2.
4. Using just the multiplication key on the calculator try to find an estimate of c. Start with numbers with one digit to the right of the decimal. Then try two digits to right of the decimal. Remember you want c^2 to be close to 2. Put a star on the number line below where

you think c should be.

5. Try to zoom in by adding more decimals. Finish labeling the number line below and put a star where you think the value of c is.

6. Keep on zooming. Label the number lines below (each one should have more decimal places shown) and put a star where you think c is.



You might be asking, "does this zooming ever stop?" The surprising answer is no! On your calculator you may find a number that seems to work, but this is due to rounding and the limits on the calculator. Even your calculator can only do so many decimals. Computers can compute with many more decimals. For example, using a computer we can find an estimate with 35 digits $c = 1.4142135623730950488016887242096980$. But still c^2 is very very very close to 2 but not **exactly equal** to 2. Another interesting thing, is that unlike numbers like $\frac{1}{3} = .3\overline{3}$, there is not even a pattern that repeats. Numbers whose decimal representations never end and do not have a repeating pattern are called *irrational* numbers. It can be shown that these numbers can not be written as fractions with whole numbers in the numerator and denominator. But we can find very close approximations to them. The square root key on your calculator uses a special algorithm to find the best approximation up to the number of decimals your calculator can handle. If we combine the set of rational numbers with the set of irrational numbers, we get the set of all *real numbers*.

PROBLEM 1

Use the $\sqrt{}$ key on your calculator to estimate the following numbers. Write down the answer up to 4 decimal places. Then multiply the number you wrote down by itself to see how it differs from the exact value.

1. $\sqrt{5}$
2. $\sqrt{8}$
3. $\sqrt{12}$
4. $\sqrt{18}$

Historical Note: Pythagoras' Student

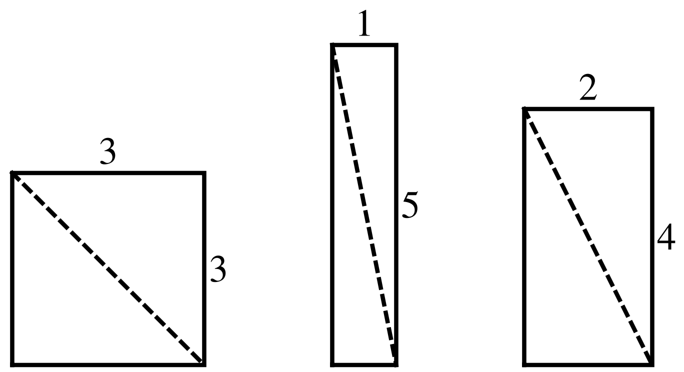
Source: http://www.mathsisfun.com/rational-numbers.html

According to legend, the ancient Greek mathematician Pythagoras (and his followers the Pythagoreans) believed that all numbers were rational (could be written as a fraction), but one of his students Hippasus proved using geometry that you could not represent the square root of 2 as a fraction, and it was therefore irrational.

However, Pythagoras could not accept the existence of irrational numbers, because he believed that all numbers had perfect values. But he could not disprove Hippasus' "irrational numbers," and so he had Hippasus thrown overboard and drowned!

EXAMPLE 1

Use the length of the diagonal to order the rectangles on the next page from smallest to largest.



SOLUTION Use the Pythagorean Theorem to determine the length of the diagonal d for each rectangle.

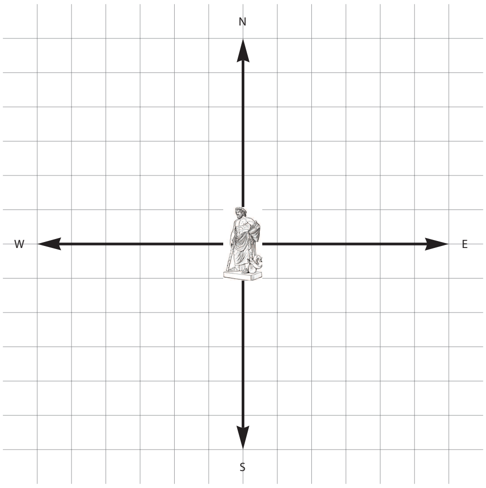
Rectangle 1: $d = \sqrt{3^2 + 3^2} = \sqrt{9}$.
Rectangle 2: $d = \sqrt{1^2 + 5^2} = \sqrt{26}$.
Rectangle 3: $d = \sqrt{2^2 + 4^2} = \sqrt{20}$.

So in terms of the length of the diagonal, $Rectangle1 < Rectangle3 < Rectangle2$.

EXPLORATION 3

Pythagorapolis is city designed by a mathematician. All the streets are laid out in a grid. Half of the streets go east-west, the other half go north-south. At the center of town, they have tall statue of Pythagoras. See the figure on the next page. Four mathematicians decide to explore the town. All four start at the center of town and then:

- Archimedes walks three blocks east and four blocks north.
- Bernoulli walks two blocks south and three blocks west.
- Cantor walks one block west and five blocks north.
- Descartes walks six blocks south and one block east.



1. Draw the path of each mathematician on the grid.
2. Put the mathematicians in order from the one who walked the least to the one who walked furthest.
3. Put the mathematicians in order according to the distance (as the crow flies) of their final location to the center of town.
4. How are the two orders different? Why did this happen?

We now explore how square roots behave.

EXPLORATION 4

Fill in the following table. What relationships do you notice between $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{ab}$?

a	b	ab	$\sqrt{a}$	$\sqrt{b}$	$\sqrt{ab}$
1	4	$1 \cdot 4 = 4$	$\sqrt{1} = 1$	$\sqrt{4} = 2$	$\sqrt{1 \cdot 4} = \sqrt{4} = 2$
4	16				
4		36			
			2		10

THEOREM 10.7: PRODUCT OF RADICALS PROPERTY
If $a > 0$ and $b > 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

PROBLEM 2

Use the $\sqrt{\quad}$ key on your calculator to check that the property from Theorem 10.7 really works for the following examples.

1. $\sqrt{10} = \sqrt{2} \cdot \sqrt{5}$
2. $\sqrt{14} = \sqrt{2} \cdot \sqrt{7}$
3. $\sqrt{12} = \sqrt{4} \cdot \sqrt{3}$
4. $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$

Theorem 10.7 can sometimes be used to *simplify* square roots. Simplifying a square root means finding an equivalent expression with a smaller number under the square root sign. In order to do this, we want to write the number under the square root sign as a product of factors where one of the factors a or b is a perfect square. A *perfect square* is a number whose square root is a whole number. For example, 9 is a perfect square because $\sqrt{9} = 3$.

EXPLORATION 5

1. Fill in the table below. In the second column, write the number from the first column as a product of two factors where one of the factors is a perfect square that is greater than 1. Note we filled in the first row for you.

Number	Product
18	$9 \cdot 2 = 3^2 \cdot 2$
12	
8	
75	

Now we are ready to simplify some square roots.

EXAMPLE 2

Simplify $\sqrt{18}$. Verify your answer with a calculator.

SOLUTION Simplifying a square root means to use Theorem 10.7 to make the number under the square root sign smaller. To do this we want to find a factor that is a perfect square. In this case 9 is a factor of 18, so by Theorem 10.7

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$$

So we have reduced the number under the square root from 18 to 2. We can use a calculator to check that 18 and $3\sqrt{2}$ are the same:

$$\sqrt{18} \approx 4.24264$$
$$3\sqrt{2} \approx 4.24264$$

PROBLEM 3

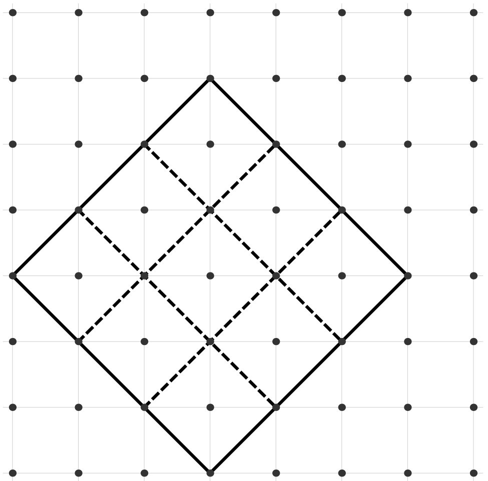
Simplify each of the following square roots. Verify your answer using the $\sqrt{\quad}$ key on your calculator.

1. $\sqrt{12}$
2. $\sqrt{8}$
3. $\sqrt{75}$

In Exploration 1 we thought of $\sqrt{18}$ as the length of a side of a square with area 18. Let's use this idea to see why $\sqrt{18} = 3\sqrt{2}$?

EXPLORATION 6

The figure below represents a square whose area is 18 square units outlined on a geoboard.



1. Verify that the big square has area 18. Hence, the length of a side of the big square is $\sqrt{18}$.
2. The big square has been divided into 9 equal smaller squares. What is the area of each of these smaller squares?
3. What is the length of the sides of each of the smaller squares?
4. Explain why this means that $\sqrt{18}$ must be the same as $3\sqrt{2}$.

Not all square roots can be simplified using Theorem 10.7.

EXAMPLE 3

Explain why $\sqrt{30}$ can not be simplified.

SOLUTION The factors of 30 are $\{1, 2, 3, 5, 6, 10, 15, 30\}$. Only 1 is a perfect square. But this does not help us simplify, because if we try to use Theorem 10.7 we get:

$$\sqrt{30} = \sqrt{1 \cdot 30} = \sqrt{1} \cdot \sqrt{30} = 1 \cdot \sqrt{30} = \sqrt{30}$$

which is right where we started. We can only simplify a square root if the number under the square root has a factor that is a perfect square greater than 1.

Theorem 10.7 is a nice tool to simplify square roots. You might wonder if there is another way. What if a number is the sum of perfect squares? Let's explore this possibility.

EXPLORATION 7

1. Fill in the following table. Use the $\sqrt{}$ key on your calculator to get approximate values when necessary. What relationships do you notice between $\sqrt{a+b}$, $\sqrt{a}+\sqrt{b}$?

a	b	$a+b$	$\sqrt{a}$	$\sqrt{b}$	$\sqrt{a+b}$
25	144	$25+144=169$	$\sqrt{25}=5$	$\sqrt{144}=12$	$\sqrt{25+144}=\sqrt{169}=13$
9	16				
36	64				
4	9				

2. Looking at the numbers in the table, do you think that $\sqrt{a+b} < \sqrt{a}+\sqrt{b}$, $\sqrt{a+b} > \sqrt{a}+\sqrt{b}$, $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ or does it depend?

EXERCISES

1. Draw a number line. Using a calculator and the same technique as in Exploration 2, estimate the following square roots to 2 decimal places. Locate your estimate in the number line. Then use the $\sqrt{}$ key on the calculator to check your answer.
   a. $\sqrt{3}$
   b. $\sqrt{10}$
   c. $\sqrt{49}$
2. In Exploration 2 we argued that $\sqrt{2}$ had to be between 1 and 2. For each of the following square roots, find the two consecutive integers the value falls between. Then use the $\sqrt{}$ key on your calculator to check your answer.
   a. $\sqrt{18}$
   b. $\sqrt{27}$
   c. $\sqrt{15}$
   d. $\sqrt{58}$

3. Simplify the following square roots. Then use the $\sqrt{}$ key on your calculator to check your answer. If a given square root cannot be simplified, explain why.
   a. $\sqrt{18}$
   b. $\sqrt{45}$
   c. $\sqrt{28}$
   d. $\sqrt{100}$
   e. $\sqrt{42}$
   f. $\sqrt{63}$
   g. $\sqrt{108}$
   h. $\sqrt{125}$
   i. $\sqrt{110}$
   j. $\sqrt{96}$

4. **Investigation:**
   In this section, we saw that some square roots are irrational; that is, we cannot write exact decimal representations for them, even if we use repeating blocks of digits. However, as we will see in this investigation, we can approximate certain square roots fairly accurately without a calculator.
   a. Consider the following amounts: $\sqrt{7}-\sqrt{6}$, $\sqrt{61}-\sqrt{60}$, and $\sqrt{601}-\sqrt{600}$. Do you think these 3 amounts are equal? If not, which one is the greatest? Make a guess without using a calculator.
   b. For each of the following expressions, use a calculator to approximate the value of the expression to 4 decimal places. Then find the unit fraction (fraction of the form $\frac{1}{n}$, where n is a positive integer) that is closest to the value you obtained. (For example, if you get 0.3286, then observe that this value is close to $\frac{1}{3} \approx 0.3333$.)
      i. $\sqrt{5}-\sqrt{4}$
      ii. $\sqrt{10}-\sqrt{9}$
      iii. $\sqrt{17}-\sqrt{16}$

iv. $\sqrt{26} - \sqrt{25}$

c. What do you notice about the expressions you just evaluated? Is there a relationship between the unit fractions you got and the numbers whose square roots we are comparing?

d. Use this pattern to estimate the difference $\sqrt{101} - \sqrt{100}$ without a calculator. (What unit fraction do you expect this difference to be close to?) How can we use this information to estimate $\sqrt{101}$?

e. Use a calculator to approximate $\sqrt{101}$ to 4 decimal places. What is the difference between your estimate for $\sqrt{101}$ from the previous part and the calculator's estimate?

5. **Investigation:**
Now we will use a different number line method to come up with an approximate value of square roots of numbers less than 225.
a. Extend the table from Exploration 4 to include the squares of 11 through 15.
b. Make a number line from 0 to 225.
c. Above each of the perfect squares (like $1, 4, 9, \ldots$) write its square root (like $1, 2, 3, \ldots$).
d. Let's use the number line to approximate $\sqrt{54}$. Since $7^2 = 49 < 54 < 64 = 8^2$, we know that $7 < \sqrt{54} < 8$. So let's look at the number line zoomed in between 49 and 64. The distance between 54 and 49 is $54 - 49 = 5$. The distance between 64 and 49 is $64 - 49 = 15$. So 54 is $\frac{5}{15} = \frac{1}{3}$ of the way between 49 and 64. So a reasonable approximation of $\sqrt{54}$ is $7 + \frac{1}{3} \approx 7.33$. Using a calculator, we can check that $\sqrt{54} \approx 7.35$. So our estimate is pretty close, but a little too small.
e. Use the method described above to estimate the following square roots: i. $\sqrt{12}$ ii. $\sqrt{20}$ iii. $\sqrt{140}$ iv. $\sqrt{210}$
f. Use $\sqrt{\ }$ key on the calculator to find the square roots above. How do the estimates using the number line method compare to the values from the calculator?

6. Order the following geometric shapes from smallest area to largest area.

A A circle with radius 1.

B A square with side of length $\sqrt{3}$.

C A rectangle with sides of length 4 and $\sqrt{\frac{1}{2}}$.

D A right triangle with legs of length $\sqrt{6}$ and $\sqrt{8}$.

7. The height of the flight of rockets can be modeled with a quadratic function. Suppose for Rocket 1 the height after x minutes is given by $f(x) = -2x^2 + 8x + 2$, and for Rocket 2 it is given by $g(x) = -x^2 + 6x - 1$. Find when each rocket will hit the ground. Which rocket will hit the ground first? *Hint: What does* $f(x) = 0$ *mean?*

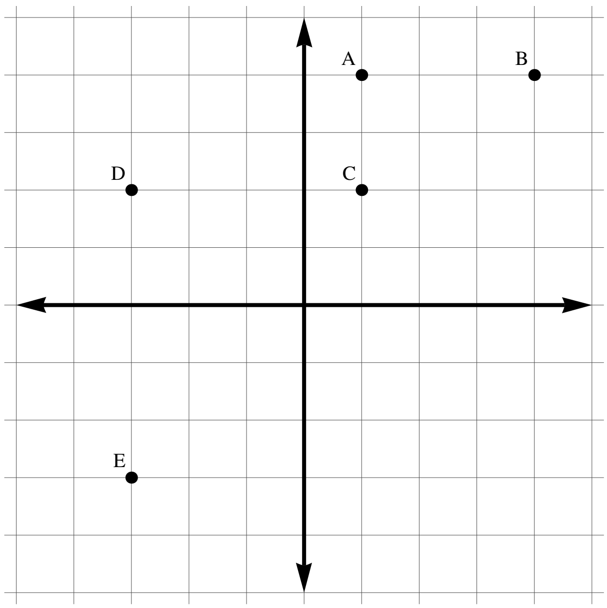
SECTION 10.3 THE COORDINATE PLANE

Distance on the Plane: Part I

As with the number line, we are interested in the distance between points on the plane. The distance between points is the length of the line segment connecting them. When the points share the same x coordinate or the same y coordinate, we can use what we learned about distance on the number line to find the distance.

EXAMPLE 1

Consider the points labeled in the graph below.



Find the distance between the following pairs of points:

1. A and B
2. A and C

3. C and D
4. D and E

SOLUTION

1. First let's determine the coordinates of the points. $A = (1, 4)$ and $B = (4, 4)$ Points A and B lie along the same grid line since the y-coordinate is 4 for each point. If we count the number spaces along the x direction between point A and point B, we see the distance is 3. We can also imagine a number line passing through the points with the point's position on the number line determined by its x coordinate. Then we use subtraction to find the distance. So the distance is then $4 - 1 = 3$. This way of thinking about the distance is helpful when the coordinates are large.
2. $A = (1, 4)$ and $C = (1, 2)$. The x-coordinate is 1 for each point. So the points lie along the same vertical grid line. Again we can imagine a number line passing through the points but with the point's position on the number line determined by its y coordinate. The distance is then $4-2 = 2$. Again we can get this by counting the number spaces between point A and point C.
3. $D = (-3, 2)$ and $C = (1, 2)$. have the same y coordinate, so the distance is $1 - (-3) = 4$. Be careful with the substraction with negative numbers.
4. $D = (-3, 2)$ and $E = (-3, -3)$ have the same x coordinate, so the distance is $2 - (-3) = 5$.

PROBLEM 1

Find the distance between each pair of points: *You may find it helpful to plot the points on a coordinate grid.*

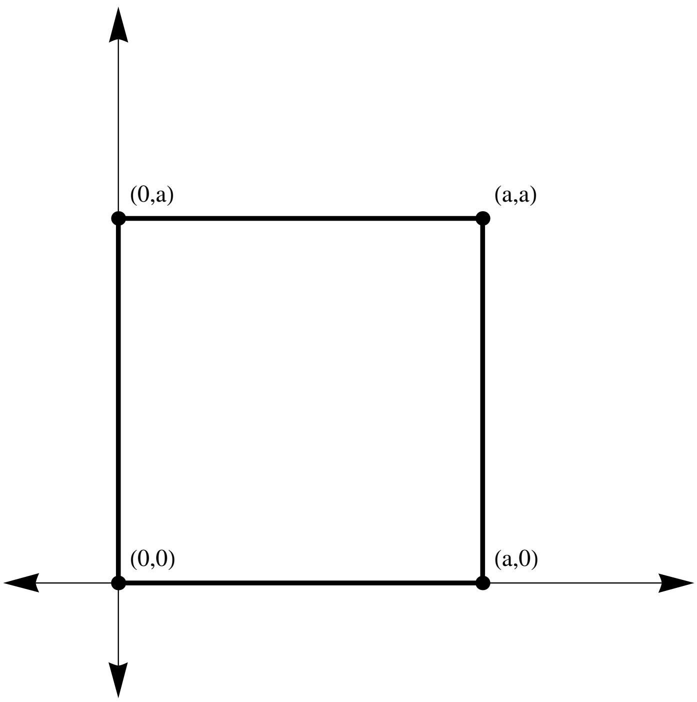
1. $(-3, 2)$ and $(-3, 6)$
2. $(-1, -3)$ and $(-4, -3)$
3. $(3, 2)$ and $(-1, 2)$

4. $(-4, -3)$ and $(1, -3)$
5. $(142, 43)$ and $(97, 43)$
6. $(53, -27)$ and $(53, 15)$

We can use variables to represent unknown values on the plane. Here we will use one variable to represent the x-coordinate and another to represent the y-coordinate. Now, let's use what you have learned about distance on the plane and geometry to explore this connection.

EXPLORATION 1

The graph below shows a square with its vertices labeled. Since the scale of the axes are not clearly marked, we do not know the value of the variable a, but we do know its sign. Is it positive or negative? How can you tell?

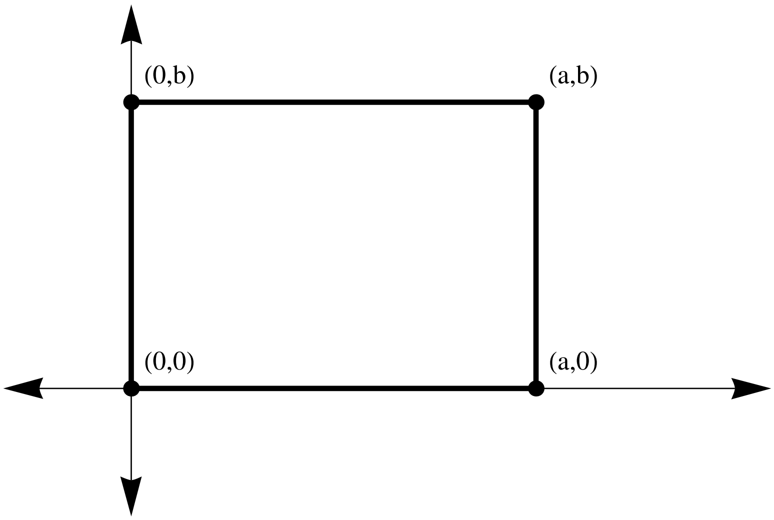


1. Write an expression using the variable a for the area of the square.

2. Write an expression using the variable a for the perimeter of the square.

PROBLEM 2

The graph below shows a rectangle with its vertices labeled.



1. Write an expression using the variables a and b for the area of the rectangle.
2. Write an expression using the variables a and b for the perimeter of the rectangle.

Distance on the Plane: Part II

So far we have discussed finding the distance between points that share the same coordinate. It is more difficult to find the distance between points for which the x coordinates are different **and** the y coordinates are different. To do this we use the Pythagorean Theorem.

EXPLORATION 2

1. Plot the points $(1, 1), (4, 5)$. Can you think of any way to find the distance between the points?
2. Can you form a right triangle and use the Pythagorean Theorem to find the distance between the points?
3. If you haven't already, plot the point $(4, 1)$. Connect each pair of points with line segments to form a triangle. What kind of triangle is it?
4. Find the length of the base and the height of the right triangle.
5. Use the Pythagorean Theorem to find the length of the hypotenuse.
6. Find the perimeter and area of the triangle.

PROBLEM 3

1. Sketch a triangle with vertices at the points $(2, 1)$, $(2, 4)$, $(6, 4)$.
2. Find the perimeter and area of the triangle.

In the next example we see that even if the coordinates of the vertices are integers, the length of the hypotenuse may not be an integer. However, we can use the Pythagorean Theorem to find it.

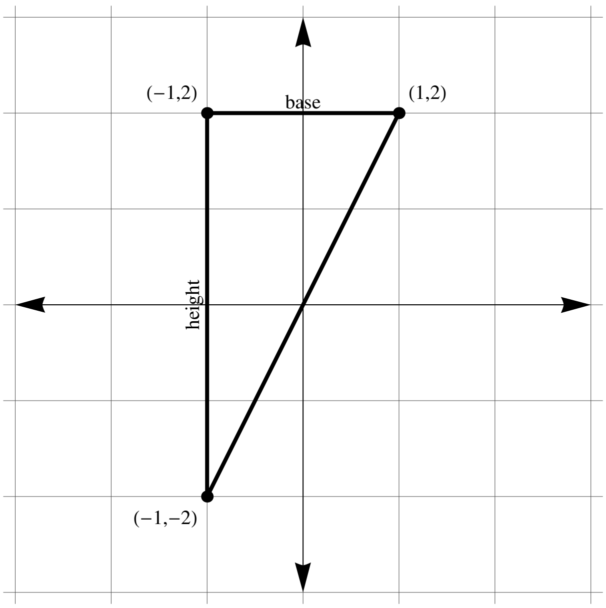
EXAMPLE 2

1. Sketch a triangle with vertices at the points $(-1, -2)$, $(-1, 2)$, $(1, 2)$.
2. Find the lengths of the base, height and hypotenuse.
3. Find the perimeter and area of the triangle.

SOLUTION

1. Call the vertical line segment between $(-1,-2)$ and $(-1,2)$ the height and the horizontal line segment between $(-1,2)$ and $(1,2)$ the base. The vertices of the height have the same x-coordinate, so we find the length from the difference of the y-coordinates: $2 - (-2) = 4$. The vertices of the base have the same y-coordinate, so we find the length from the difference of the x-coordinates: $1 - (-1) = 2$. Be careful! Remember that distance is always positive, so we always subtract the lesser from the greater. The hypotenuse of the triangle is the line segment between $(-1,-2)$ and $(1,2)$. Both the x-coordinates and the y-coordinates of the hypotenuse are different. So we find its length using the Pythagorean Theorem:

$$\begin{aligned}(\text{hypotenuse})^2 &= (\text{base})^2 + (\text{height})^2 \\ &= (1-(-1))^2 + (2-(-2))^2\end{aligned}$$



We need to take the square root to find the length of the hypotenuse. So

$$\text{hypotenuse} = \sqrt{(1-(-1))^2 + (2-(-2))^2}$$

$$= \sqrt{2^2 + 4^2}$$
$$= \sqrt{20}$$

We can leave the answer as $\sqrt{20}$, or use a calculator to find $\sqrt{20} \approx 4.47213595$.

2. The area of the triangle is $\frac{1}{2}$ base time height: $\frac{1}{2} \cdot 2 \cdot 4 = 4$ square units. The perimeter is the sum of the lengths of the sides: $2 + 4 + 2\sqrt{5} = 6 + 2\sqrt{5}$ units. Using a calculator, we get 10.472136 units. Notice that even though the area is a nice whole number, the perimeter is not a whole number.

What if we are simply given 2 points and not a triangle? Can we find the distance between the 2 points? In the next exploration we see it is possible to pick a third point to make a right triangle. The length of the hypotenuse of the triangle will be the distance between the original pair of points. This way we can use the Pythagorean Theorem to find the distance.

EXPLORATION 3

1. Plot the points $(2, 3)$ and $(5, 1)$.
2. Plot another point to form a right triangle.
3. Use Pythagorean Theorem to find the distance between $(2, 3)$ and $(5, 1)$.

PROBLEM 4

Find the distance between $(1, 1)$ and $(3, 2)$.

Distance on the Plane: The Distance Formula

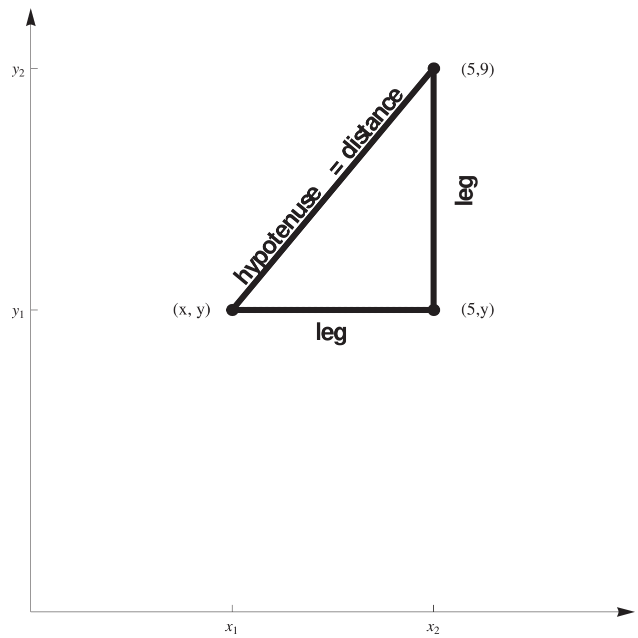
Now we develop a formula for finding the distance between any two points on the plane. This formula is derived using the same idea as in Exploration 3: choose a third point to make a right triangle and use the

Pythagorean Theorem to find the distance.

EXAMPLE 3

Find the distance between the points (x, y) and $(5, 9)$.

SOLUTION We begin by making a sketch of the points. To make a sketch, we need to choose where to put (x, y). In our sketch, we chose $x < 5$, and $y < 9$. Try to make another choice and see what changes. Now we choose a third point, $(5, y)$, to form a right triangle. (Note we could have also chosen $(x, 9)$. How would the triangle look different?) The line segment between (x, y) and $(5, 9)$ is the hypotenuse of the triangle.



To find its length, we first find the length of the legs:

- the length of the first leg $= |5 - x| = 5 - x$
- the length of the second leg $= |9 - y| = 9 - y$

Now we use the Pythagorean Theorem:

$$(\text{hypotenuse})^2 = (\text{first leg})^2 + (\text{second leg})^2$$

so

$$(\text{hypotenuse})^2 = (5 - x)^2 + (9 - y)^2.$$

We take the square root to find the length of the hypotenuse:

$$\sqrt{(5 - x)^2 + (9 - y)^2}.$$

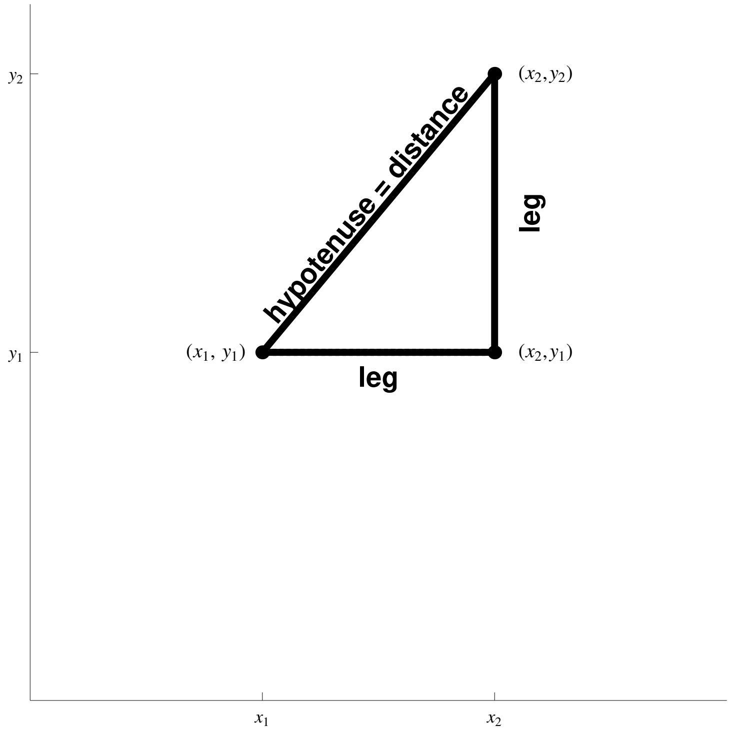
In the last example, we were given specific numbers for coordinates of one of the points. Next we develop a general formula for any pair of points.

EXAMPLE 4

Find the distance between the points (x_1, y_1) and (x_2, y_2) where $x_1 \neq x_2$ and $y_1 \neq y_2$.

SOLUTION We begin by making a sketch of the points. We choose a third point, (x_2, y_1), to form a right triangle. (Note we could have also chosen (x_1, y_2). How would the triangle look different?) The line segment between (x_1, y_1) and (x_2, y_2) is the hypotenuse of the triangle. To find its length, we first find the length of the 2 legs:

- the length of the first leg $= |x_2 - x_1|$
- the length of the second leg $= |y_2 - y_1|$



Now we use the Pythagorean Theorem:

$$\text{(hypotenuse)}^2 = \text{(first leg)}^2 + \text{(second leg)}^2$$

so

$$\text{(length of hypotenuse)}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

We take the square root to find the length of the hypotenuse:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This leads to the general formula:

THEOREM 10.8: DISTANCE BETWEEN TWO POINTS

For any points on the plane (x_1, y_1) and (x_2, y_2) the *distance* between the points is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For example, the distance between $(1, 2)$, $(4, 6)$ is

$$\sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

EXAMPLE 5

Find the distance between each pair of points:

1. $(0, 3)$ and $(4, 5)$
2. $(-1, -2)$ and $(2, 1)$
3. $(1, 3)$ and $(3, -2)$

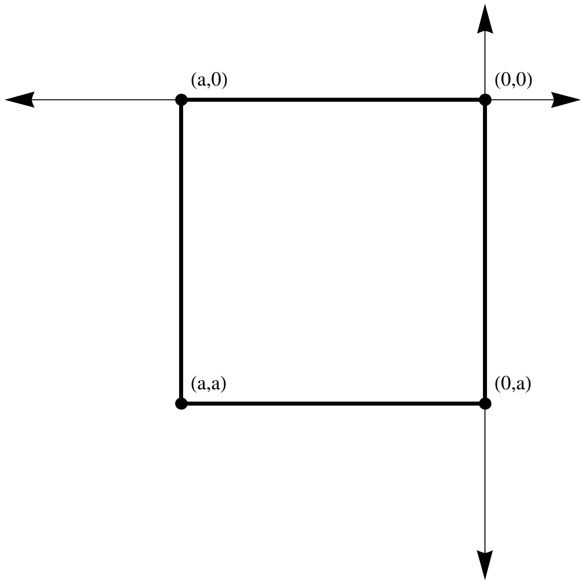
SOLUTION

1. $\sqrt{(0-4)^2 + (3-5)^2} = \sqrt{16+4} = \sqrt{20}$
2. $\sqrt{(-1-2)^2 + (-2-1)^2} = \sqrt{9+9} = \sqrt{18}$
3. $\sqrt{(1-3)^2 + (3-(-2))^2} = \sqrt{4+25} = \sqrt{29}$

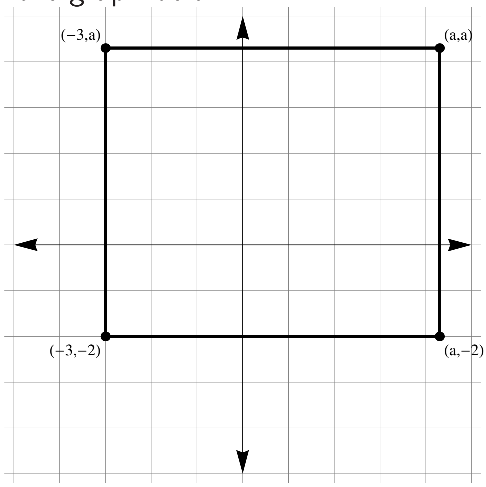
EXERCISES

1. Plot each pair of points and find the distance between them:
   a. $(-2, 2)$ and $(-2, 5)$
   b. $(3, -1)$ and $(3, 4)$
   c. $(-5, 1)$ and $(-2, 1)$
   d. $(2, 3)$ and $(-4, 3)$

2. The vertices of a square are labeled in the graph below. What is the sign of a? Find an expression in terms of a for the area and the perimeter of the square.



3. Find an expression in terms of a for the area and the perimeter of the rectangle in the graph below.



4. Form the triangle with vertices $(1, 2)$, $(1, 1)$ and $(5, 2)$. Find the lengths of each of the 3 sides.

5. Form the triangle with vertices $(1, 1)$, $(-1, -2)$ and $(-1, 1)$. Find the perimeter.

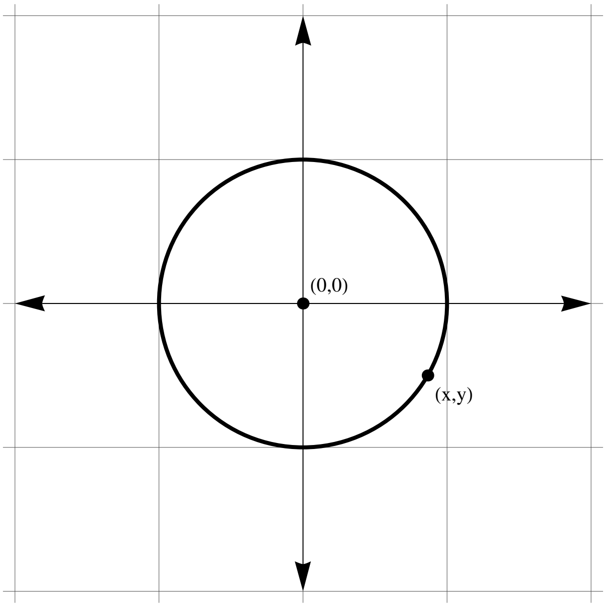
6. a. Plot the points $(2, 5)$ and $(1, 3)$.
   b. Plot a third point, so that the points are the vertices of a right triangle.
   c. Find the lengths of the legs of the right triangle you formed. Use Pythagorean Theorem to find the length of the hypotenuse.
7. a. Plot the points $(2, 5)$ and $(1, 3)$.
   b. Plot a third point, so that the points are the vertices of a right triangle, but choose a different point than you did in exercise 6.
   c. Find the lengths of the sides of the triangle you formed. Are the lengths the same or different than what you found in exercise 6?
8. a. Plot the points $(-1, 5)$ and $(2, 3)$.
   b. Plot a third point, so that the points are the vertices of a right triangle.
   c. Find the lengths of the sides of the triangle you formed.
9. Find the distance between each of the following pairs of points.
   a. $(-2, 1)$ and $(-2, 6)$
   b. $(3, 1)$ and $(3, 4)$
   c. $(2, 3)$ and $(5, 7)$
   d. $(-5, 1)$ and $(-2, 3)$
   e. $(2, 3)$ and $(-4, -1)$
10. **Investigation:**
   a. Plot the point $(3, 1)$ and $(1, 3)$ on the same coordinate plane.
   b. Calculate the distance from $(3, 1)$ to the origin $(0, 0)$. Calculate the distance from $(1, 3)$ to the origin. Compare the distances.
   c. Sketch the general situation: plot a point (a, b) and the point (b, a) on the same coordinate plane. Calculate the distance between (a, b) and the origin. Calculate the distance between (b, a) and the origin. Compare the distances.
11. Kate and Ally are asked to find the distance between $(2, 6)$ and $(4, 12)$. Kate labels the points $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (4, 12)$ and computes the distance as:

$$\begin{aligned}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} &= \sqrt{(2-4)^2 + (6-12)^2} \\ &= \sqrt{4 + 36} = \sqrt{40}.\end{aligned}$$

Ally decided to label the points as $(x_1, y_1) = (4, 12)$ and $(x_2, y_2) = (2, 6)$. If Ally uses the distance formula, will she get the same answer as Kate? Explain.

12. **Investigation:**
In this problem we use the Pythagorean theorem to investigate a special collection of points.
    a. Plot the points $(3, 4)$, $(-3, 4)$, $(4, -3)$, $(-4, -3)$, $(0, 5)$, $(-5, 0)$, $(5, 0)$.
    b. For each of the points find the distance from the point to the origin $(0, 0)$. What do you notice?
    c. Look at the pattern in the points, and think about how you compute distance. Can you find other points that are the same distance from the origin? Plot these points as well.
    d. If we connect the dots, what shape do the points seem to form? (Hint: You don't have to connect the dots with straight line segments.)

13. **Ingenuity:**
The unit circle is a special circle graphed on the coordinate plane with center $(0, 0)$ and radius 1. One definition of the unit circle is the "the collection of all points that are distance 1 away from the origin $(0, 0)$." Another definition is "the collection of all points (x, y) so that $x^2 + y^2 = 1$." The graph on the next page shows the unit circle with one point on the circle labeled (x, y).
    a. Draw the right triangle with vertices $(0, 0)$, $(0, y)$ and (x, y). Explain why the hypotenuse of the triangle is a radius of the unit circle. Note that this means the length of the hypotenuse must be 1.
    b. Find the lengths of all the sides of the right triangle.
    c. Explain why $x^2 + y^2 = 1$.
    d. Explain why this shows that the 2 definitions of the unit circle are really the same.


(0,0)
(x,y)

SECTION 10.4 Translations and Reflections

In this section, we will review two ways we can transform figures.

Translations

EXPLORATION 1

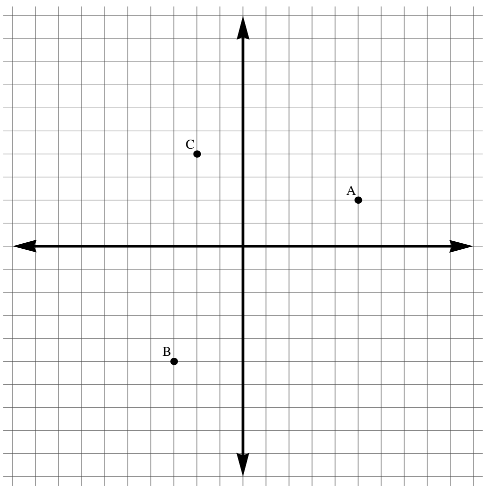
On a piece of grid paper, locate and label the point P with coordinates $(4, -2)$.

1. Starting at P, move 3 units left and 2 units down. Label this new location P'. What are its coordinates?
2. Starting at P', move 4 units left and 1 unit up. Label this new location P''. What are its coordinates?
3. If you wanted to tell someone how to move from P to P'' directly, what would you say?

PROBLEM 1

Look at the grid below. Using only the directional terms up down, left and right, describe how to move from:

1. A to B
2. B to C
3. A to C



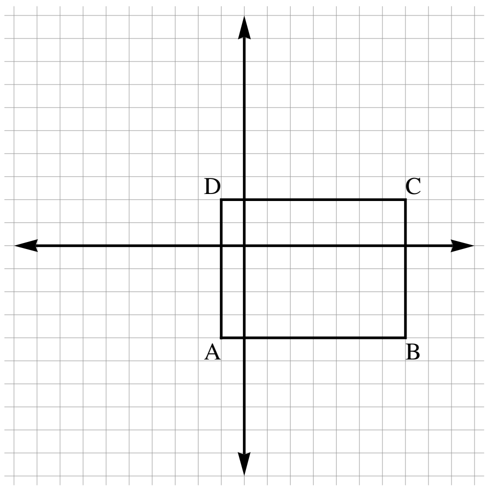
EXPLORATION 2

We can also describe how the coordinates change. For example, you can move from A to D by adding 3 to the x-coordinate and -2 to the y-coordinate. We write this algebraically as $(x, y) \rightarrow (x+3, y-2)$. Using the algebraic notation, describe the move from:

1. A to B
2. B to C
3. A to C

TRANSLATION
A *translation* is a transformation that represents a sliding motion. Each point is moved the same distance in the same direction. We usually describe the translation in terms of its movement in the horizontal direction and in the vertical direction.

EXAMPLE 1

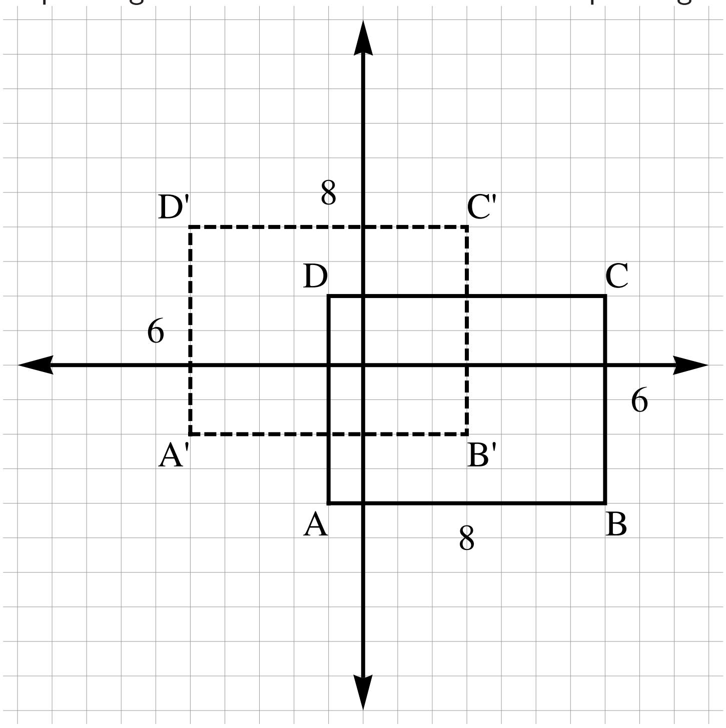


1. Use the rule $(x, y) \rightarrow (x-4, y+2)$ to translate the rectangle $ABCD$.

2. Describe the shape of $A'B'C'D'$.
3. Find the area of the rectangle $ABCD$. Find the area of the $A'B'C'D'$. What do we notice?
4. Find the lengths of the sides of $ABCD$ and of $A'B'C'D'$.

SOLUTION

1. When we transform a figure, it is important to be careful with our labels. When we label the transformed figure, A' is the result of transforming A, B' is the result of transforming B. We call A and A' corresponding vertices and $\overline{AB}$ and $\overline{A'B'}$ corresponding sides.

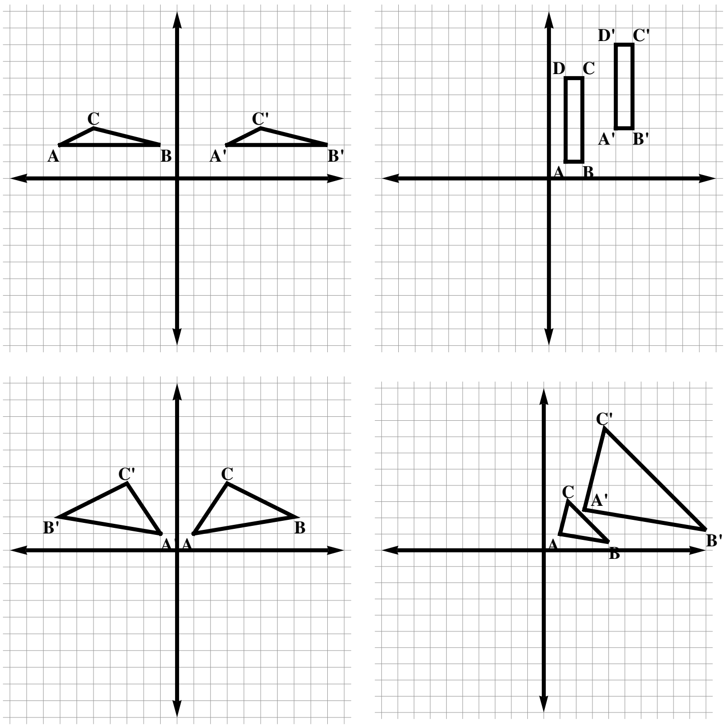


2. The shape $A'B'C'D'$ is a rectangle.
3. The area of $ABCD$ is $bh = 8 \times 6 = 48$. The area of $A'B'C'D'$ is also 48.
4. The lengths of the sides are labeled on the figure. Notice that corresponding sides have the same length. For example, $m\overline{AB} = m\overline{A'B'} = 8$.

In Example 1, $ABCD$ and the result of the translation $A'B'C'D'$ are the same size and shape. Only the location has changed. Two figures are *congruent* if their size and shape are the same. Translations always produce congruent figures. Why does this make sense? When we use the word translation to describe a transformation, it is important that all the points in a figure are moved in the same direction and same distance.

PROBLEM 2

For each of the following determine if the transformation is a translation. Explain.

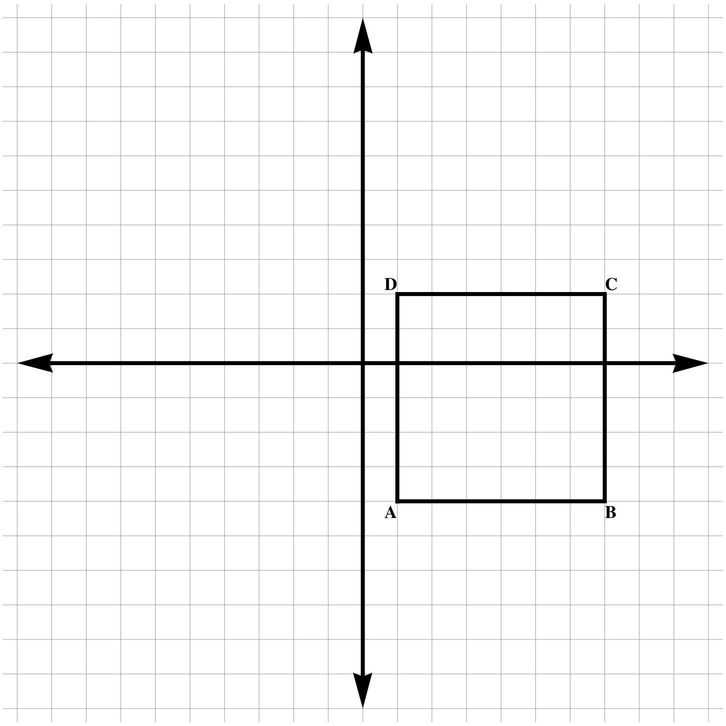


Reflections

Now we will explore reflections. A reflection about a line can be thought of in terms of folding the grid along a line.

EXPLORATION 3

Copy the following onto a separate piece of grid paper.



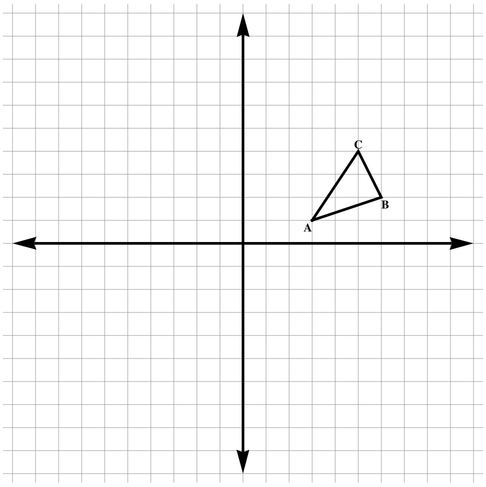
1. Fold the paper along the y-axis, what do you notice about the figures $ABCD$ and $A'B'C'D'$?
2. Find the coordinates of all the points. What do you notice about the x-coordinates of A and A'? How about y-coordinates of A and A'? Compare the coordinates of each pair B and B', C and C' and D and D'.
3. Find the distance from the point A to the y-axis. Repeat for A'. What do you notice? How this compare for other pairs of points.

REFLECTION
A *reflection* is a transformation that represents a flipping motion. Points are flipped across the line of reflection. This creates the mirror image of the points on the other side of the line.

Just like translations, reflections always produce congruent figures. Why does this make sense? Also, a point and its flipped image (like A and A' above) are the same distance from the line of reflection.

We can reflect a figure or set of points around any line, but we most often do this with x-axis or y-axis. When the line of reflection is an axis, the transformation can represented algebraically very simply.

EXAMPLE 2

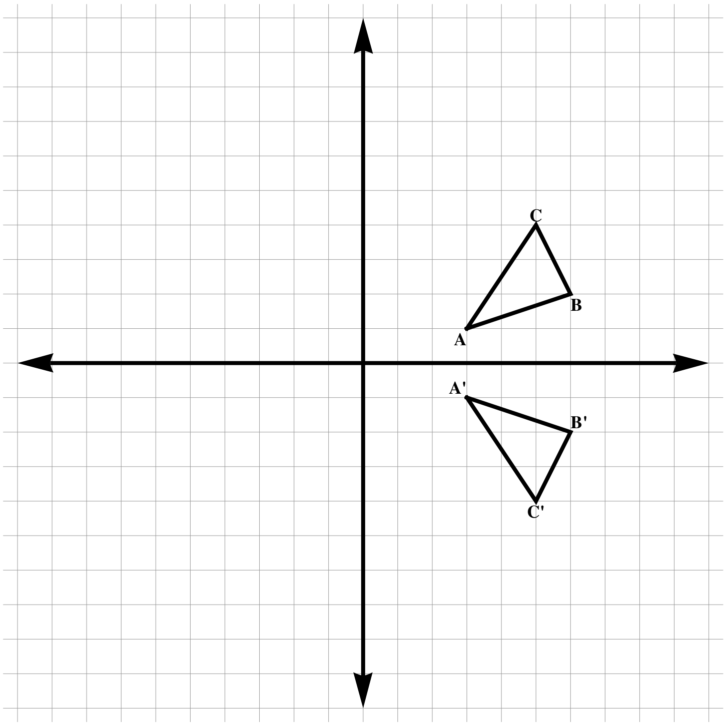


1. Reflect $\triangle ABC$ across the x-axis.
2. Find the coordinates of A, B and C and their reflections A', B', C'. Compare the corresponding x and y coordinates.
3. Find the lengths of the sides of $\triangle ABC$ and $\triangle A'B'C'$. What do you notice?
4. Now reflect $\triangle ABC$ across the y-axis.

5. Find the coordinates of A, B and C and their reflections A', B', C'. Compare the corresponding x and y coordinates.
6. Find the lengths of the sides of $\triangle ABC$ and $\triangle A'B'C'$. What do you notice?

SOLUTION

1. To reflect $\triangle ABC$, we can reflect each of the vertices and then draw $\triangle A'B'C'$. Since $\triangle ABC$ is in Quadrant I, its reflection is in Quadrant IV.



2. Make a table of the coordinates. Compare the coordinates of the corresponding vertices. The x-coordinate is the same, but the y-coordinate changed sign. So the reflection follows the rule $(x, y) \to (x, -y)$.

Point	Coordinates	Point	Coordinates
A	$(3, 1)$	A'	$(3, -1)$
B	$(5, 4)$	B'	$(5, -4)$
C	$(6, 2)$	C'	$(6, -2)$

3. Use Pythagorean Theorem or the distance formula to find the lengths of the sides. This verifies that corresponding sides have the same length.

Side	Length
$\overline{AC}$	$\sqrt{(3-5)^2+(1-4)^2}=\sqrt{13}$
$\overline{BC}$	$\sqrt{(5-6)^2+(4-2)^2}=\sqrt{5}$
$\overline{AB}$	$\sqrt{(3-6)^2+(1-2)^2}=\sqrt{10}$
$\overline{A'C'}$	$\sqrt{(3-5)^2+(-1-(-4))^2}=\sqrt{13}$
$\overline{B'C'}$	$\sqrt{(5-6)^2+(-4-(-2))^2}=\sqrt{5}$
$\overline{A'B'}$	$\sqrt{(3-6)^2+(-1-(-2))^2}=\sqrt{10}$

4. Now reflect $\triangle ABC$ across the y-axis. The reflection should be in Quadrant II.

Point	Coordinates	Point	Coordinates
A	$(3,1)$	A'	$(-3,1)$
B	$(5,4)$	B'	$(-5,4)$
C	$(6,2)$	C'	$(-6,2)$

5. Again we make a table of the coordinates. This time the x-coordinate has changed sign, but the y-coordinate is unchanged. The rule is $(x,y)\rightarrow(-x,y)$.
6. You can verify that the corresponding sides have the same length.

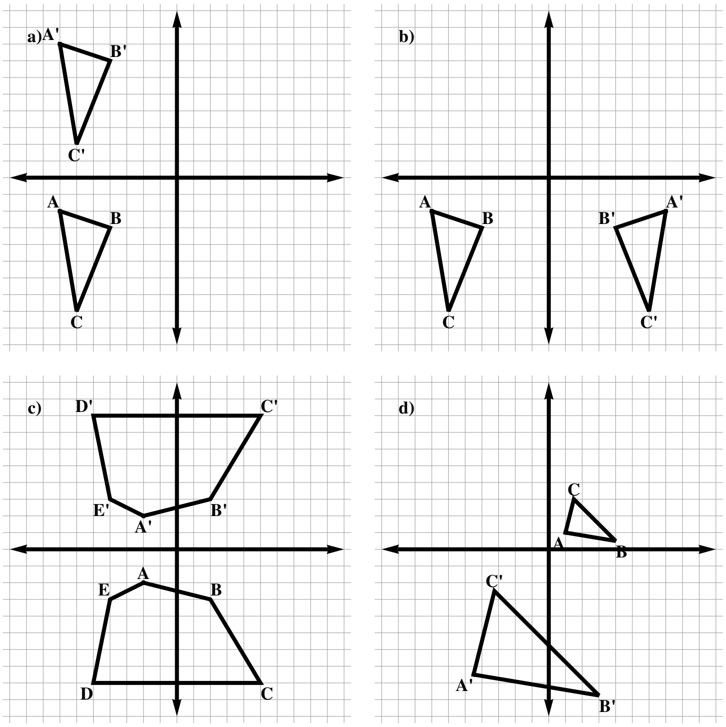
Summary

Transformation	Rule
Translation	$(x,y)\rightarrow(x+a,y+b)$
Reflection about the x-axis	$(x,y)\rightarrow(x,-y)$
Reflection about the y-axis	$(x,y)\rightarrow(-x,y)$

EXERCISES

1. Plot the following points and translate each using the rule $(x,y)\rightarrow(x+2,y-2)$.
   a) $(3,3)$ b) $(-2,4)$ c) $(6,0)$ d) $(\frac{1}{2},\frac{3}{2})$
2. Plot the following points and translate each using the rule $(x,y)\rightarrow(x-3,y+1)$.
   a) $(2,-1)$ b) $(1,3)$ c) $(-1,-5)$ d) $(-3,1)$

3. Plot the following points and reflect each about the y-axis.
   a) $(2, 1)$ b) $(-3, 2)$ c) $(-1, -2)$ d) $(5, -1)$
   e) (x, y) f) $(x, 2x)$
4. Reflect the points from Exercise 3 about the x-axis.
5. Draw a triangle with vertices $A(2, 2)$, $B(3, 5)$ and $C(5, 1)$.
   a. Translate $\triangle ABC$ using the rule $(x, y) \rightarrow (x - 4, y + -2)$. What are the coordinates of the new vertices?
   b. Reflect $\triangle ABC$ about the x-axis. What are the coordinates of the new vertices?
   c. Reflect $\triangle ABC$ about the y-axis. What are the coordinates of the new vertices?
6. Find the lengths of the sides of triangle ABC from Exercise 5.
   a. Use the distance formula to find the lengths of the sides of the translated triangle you found in 5a?
   b. Without measuring or computing anything, predict the lengths of the reflected triangles in 5b and 5c. Explain.
7. Often we perform two or more transformations in a row. Draw a quadrilateral with vertices $A(2, 1)$, $B(5, 1)$,$C(5, 4)$ and $D(2, 3)$. For each of the following, locate and label the final transformed figure.
   a. Reflect $ABCD$ about the x-axis. Then reflect the result around the y-axis.
   b. Reflect $ABCD$ about the x-axis. Then reflect the result about the x-axis.
   c. Reflect $ABCD$ about y-axis. Then translate the result using the rule $(x, y) \rightarrow (x, y - 5)$.
8. For each of the following transformations determine if it is a translation, reflection or neither. Write the rule for the translations and reflections.

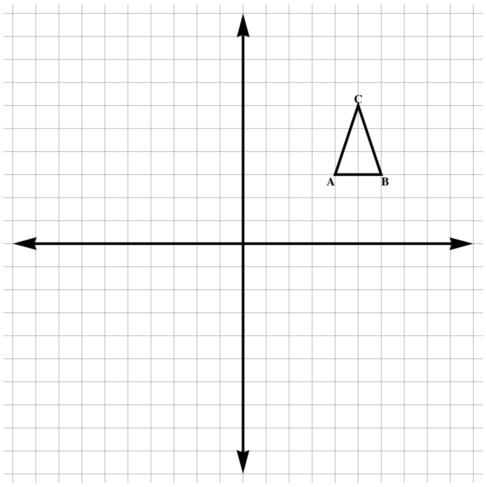


9. **Investigation:**
Sometimes it is helpful to think of all linear functions as a family: while they look alike, they are not all identical. This family also has a parent. For linear functions, the parent function is $f(x) = x$. We can find other functions in the family by transforming the parent function.
   a. Sketch the graph of $f(x) = x$.
   b. Translate the graph of $f(x)$ two units up. Find the formula for the new function.
   c. Translate the graph of $f(x)$ to the left two units. What do you notice?
   d. Graph the function $g(x) = x + 3$.
   e. Describe two ways you can translate the graph of $f(x)$ to move it on top of $g(x)$.

SECTION 10.5 ROTATIONS

In the last section, you explored two types of transformations: translations and reflections. In this section, you will explore another kind of transformation.

EXPLORATION 1



1. Label the coordinates of the vertices of triangle ABC.
2. Transform triangle ABC using the rule $(x, y) \to (-y, x)$. Label the transformed vertices A', B' and C'. Make sure you label the vertices so that A' is the result of transforming the point A, B' is the result of transforming the point B and C' is the result of transforming point C. Imagine moving $\triangle ABC$ and putting it on top of $\triangle A'B'C'$. How would you describe the action? What real life experience uses the same term?
3. Transform triangle ABC using the rule $(x, y) \to (-x, -y)$. Label the corresponding vertices A'', B'', C''. What action describes this transformation? How is movement of triangle $\triangle A''B''C''$ different from that of triangle $\triangle A'B'C'$?
4. Transform triangle ABC using the rule $(x, y) \to (y, -x)$. Label the

corresponding vertices A''', B''', C'''. What action describes this transformation? Compare this to the original transformation.

5. With a compass, draw a circle whose center is at the origin (0,0) that passes through point B. What do you notice about the circle? Explain why this occurred.
6. What is the distance from the origin to point B? From the origin to point B'? From the origin to point B''? From the origin to point B'''? What do you notice? Explain the reason for your discovery.

PROBLEM 1

1. Find the areas of triangles ABC and $\triangle A'B'C'$.
2. Find the lengths of the sides of ABC and $\triangle A'B'C'$.
3. What do you think stays the same when figures are rotated? What is the technical name for your observation? Generalize what you discovered about rotations.
4. Without measuring or computing, predict the area and perimeter of $\triangle A''B''C''$. Explain why the prediction is possible.

In Exploration 1, the triangle is turning or rotating about the origin. It is easy to imagine attaching a string with one end at the origin and the other on $\triangle ABC$ and then spinning the triangle around in a circular motion around the origin. As it spins, the triangle passes through the transformed triangles you drew. Just like translations and reflections, rotations maintain the size and shape of the figures. So all the figures are congruent.

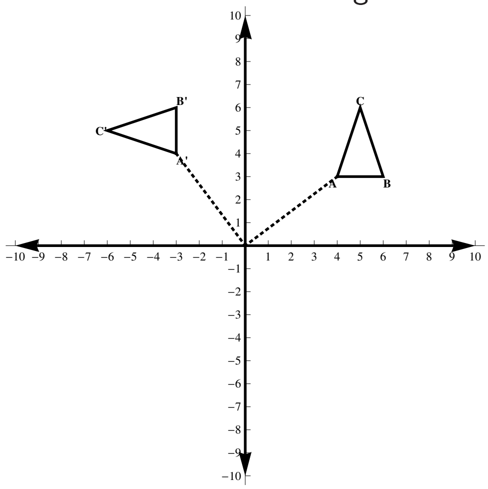
ROTATION
A *rotation* is a transformation that represents turning about a point. The point is called the center of the rotation. Each point in the figure is rotated by the same angle, and its distance to the center stays the same. It is possible to rotate a figure about any center point, but often the center is the origin $(0, 0)$.

EXAMPLE 1

In Exploration 1, $\triangle ABC$ was rotated around the origin. For each rotation determine the angle of rotation.

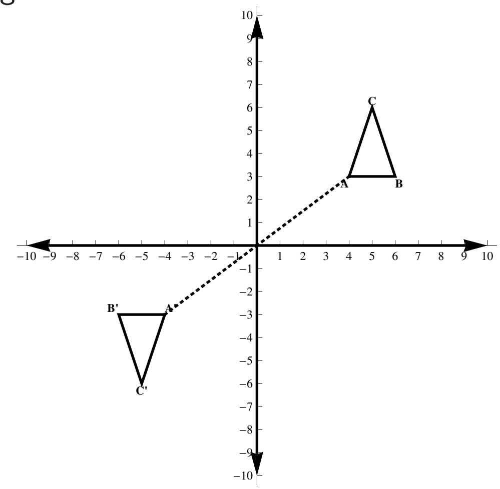
SOLUTION

1. To find the angle of rotation, extend a dotted line from the origin to a point on the original figure. Then another from the origin to the corresponding point on the transformed figure. These two dotted lines form the angle of rotation. The angle of rotation is represented by $\angle AOA'$ in the figure below. In this case, it is easy to see that the measure of $\angle AOA' = 90°$. So the transformation from $\triangle ABC$ to $\triangle A'B'C'$ is a 90° rotation around the origin.

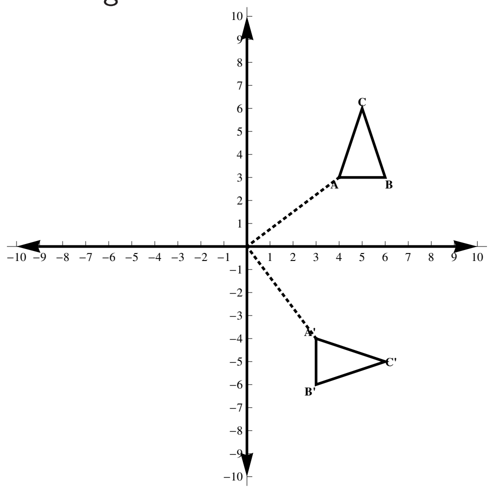


2. The transformation from $\triangle ABC$ to $\triangle A''B''C''$ is a 180° rotation

around the origin.

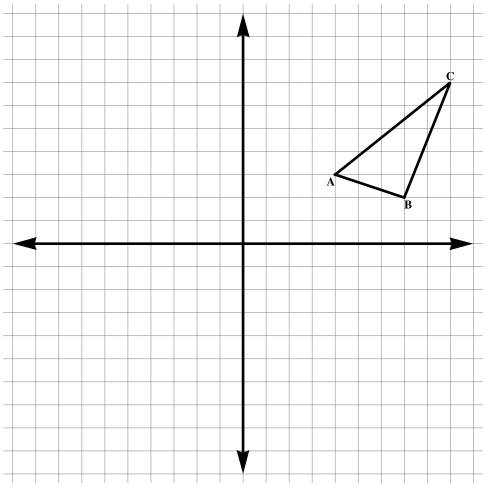


3. The transformation from $\triangle ABC$ to $\triangle A'''B'''C'''$ is shown below. When determining the angle of rotation, think of the rotation in the counter clock-wise direction. This is true, even if rotating it in the clockwise direction is shorter. So the transformation is a 270° rotation around the origin.



Now let's explore what happens with a repeated use of the same rotation.

EXPLORATION 2



1. Without finding the coordinates, predict what will happen if Triangle DEF is rotated by 90 degrees around the origin. What will $\triangle D'E'F'$ look like?
2. Predict what will happen if $\triangle D'E'F'$ rotates by 90° once. Mark that rotation as $\triangle D''E''F''$. What happens if $\triangle D'E'F'$ rotates two more times?
3. To verify your prediction, find the coordinates of the rotated triangles. Check your prediction by using the transformation rule $(x, y) \rightarrow (-y, x)$ each time.

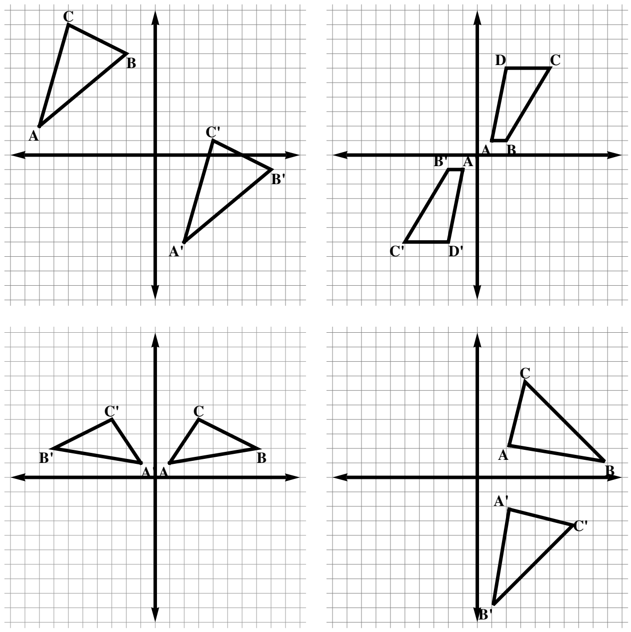
PROBLEM 2

Describe the effect of each of the following:

1. a 360° rotation about the origin.
2. a 450° rotation about the origin.

PROBLEM 3

Four transformations are shown below. Identify each as a translation, reflection or rotation.



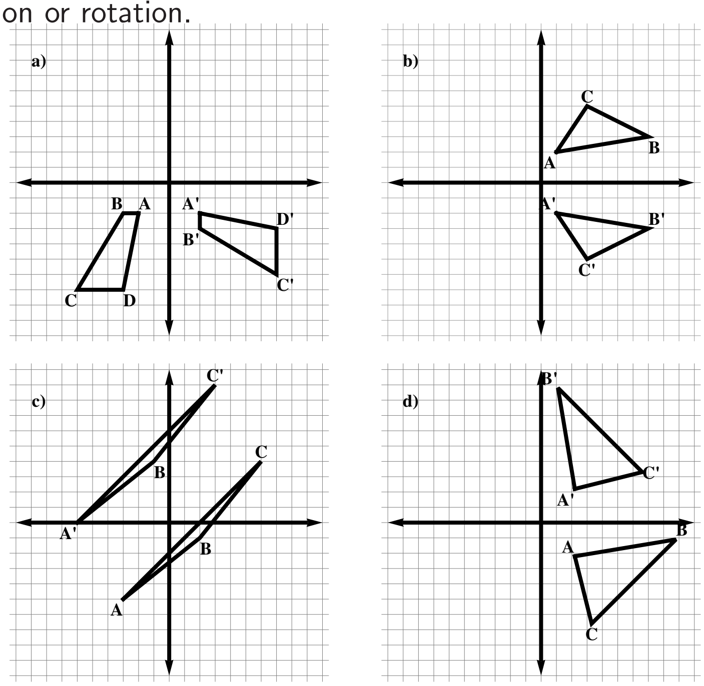
EXPLORATION 3

Get into your groups. Describe in words what happens when you rotate a figure about the origin. Determine a rule for each of the rotations in the table below.

Angle	Description	Algebraic Rule
90°		$(x, y) \rightarrow$
180°		$(x, y) \rightarrow$
270°		$(x, y) \rightarrow$
360°		$(x, y) \rightarrow$

EXERCISES

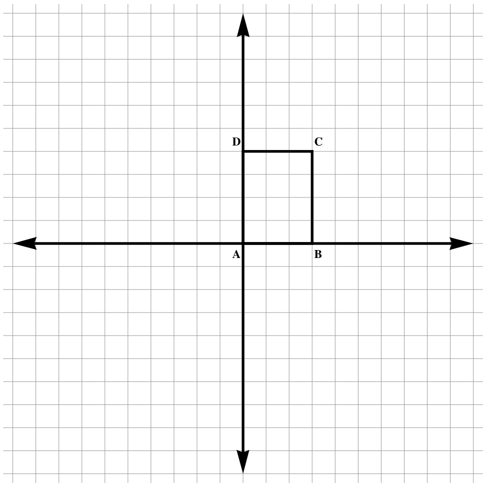
1. Draw the line segment from the origin $O(0,0)$ to the point $A(3,4)$.
   a. Find the length of $\overline{OA}$.
   b. Rotate $\overline{OA}$ 90° about the origin.
   c. Rotate $\overline{OA}$ 180° about the origin.
   d. Rotate $\overline{OA}$ 270° about the origin.
   e. In each case, what is the length of the transformed segment? Explain.
   f. Describe what each rotation does to the point.
2. What does a rotation of 360° do? Explain.
3. Draw label a quadrilateral with vertices: $A(3,2)$, $B(5,2)$, $C(2,-2)$, $D(5,-2)$.
   a. Find the perimeter of $ABCD$.
   b. Find the area of $ABCD$.
   c. Rotate $ABCD$ around the origin 180°. What are the coordinates of the rotated figure? What are the area and perimeter of the rotated figure? Explain your reasoning.
4. Four transformations are shown below. Identify each as a translation, reflection or rotation.



5. Draw and label the quadrilateral with vertices: $A(2,2)$, $B(-2,2)$, $C(-2,-2)$, $D(2,-2)$.
   a. Rotate the figure by 90°. What are the coordinates of the transformed figure? What do you notice?
   b. Predict what will happen if you rotate $ABCD$ by 180°? 270°? Explain.
6. Graph the line $y = 3x$.
   a. Rotate the line 90°.
   b. Find the equation of the rotated line. $y = -\frac{1}{3}x$
   c. How are the slopes of the rotated line and the original line related? Why does this make sense?
7. Let $a > 0$. Choose a value for a and sketch a square with vertices, $(0,0)$, $(a,0)$, $(0,a)$ and (a,a).
   a. Find expressions for the area and the perimeter of the square.
   b. Rotate the square about the origin by 270°. What are the coordinates of the rotated figure?
8. If we rotate a figure clockwise (instead of counter clockwise) about the center point, the angle of rotation is negative.
   a. Explain why a 270° and a $-90°$ degree rotation produce the same transformed figure.
   b. What negative angle of rotation would produce the same transformed figure as a $-90°$ rotation?

SECTION 10.6 Dilations

EXPLORATION 1



For each rule below, transform the rectangle ABCD and find the coordinates of the transformed figure.

1. $(x, y) \rightarrow (2x, 2y)$.
2. $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$.
3. $(x, y) \rightarrow (3x, 3y)$.
4. $(x, y) \rightarrow (\frac{1}{10}x, \frac{1}{10}y)$.
5. Ask your neighbors to describe in their own words the effect of each transformation above?
6. Compare the original figure and the transformed figure.
   a. List two properties of the figure that did not change.
   b. List two properties of the figure that did change.
7. Write the definition of congruent figures. Recall all the transformations studied in previous sections produced congruent figures. How about the transformations above? Did these transformations above produce congruent figures? Why or why not?

EXPLORATION 2

Each of the rules in Exploration 1 are of the form $(x, y) \to (kx, ky)$ for some number k. Let P_1= Perimeter of $ABCD$, P_2 = Perimeter of $A'B'C'D'$, A_1= Area of $ABCD$, and A_2 = Area of $A'B'C'D'$.

1. Find k for each of the rules and complete the following table.

Rule	k	P_1	P_2	$\frac{P_2}{P_1}$
$(x, y) \to (2x, 2y)$				
$(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$				
$(x, y) \to (3x, 3y)$				
$(x, y) \to (\frac{1}{10}x, \frac{1}{10}y)$				

2. Describe the relationship between k and the perimeters of the figures.
3. Find the areas of the figures and complete the following table.

Rule	k	A_1	A_2	$\frac{A_2}{A_1}$
$(x, y) \to (2x, 2y)$				
$(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$				
$(x, y) \to (3x, 3y)$				
$(x, y) \to (\frac{1}{10}x, \frac{1}{10}y)$				

4. Describe the relationship between k and the areas of the figures.

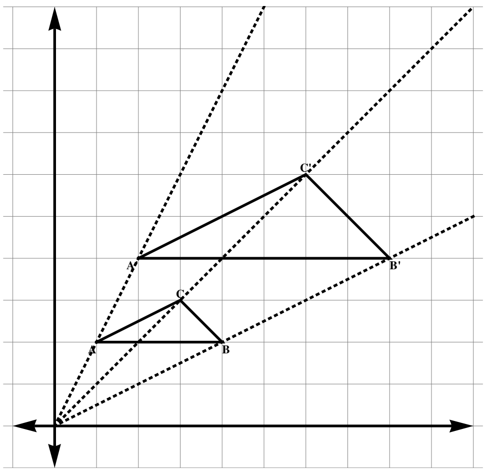
PROBLEM 1

1. Draw and label the triangle with $A(-1, 1), B(-4, 1), C(-1, 5)$.
2. Find the area of $\triangle ABC$.
3. Predict what will happen if $\triangle ABC$ is transformed using the rule $(x, y) \to (4x, 4y)$. What will be the area of the transformed figure?
4. To verify your prediction, find the coordinates of the transformed figure. Compute its area.

In Exploration 1, the rectangle is changing in size but its shape stays the same. If an object is shrunk or enlarged in such a way that all of its dimensions preserve the proportions within the original object, then we say the objects are *similar*. In Exploration 1 we saw that the rectangle was enlarged when $k > 1$ and was reduced when $k < 1$.

DILATION
A *dilation* is a transformation that represents a change in size while maintaining the shape. The figure is enlarged or reduced by a *scale factor* k around a given point. The point is called the center of the dilation. The distance of each point to the center is multiplied by the scale factor k. It is possible to dilate a figure about any center point, but most often the center is the origin $(0, 0)$.

EXAMPLE 1



Dilate the triangle with vertices $A(1, 2), B(4, 2), C(3, 3)$ by a factor of $k = 2$ about the origin using the rule $(x, y) \to (2x, 2y)$.

1. Verify that the vertices of the transformed triangle are twice as far from the origin as the original triangle.
2. Verify that the sides of the transformed triangle are twice as long as

the original triangle.

SOLUTION One way to think about a dilation is to imagine a flashlight at the center of the dilation shining towards the figure. A dilation is concerned with the distance from the center to points on the figure. If we draw dotted lines from the origin through the vertices, we can visualize the distance to the center and can imagine multiplying this distance by a factor of k by sliding the figure along the dotted lines. See the figure below.

Label the origin O. Then the distance to the origin of any point D is the length of the segment OD. Using Pythagorean Theorem or the distance formula we can fill in the following table:

Segment	Length
$\overline{OA}$	$\sqrt{1^2+2^2}=\sqrt{5}$
$\overline{OB}$	$\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$
$\overline{OC}$	$\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$
$\overline{OA'}$	$\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}$
$\overline{OB'}$	$\sqrt{8^2+4^2}=\sqrt{80}=4\sqrt{5}$
$\overline{OC'}$	$\sqrt{6^2+6^2}=\sqrt{72}=6\sqrt{2}$
$\overline{AB}$	3
$\overline{BC}$	$\sqrt{(4-3)^2+(2-3)^2}=\sqrt{2}$
$\overline{AC}$	$\sqrt{(1-3)^2+(2-3)^2}=\sqrt{5}$
$\overline{A'B'}$	6
$\overline{B'C'}$	$\sqrt{(8-6)^2+(4-6)^2}=\sqrt{8}=2\sqrt{2}$
$\overline{A'C'}$	$\sqrt{(2-6)^2+(4-6)^2}=\sqrt{32}=4\sqrt{2}$

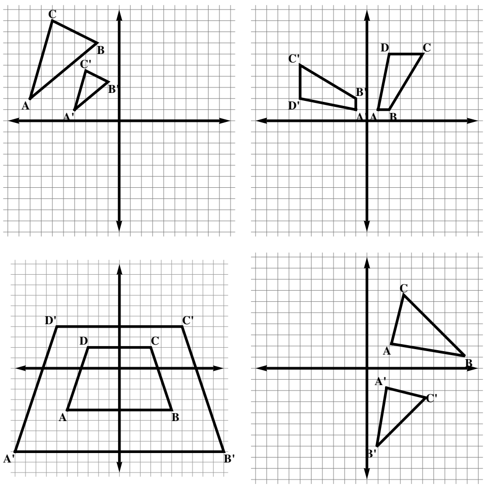
1. Looking at the table we can see that the distance to the origin doubled. For example the distance for vertex A is $\sqrt{5}$ and for vertex A' it is $2\sqrt{5}$.
2. Also the length of the sides of the triangle doubled. For example, $m\overline{AB}=3$ and $m\overline{A'B'}=6$.

EXPLORATION 3

Get into your groups. Describe in words what happens when we transform a figure. Determine a rule for each of the transformations in the table below.

Transformation	Description	Algebraic Rule
Translation		$(x, y) \to$
Reflection about x axis		$(x, y) \to$
Reflection about y axis		$(x, y) \to$
90° Rotation		$(x, y) \to$
Dilation		$(x, y) \to$

PROBLEM 2

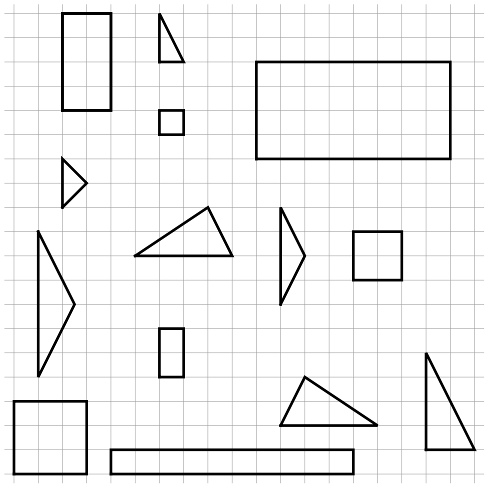


Four transformations are shown above. Identify which transformations are dilations. For each dilation, determine the value of the scale factor k.

Similarity Similarity is an important concept in geometry. Once we determine that two figures are similar, it is easy to compare the sizes of corresponding parts of the figures.

EXPLORATION 4

Intuitively, two figures are similar if they have the same shape. But it is not so easy to define what we mean by "have the same shape".



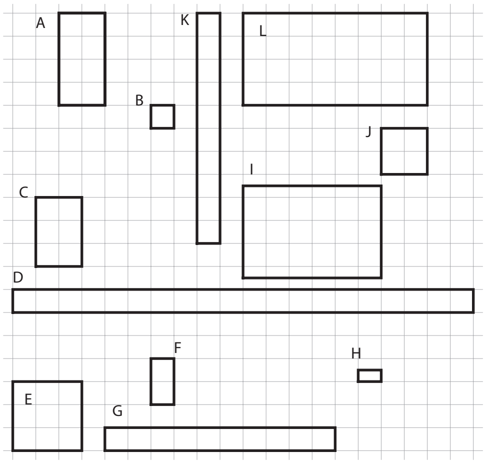
Split the set of figures above into groups of similar figures. Compare with your neighbor. Did they form the same groups? Discuss in your group what properties of the figure help you determine the shape?

It is possible to discuss the similarity of any shape. For example, all circles are similar. However, in geometry we often focus on similar polygons. For polygons, it is possible to give a concise definition for similarity.

SIMILAR POLYGONS

Two polygons are similar if corresponding angles are congruent **and** their corresponding sides have the same ratio.

EXAMPLE 2



Look at the rectangles in the figure above.

1. Find groups of similar rectangles.
2. Use the definition to justify your answer.

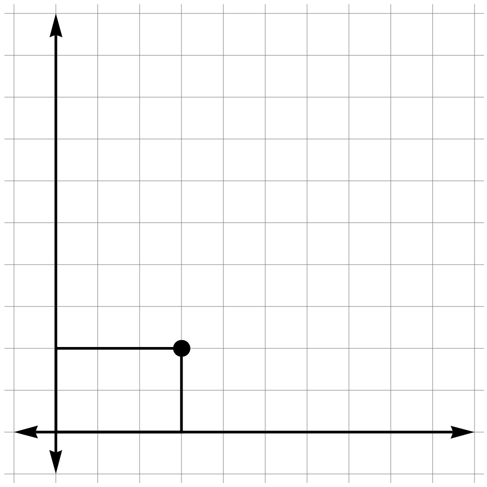
SOLUTION

1. The groups are Group 1 = $\{A, F, H, L\}$, Group 2 = $\{B, E, J\}$, Group 3 = $\{D, G, K\}$, Group 4 = $\{D, G, K\}$
2. Since all the figures are rectangles, all of the angles are right angles and each figure has two pairs of congruent sides. So to prove two rectangles are similar, we need only check that the ratio between the longer sides is equal to ratio of the shorter. For example, Rectangle A and Rectangle F are similar because

$$\frac{\text{Length of Long Side of A}}{\text{Length of Long Side of F}} = \frac{4}{2} = \frac{2}{1} = \frac{\text{Length of Short Side of A}}{\text{Length of Short Side of F}} = 2.$$

Check that this works for each pair of rectangles in Group 1.

EXPLORATION 5



In the figure above a 2 by 3 rectangle has been drawn in a special way.

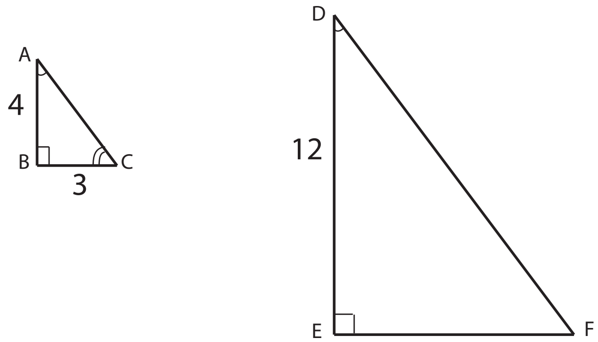
- One vertex is placed at the origin.
- One of the longer sides is placed along the x-axis.
- One of the short sides is placed along the y-axis.
- A point is placed at the vertex diagonal to origin. In this case, this vertex is at the point $(3, 2)$.

1. Draw four more rectangles with dimensions 4×6, 6×9, 1×1.5 and 10×15 on the graph. Draw each rectangle using the same rules as the one above (one vertex at the origin, etc.).
2. Use the definition of similarity to verify that all the rectangles are similar. What is the common ratio of the corresponding sides?
3. Notice that the points you marked on the rectangles lie on a line. Draw the line passes through the points. Find the equation of the line. How does the slope compare to the common ratio you found above?
4. Predict what would happen if you produced the same kind of graph for a bunch of squares. What would be the slope of the line? Explain.

Similar triangles play an especially important role in geometry. Many problems can be modeled with triangles. The following theorem shows that proving two triangles are similar requires less than the general polygon. You need only consider either two of the angles or two of the sides for each triangle.

THEOREM 10.9: SIMILAR TRIANGLES
Two triangles are similar if two of corresponding angles are congruent **or** their corresponding sides have the same ratio.

EXAMPLE 3



Consider the two triangles shown above.

1. Prove that the two triangles are similar.
2. Find the lengths of the sides of $\triangle DEF$.

SOLUTION

1. As indicated $m\angle B = m\angle E = 90^\circ$ and $m\angle A = m\angle D$. So two of the corresponding angles are congruent. Hence $\triangle ABC$ and $\triangle DEF$ are similar by the Similar Triangle Theorem.
2. By Pythagorean Theorem $m\overline{AC} = 5$. Since the ratio of corresponding sides must be equal we can set up the follow proportions:

$$\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{EF}}{m\overline{BC}} \qquad \frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{DF}}{m\overline{AC}}$$

Substituting the known values produces:

$$\frac{12}{4} = \frac{m\overline{EF}}{3} \qquad \frac{12}{4} = \frac{m\overline{DF}}{5}$$

Solving for the unknown lengths, gives $m\overline{EF} = 9$ and $m\overline{DF} = 15$.

PROBLEM 3

In the afternoon, a tree casts a shadow of 28 feet and a 5 foot post casts a 4 foot shadow. See the figure below.

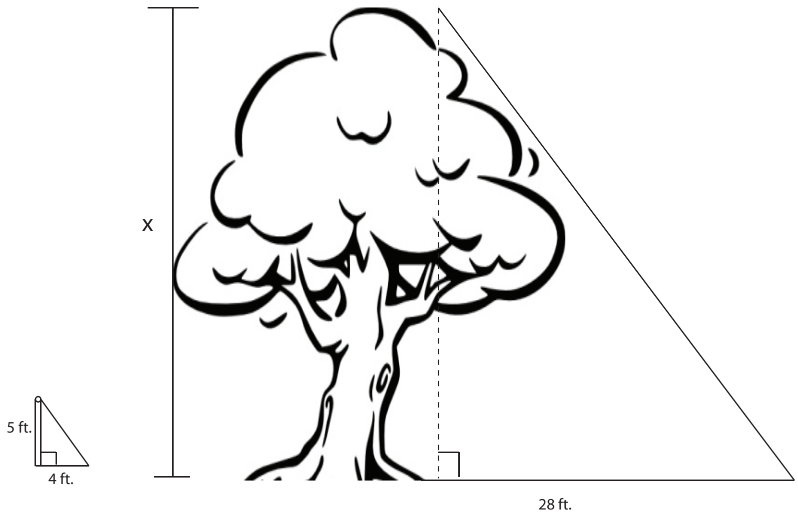


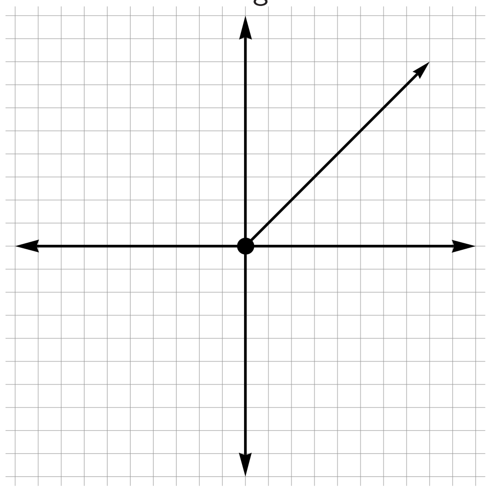
Figure A

Figure B

1. Explain why the two triangles in the figure are similar.
2. Find the height of the tree.

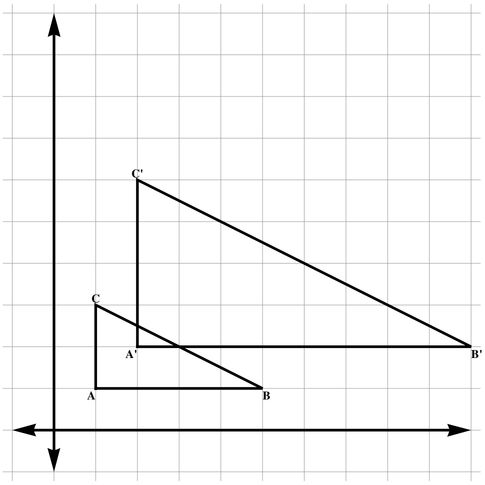
EXERCISES

1. In this chapter you have studied four transformations: translations, reflections, rotations, and dilations. Which of the transformations produce congruent figures? Explain.
2. Consider the arrow shown in the figure below.



a. Imagine the arrow was on a map. Describe the direction the arrow is pointing like North, South, etc..
b. Explain why if you apply any of the four transformations to the arrow, the transformed figure would also be an arrow.
c. Which of the transformations would produce a arrow pointing in the same direction? Explain.

3. The figure below shows $\triangle ABC$ and the result of a transformation.



a. Find the rule of the transformation.
b. Find the area and perimeter of the two figures.

4. Draw and label a quadrilateral with vertices: $A(3, 2)$, $B(5, 2)$, $C(2, -2)$, $D(5, -2)$.
   a. Find the perimeter of $ABCD$.
   b. Find the area of $ABCD$.
   c. Dilate $ABCD$ about the origin with a scale factor of $k = 3$. What are the coordinates of the dilated figure? What are the area and perimeter of the dilated figure? Explain your reasoning.
   d. Use the definition to prove that $ABCD$ and its dilation are similar.

5. Draw a triangle with vertices $A(2, 2)$, $B(3, 5)$ and $C(5, 1)$.
   a. Find the lengths of the sides of $\triangle ABC$.
   b. Dilate $\triangle ABC$ by a scale factor of 3. Locate and label $\triangle A'B'C'$.
   c. Find the lengths of the sides of $\triangle A'B'C'$.

6. Draw a triangle with vertices $A(2,2)$, $B(3,5)$ and $C(5,1)$.
   a. Find the distance from the origin to each of the vertices of $\triangle ABC$.
   b. Dilate $\triangle ABC$ by a scale factor of 4. Locate and label $\triangle A'B'C'$.
   c. Find the distance from the origin to each of the vertices of $\triangle A'B'C'$.
7. Draw a rectangle on a grid. Label the vertices A, B,C D.
   a. Draw another rectangle on the grid that is similar to $ABCD$. Explain why the two rectangles are similar using the definition of similar polygons.
   b. Draw another rectangle on the grid. This time draw rectangle that is not similar to $ABCD$. Explain why the two rectangles are not similar using the definition of similar polygons.
8. Kate says she has an example of three triangles where the following is true:
   - $\triangle A$ is similar to $\triangle B$
   - $\triangle B$ is similar to $\triangle C$
   - but $\triangle A$ is not similar to $\triangle C$

   Ally says this is impossible. Use definition of similarity to determine who is correct.

SECTION 10.7 CHAPTER REVIEW

Key Terms

congruent	rotation
dilation	scale factor
irrational number	similar
radical	translation
reflection	

Formulas Pythagorean Theorem: For a right triangle with legs of length a and b and hypotenuse of length c:

$$a^2 + b^2 = c^2$$

Distance between (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Transformation	Algebraic Rule
Translation	$(x, y) \to (x + a, y + b)$
Reflection about x axis	$(x, y) \to (x, -y)$
Reflection about y axis	$(x, y) \to (-x, y)$
90° Rotation	$(x, y) \to (-y, x)$
180° Rotation	$(x, y) \to (-x, -y)$
270° Rotation	$(x, y) \to (y, -x)$
360° Rotation	$(x, y) \to (x, y)$
Dilation	$(x, y) \to (kx, ky)$

Practice Problems

1. A right triangle has legs of 4 and 12. Find the length of the hypotenuse of the triangle. Calculate the perimeter and area of the triangle.
2. A right triangle has a hypotenuse of 40 and a leg of 25. Calculate the length of the remaining leg.

3. A 20 ft. tall tree casts a 10 ft. shadow. Find the distance from the tip of the shadow to the top of the tree.

4. Find which 2 consecutive integers the following values fall between. For example, $\sqrt{2}$ is between 1 and 2.
   a. $\sqrt{24}$ c. $\sqrt{35}$ e. $\sqrt{150}$
   b. $\sqrt{74}$ d. $\sqrt{113}$

5. Find the distance between each of the following pairs of points
   a. $(3, -2), (3, -7)$ c. $(6, -5), (8, 2)$
   b. $(-2, 6), (15, 6)$ d. $(15, 2), (13, -1)$

6. Sketch the triangle with vertices: $A(-3, 2)$, $B(5, 2)$, $C(5, -5)$. Find its area and perimeter.

7. Apply each of the following transformations to $\triangle ABC$ from Problem 6. Sketch the transformed figure and determine its area and perimeter.
   a. Use the rule $(x, y) \to (x - 3, y + 2)$.
   b. Reflect about the x-axis.
   c. Rotate 270° about the origin.
   d. Dilate with a scale factor of $k = \frac{3}{2}$.

8. For each of the following rules, determine if the transformation is translation, reflection, rotation or dilation. For each describe the effect of the transformation in words.
   a. $(x, y) \to (1.1x, 1.1y)$
   b. $(x, y) \to (-y, x)$
   c. $(x, y) \to (-x, y)$
   d. $(x, y) \to (-x, -y)$
   e. $(x, y) \to (x + 3, y - 2)$

9. Esequiel applied a transformation to a figure. After the transformation, he noticed that the resulting figure was the same size, shape and pointed in the same direction, just the location had changed. What kind of transformation did he apply?

RADICAL EXPRESSIONS

11

SECTION 11.1 THE SQUARE ROOT FUNCTION

In Chapter 8 we investigated the properties of quadratic functions like $f(x) = x^2$. In this section we will formally define the square root function, see its relationship to quadratic functions and investigate its properties.

EXPLORATION 1

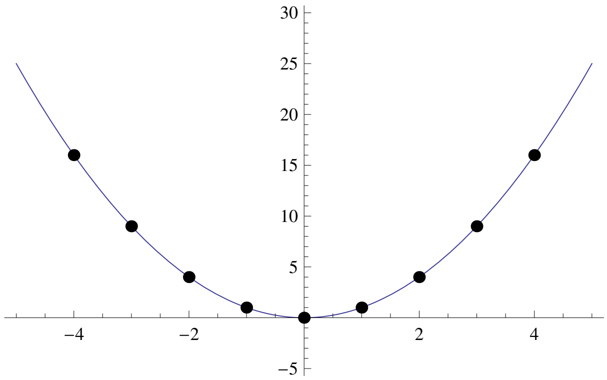
1. Compute each of the following. What do you notice?
   a. $\sqrt{3^2}$ and $(\sqrt{3})^2$
   b. $\sqrt{5^2}$ and $(\sqrt{5})^2$
   c. $\sqrt{6^2}$ and $(\sqrt{6})^2$
2. If $a > 0$, what is $\sqrt{a^2}$? What is $(\sqrt{a})^2$?
3. What happens if we square a positive number and then take its square root?
4. What happens if we square the square root of a positive number?

In Exploration 1 we saw that taking the square root "undoes" what the squaring operation does to a positive number and vice versa. When this happens, we call the operations *inverses of one another*. We want to use this idea of inverse to formally define the square root function. In the next exploration, we will find that we must be careful when we consider negative numbers.

We begin with the function $f(x) = x^2$.

input x	output $f(x) = x^2$
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

Graph of f(x) = x^2



EXPLORATION 2

1. If f takes us from x to x^2, then the inverse function should take us back again, that is from x^2 to x. In tabular form, this means that the outputs and inputs get exchanged. Exchange the outputs and inputs from above into the following table:

new input x	new output y
16	
9	
4	
1	-1
0	0
1	1
4	
9	
16	

2. Plot the points in the table from part 1 and connect the points to form the graph.
3. In Chapter 2 we gave the following definition of a function: A function is a rule that assigns **one and only one output value** to each element of a given set of inputs. Do the table from part 1 and the graph from part 2 represent a function? Why or why not?

In Exploration 2 we saw that simply exchanging the inputs and outputs for the function $f(x) = x^2$ does not produce a function. The basic problem is that since $x^2 = (-x)^2$, we are not sure whether $\sqrt{x^2}$ should be x or $-x$. So when we want to define the square root function, we must make a choice. Mathematicians have decided that the output of the square root function is always nonnegative. Hence the $\sqrt{25} = 5$, not -5 and not ± 5.

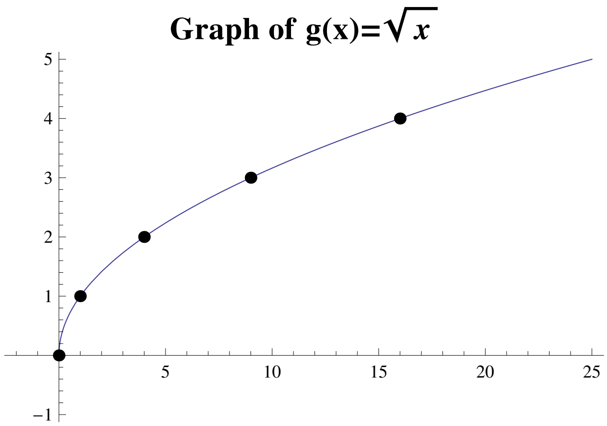
SQUARE ROOT
For every nonnegative number t, $s = \sqrt{t}$ is the nonnegative number s such that $s^2 = t$. The value s is called the *square root* of t.

EXAMPLE 1

Let $g(x) = \sqrt{x}$. What is the domain of g? What is the range of g? Make a graph of g.

SOLUTION Recall that the domain is the set of inputs. In many cases the domain was determined by the context of the problem. Here, however, the domain is restricted by the definition of square root. Since we can not take the square root of a negative number, the domain is the set of nonnegative numbers. The range is the set of possible outputs. From the definition of square root, the outputs must be nonnegative as well. We can use the values from the table in Exploration 2 to get a list of points and plot them to form the graph:

x	$g(x) = \sqrt{x}$
0	0
1	1
4	2
9	3
16	4



Notice that the graph looks like the top half of the graph from part 2 from Exploration 2 and now satisfies the definition of a function: each input has one and only one output.

The square root sign $\sqrt{\quad}$ is sometimes called the *radical*. The value inside the radical is called the *radicand*. Functions involving the square root are called *radical functions*. As we did with quadratic functions, we can form radical functions by thinking about vertical and horizontal shifts of a parent function, in this case $g(x) = \sqrt{x}$.

EXPLORATION 3

1. Using a graphing calculator, graph these radical functions on the same coordinate plane. *Set the window so that x goes from -2 to 16 and y from -5 to 10.*
   a. $f(x) = \sqrt{x}$
   b. $r(x) = \sqrt{x} + 1$
   c. $r(x) = \sqrt{x} + 3$
   d. $r(x) = \sqrt{x} - 2$
2. Each of the radical functions has the form $r(x) = \sqrt{x} + k$. What does changing k do to the graph? Predict without graphing what the graph of $r(x) = \sqrt{x} + 100$ would look like.

We can think about horizontal shifts as well. However, we need to be careful about the domain of the function, always remembering that we do not allow negative numbers under the square root.

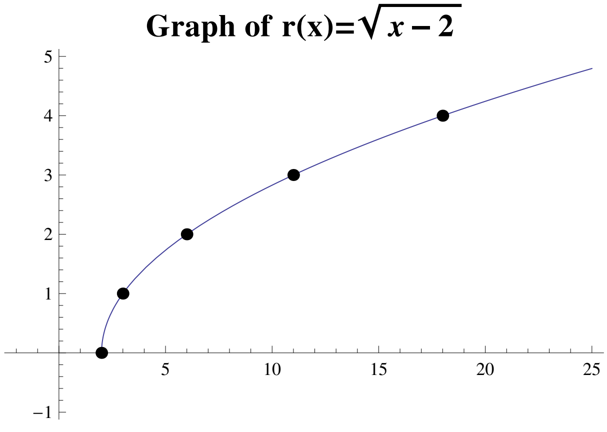
EXAMPLE 2

Let $r(x) = \sqrt{x - 2}$. What is the domain of r? Make a graph of $r(x)$.

SOLUTION Since $r(x) = \sqrt{x - 2}$, we must have $x - 2 \geq 0$. So the domain of r is all numbers x such that $x \geq 2$. If we make a table of values we get:

x	$g(x)$	$= \sqrt{x-2}$
2	$g(2)$	$= \sqrt{2-2} = \sqrt{0} = 0$
3	$g(3)$	$= \sqrt{3-2} = \sqrt{1} = 1$
4	$g(4)$	$= \sqrt{4-2} = \sqrt{2}$
5	$g(5)$	$= \sqrt{5-2} = \sqrt{3}$
6	$g(6)$	$= \sqrt{6-2} = \sqrt{4} = 2$
11	$g(5)$	$= \sqrt{11-2} = \sqrt{9} = 3$
18	$g(18)$	$= \sqrt{18-2} = \sqrt{16} = 4$

Notice that our list of inputs starts at $x = 2$. Do you see why we start skipping for some of the other inputs like $x = 11$ and $x = 18$? We now plot the points and make the graph.



As we expected, the graph of $r(x) = \sqrt{x-2}$ looks like the graph of $g(x) = \sqrt{x}$ shifted to the right by 2 units.

PROBLEM 1

Determine the domain of each of the following radical functions, and then sketch a graph.

1. $r(x) = \sqrt{x+3}$
2. $r(x) = \sqrt{x-4}$
3. $r(x) = \sqrt{x+3} + 2$

EXPLORATION 4

Consider the horizontal lines given by $y = 4$, $y = 1$ and $y = -2$.

1. Sketch a graph of $r(x) = \sqrt{x}+3$. How many times will $r(x)$ intersect each of the horizontal lines?
2. Sketch a graph of $r(x) = \sqrt{x+3}$. How many times will $r(x)$ intersect each of the horizontal lines?
3. Sketch a graph of $r(x) = \sqrt{x+3} + 2$. How many times will $r(x)$ intersect each of the horizontal lines?
4. How many solutions will the equation $\sqrt{x+3} + 2 = 4$ have?
5. How many solutions will the equation $\sqrt{x+3} + 2 = 1$ have?

PROBLEM 2

Predict the number of solutions for each of the following equations.

1. $\sqrt{x} + 5 = 7$
2. $\sqrt{x} - 3 = -2$
3. $\sqrt{x-3} = 10$
4. $\sqrt{x-2} + 5000 = 1000$

EXERCISES

1. For each of the following, determine the domain and sketch its graph:
   a. $r(x) = \sqrt{x} - 4$.
   b. $r(x) = \sqrt{x} + 3$.
   c. $r(x) = \sqrt{x-3}$.
   d. $r(x) = \sqrt{x-1} + \frac{3}{2}$.
   e. $r(x) = \sqrt{x + \frac{2}{3}} - 4$.
2. Predict the number of solutions for each of the following equations. Explain.
   a. $\sqrt{x-5} = 3$
   b. $\sqrt{x+7} - 2 = -\frac{1}{2}$
   c. $\sqrt{x-8} + 3 = 2$
3. Let $r(x) = 3 - \sqrt{x}$.
   a. What is the domain of r?
   b. What is the range of r?
   c. Make a table of at least 4 points of the graph of $r(x)$.
   d. Graph $r(x)$.
   e. Without making a table, predict what the graph of $s(x) = r(x+2) = 3 - \sqrt{x+2}$ would look like.
4. Let $r(x) = \sqrt{2x+2}$.
   a. What is the domain of r?
   b. What is the range of r?
   c. Make a table of at least 4 points of the graph of $r(x)$.
   d. Graph $r(x)$.
   e. Without making a table, predict what the graph of $s(x) = r(x) + 3 = \sqrt{2x+2} + 3$ would look like.
5. Let $r(x) = \sqrt{x+3} + 2$ and $h(x) = x - 1$.
   a. What is the domain of r? What is the domain of h?
   b. Make a table of at least 4 points of the graph of $r(x)$.
   c. Graph $r(x)$ and $h(x)$ on the same coordinate grid.
   d. How many times does $h(x)$ intersect with $r(x)$? Use the graph to estimate the coordinates of the points of intersection.
   e. Use what you found in the graph to determine a solution to the equation $r(x) = h(x)$. Check that it works.

6. A right triangle has legs of length 1 and x units.
   a. Sketch and label the triangle.
   b. Write an expression for the length of the hypotenuse in terms of x.
   c. Let $h(x)$ be the length of the hypotenuse for a given x. Sketch a graph of $h(x)$.
7. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$.
   a. What is the range of f?
   b. What is the domain of g?
   c. What is the range of g?
   d. Let $h(x) = g(f(x)) = \sqrt{x^2}$. Fill in the following table of points. Pay close attention to the signs.

input x	output $h(x)$
-4	$h(-4) = \sqrt{(-4)^2} = \sqrt{16} = 4$
-3	
-2	
-1	
0	
1	
2	
3	
4	

   e. Plot and connect the points to form the graph of $h(x)$.
   f. Describe what the function h does if the input x is negative. What does it do if the input is positive?

SECTION 11.2 OPERATIONS WITH RADICALS

In Section 10.2 we explored the properties of square roots of numbers. These same properties apply to the square root function and to expressions involving square roots.

Properties of Square Roots Let $a \geq 0$ and $b \geq 0$ be non negative numbers.

- **Multiplication:** $\sqrt{a}\sqrt{b} = \sqrt{ab}$
- **Division:** $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- **Inverse:** $\sqrt{a^2} = a$

We can use these properties to simplify expressions involving square roots. To simplify a square root, you "take out" anything that is a "perfect square"; that is, you take out anything that has 2 copies of the same factor: $\sqrt{9} = \sqrt{3^2} = 3$ or $\sqrt{8} = \sqrt{2^3} = \sqrt{2^2 \cdot 2} = \sqrt{2^2}\sqrt{2} = 2\sqrt{2}$. This simplifies the square root by reducing the number under the square root sign to as much as possible or by removing the need for the square root sign altogether.

EXPLORATION 1

1. Simplify each square root.
   a. $\sqrt{36}$
   b. $\sqrt{20}$
   c. $\sqrt{6} \cdot \sqrt{10}$
   d. $\sqrt{\frac{40}{8}}$
   e. $\frac{\sqrt{20}}{\sqrt{5}}$
2. Use the patterns above to simplify.
   a. $\sqrt{a^2b^2}$
   b. $\sqrt{x^2y}$
   c. $\sqrt{xy} \cdot \sqrt{xz}$

d. $\sqrt{\frac{x^3y}{x^3}}$

e. $\frac{\sqrt{x^2y}}{\sqrt{y}}$

3. In order for the expressions above to make sense, what restrictions must be placed on the variables a, b, x, y and z? For example can $a = 0$? Can $x = 0$? Can $y = -1$?

PROBLEM 1

For each of the following, specify the conditions that must be placed on the variables. Then simplify.

1. $\sqrt{a^2b^3c}$
2. $\sqrt{5ab^2}\sqrt{10a^3}$
3. $\sqrt{\frac{6x^3}{10xy^2}}$
4. $\sqrt{(x+1)^2}$

Now we look at the relationship between square roots and exponents. Recall the rules of exponents from Chapter 6.

Rules of Exponents Let $x \geq 0$ and $y \geq 0$ be two non negative numbers and r and s be real numbers.

Product of Powers: $(x^r)(x^s) = x^{(r+s)}$

Power of Power: $(x^r)^s = x^{rs}$

Quotient of Powers: $\frac{x^r}{x^s} = x^{(r-s)}$

Distributive Property: $x^r y^r = (xy)^r$

Let's think about what effect taking the square root has on the exponent.

EXPLORATION 2

Suppose $a \geq 0$ is a real number and n is a whole number. Simplify each of the following. How would you describe what the square root does to the exponent?

1. $\sqrt{2^2}$
2. $\sqrt{2^4}$
3. $\sqrt{3^4}$
4. $\sqrt{13^{20}}$
5. $\sqrt{a^2}$
6. $\sqrt{a^6}$
7. $\sqrt{a^{2n}}$

In Exploration 2 we noticed that taking the square root divides the exponent by 2. This suggests that square roots have something to do with the fraction $\frac{1}{2}$. Let's investigate further.

EXPLORATION 3

Let $x > 0$ be a real number.

1. Simplify $\sqrt{x}\sqrt{x}$.
2. Use the Product of Powers rule to compute $(x)^{\frac{1}{2}} \cdot (x)^{\frac{1}{2}}$.
3. Use the Power of a Power rule to compute $(x^2)^{\frac{1}{2}}$.

In Exploration 3 we see that it is natural to use the exponent of $\frac{1}{2}$ to represent a square root. So for example, we write $\sqrt{25} = \sqrt{5^2} = (5^2)^{\frac{1}{2}} = 5$. The rules of exponents can then be used to simplify expressions involving square roots.

EXAMPLE 1

Assume $x > 0$. Write each of the following using an exponent of $\frac{1}{2}$ in place of the square root sign. Then simplify.

1. $\sqrt{x^{100}}$

2. $\sqrt{25x^4}$
3. $\sqrt{x^9}$

SOLUTION For the first two, we simply use the rules of exponents to find the answer.

1. $\sqrt{x^{100}} = (x^{100})^{\frac{1}{2}} = x^{\frac{1}{2}\cdot 100} = x^{50}$
2. $\sqrt{25x^4} = (5^2x^4)^{\frac{1}{2}} = 5x^2$
3. We begin the same way: $\sqrt{x^9} = (x^9)^{\frac{1}{2}} = x^{\frac{9}{2}}$. Now we need to interpret the $\frac{9}{2}$ in the exponent. Since $\frac{9}{2} = 4 + \frac{1}{2}$, then:
$$x^{\frac{9}{2}} = x^{(4+\frac{1}{2})} = x^4x^{\frac{1}{2}} = x^4\sqrt{x}.$$
Depending on the situation, it may be more convenient to use the square root sign or to use exponent of $\frac{1}{2}$.

PROBLEM 2

Assume $x > 0$ and $y > 0$. Write each of the following using an exponent of $\frac{1}{2}$ in place of the square root sign. Then simplify.

1. $\sqrt{x^{20}}$
2. $\sqrt{x^4y^8}$
3. $\sqrt{x^2y^5}$

In some situations we have a polynomial expression under the square root. In these cases, we often must factor before we can simplify.

EXAMPLE 2

Simplify.

1. $\sqrt{(x+1)^4}$.
2. $\sqrt{x^2+8x+16}$.
3. $\sqrt{4x^2+12x^2y}$

SOLUTION Once we write the expression under the square root sign as a product of factors with exponents, we can use the same rules as before to simplify. Recall that the inputs and outputs of the square root function must be nonnegative. We must be careful, however, to make sure any expression we leave under the square root and any expression we take out from the square root is nonnegative.

1. In this case, we already know what to do:

$$\sqrt{(x+1)^4} = (x+1)^2.$$

   Because of the square, we know that $(x+1)^2 \geq 0$.

2. First we must write the expression under the radical as a product of factors: $x^2 + 8x + 16 = (x+4)^2$. So:

$$\sqrt{x^2 + 8x + 16} = \sqrt{(x+4)^2} = |x+4|.$$

   We do not know the value of x, so we need to write the absolute value to make sure $|x+4| \geq 0$.

3. If we take out the common factor, we obtain: $4x^2(1+3y)$. So:

$$\sqrt{4x^2 + 12x^2y} = \sqrt{4x^2(1+3y)} = 2|x|\sqrt{1+3y}.$$

   Again, we don't know the value of x, so we write the answer using absolute value. We also don't know the value of y, so we must add a condition on y. We need to solve the inequality $1 + 3y \geq 0$ to find the right condition.

$$\begin{aligned} 1 + 3y &\geq 0 \\ 3y &\geq -1 \\ y &\geq -\frac{1}{3} \end{aligned}$$

   So $\sqrt{4x^2 + 12x^2y} = 2|x|\sqrt{1+3y}$ as long as $y \geq -\frac{1}{3}$.

PROBLEM 3

Simplify:

1. $\sqrt{(x-2)^3}$.
2. $\sqrt{x^2-6x+9}$.
3. $\sqrt{18x^3y^4-9x^2y^5}$.

The cube root

In some applications, it is necessary to find other roots, in particular the cube root. We can define this using our radicals as before. The cube root is written:

$$\sqrt[3]{t} = x \text{ means that } x \text{ satisfies } x^3 = t.$$

The radical expression $\sqrt[3]{t}$ is called the cube root of t. We call t the *radicand*. We always write the 3 just outside the radical when we are referring to the cube root. We could put a 2 to represent the square root as well, but we traditionally leave it out. So $\sqrt[2]{t} = \sqrt{t}$ is the square root of t and $\sqrt[3]{t}$ is the cube root of t.

All the properties of square roots listed at the beginning of this section hold true for cube roots, except that the inverse property now becomes:

$$\sqrt[3]{t^3} = t$$

and t can now be a negative number. As we did for square roots we can use these properties to simplify expressions.

EXAMPLE 3

Simplify each of the following.

1. $\sqrt{8}$

2. $\sqrt[3]{8}$
3. $\sqrt[3]{a^6b^3}$

SOLUTION We must pay attention to which root we are working with. When we are using the cube root, we want to write the radicand in powers of 3.

1. This is the square root. As we have done before, $\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$.
2. Now it is a cube root, so $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$.
3. $\sqrt[3]{a^6b^3} = \sqrt[3]{(a^2b)^3} = a^2b$.

We can use fractional exponents to represent cube roots as well.

$$\sqrt[3]{t} = t^{\frac{1}{3}}.$$

EXAMPLE 4

Write each of the following using a fractional exponent and then simplify.

1. $\sqrt[3]{x^3y^{12}}$
2. $\sqrt[3]{27x^9y^{21}}$
3. $\sqrt[3]{x^4}$

SOLUTION

1. $\sqrt[3]{x^3y^{12}} = (x^3y^{12})^{\frac{1}{3}} = x^{(3\cdot\frac{1}{3})} \cdot y^{12\cdot\frac{1}{3}} = xy^4$
2. $\sqrt[3]{27x^9y^{21}} = (3^3x^9y^{21})^{\frac{1}{3}} = 3x^3y^7$
3. We begin the same way: $\sqrt[3]{x^4} = (x^4)^{\frac{1}{3}} = x^{\frac{4}{3}}$. Now we need to interpret the $\frac{4}{3}$ in the exponent. Since $\frac{4}{3} = 1 + \frac{1}{3}$, then

$$x^{\frac{4}{3}} = x^{(1+\frac{1}{3})} = xx^{\frac{1}{3}} = x\sqrt[3]{x}.$$

or

$$\sqrt[3]{x^4} = \sqrt[3]{x^3}\sqrt[3]{x} = x\sqrt[3]{x}$$

PROBLEM 4

Simplify.

1. $\sqrt[3]{27x^3}$
2. $\sqrt[3]{x^8y^{24}}$
3. $\sqrt[3]{9x^5y^9}$

Once we start using fractional exponents, it is important to carefully interpret what each part of the exponent means and to recognize there are many equivalent ways to write a radical expression.

EXPLORATION 4

1. What does $x^{\frac{5}{3}}$ mean? Write at least 4 expressions that are equivalent to $x^{\frac{5}{3}}$. Interpret each one.
2. What does $x^{-\frac{5}{3}}$ mean?

EXERCISES

1. Simplify each of the following.
   a. $\sqrt{20}$
   b. $\sqrt{72}$
   c. $\sqrt{27}$
   d. $\sqrt{52}$
   e. $\sqrt{\frac{72}{8}}$
   f. $\frac{2\sqrt{50}}{3\sqrt{2}}$
2. For each of the following, specify what conditions must be placed on the variables, then simplify.
   a. $\sqrt{x^2y^3}$

b. $\sqrt{ab^2c}$

c. $\sqrt{p^4q^3r^5}$

d. $\sqrt{12(x-1)^3}$

e. $\sqrt{\frac{(x-15)^3}{(x-15)}}$

f. $\sqrt{\frac{12x^5y^3z}{3xyz}}$

3. Rewrite each of the following by replacing the radical with a fractional exponent. Then simplify.

a. $\sqrt{x^2y^4z^8}$

b. $\sqrt{x^4y^5}$

c. $\sqrt[3]{x^3y^{21}}$

4. Simplify.

a. $\sqrt{54}$

b. $\sqrt[3]{54}$

c. $\sqrt{125}$

d. $\sqrt[3]{24}$

e. $\sqrt[3]{48}$

5. Simplify.

a. $\sqrt[3]{x^3y^4}$

b. $\sqrt[3]{x^{15}y^{10}}$

c. $\sqrt{(x+2)^2}$

6. Use factoring to simplify.

a. $\sqrt{x^2y+x^2z}$

b. $\sqrt{x^2+6x+9}$

c. $\sqrt[3]{8a^3b-8a^3c}$

SECTION 11.3 SOLVING RADICAL EQUATIONS

In Section 11.1, we explored the graph of the square root function given by the formula $f(x) = \sqrt{x}$. What are the possible outputs for this function? We will explore this question in the following:

EXPLORATION 1

Use the graph of the square root function to find input values that will yield the following output values:

a. 4
b. 3
c. 2
d. 1.5
e. 0
f. -2

To find an input value that gives us the output 4, it is useful to reframe the question as follows: for what numbers x is the output $f(x)$ equal to 4? In other words, for what numbers x is $\sqrt{x} = 4$? This second question states the problem as an equation to be solved:

$$\sqrt{x} = 4.$$

The point $(16, 4)$ is on the graph, which corresponds to the fact that the input 16 yields the output 4, or $f(16) = \sqrt{16} = 4$. Another way to think of this problem is to see that if $\sqrt{x} = 4$, then:

$$\sqrt{x} \cdot \sqrt{x} = 4 \cdot 4 = 16.$$

The left hand side of the equation, $\sqrt{x} \cdot \sqrt{x}$, is equal to x.

A compact way to write this solution is:

$$\begin{aligned} \sqrt{x} &= 4 \\ (\sqrt{x})^2 &= 4^2 \\ x &= 16. \end{aligned}$$

We see that squaring both sides of an equation allows us to isolate the variable x on the left side of the equation.

A *radical equation* is an equation that contains variables under the radical. In order to solve this type of equation, we perform the following steps:

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the n^{th} power, where n is the index of the radical. (In other words, if the radical is a square root, then square both sides of the equation. If the radical is a cube root, then cube both sides.) This will eliminate the radical.
3. Solve for the unknown.

We can quickly check the other output-input problems from Exploration 1. Applying the technique of squaring both sides of the equation, we get the following results:

$$\begin{aligned} \sqrt{x} &= 3 & \Rightarrow \quad x &= 9 \\ \sqrt{x} &= 2 & \Rightarrow \quad x &= 4 \\ \sqrt{x} &= 1.5 & \Rightarrow \quad x &= 2.25 \\ \sqrt{x} &= 0 & \Rightarrow \quad x &= 0 \end{aligned}$$

Take a closer look at the equation $\sqrt{x} = -2$. From before, we know that for any input x, $\sqrt{x} \geq 0$. So this equation has no solution. If, however, we square both sides of this equation, we get $x = 4$. Of course, if we try to check this solution by substituting $x = 4$ into the original equation, we get $\sqrt{4} = -2$ which is not true. Why does this happen?

This example shows that we must be careful when squaring both sides of an equation. When we do this, we may introduce *extraneous solutions* – that is, values that appear to be solutions, even though they are not. So when we solve a radical equation, we should always check the solution candidates we get in order to make sure that they satisfy the original equation.

EXAMPLE 1

Solve each of the following radical equations:

1. $\sqrt{x} - 4 = 2$
2. $\sqrt{x-4} = 2$
3. $\sqrt{x} + 5 = 2$

SOLUTION

1. We'll start by isolating the radical; we do this by adding 4 to both sides of the equation to get $\sqrt{x} = 6$. We can then square each side to get $x = 36$. When we substitute this into the original equation, we get $\sqrt{36} - 4 = 2$, which is a true statement. So $x = 36$ is the solution to this equation.
2. This time, the radical is already isolated, so we'll start by squaring both sides. This gives the equation $x - 4 = 4$, which yields $x = 8$. Substituting $x = 8$ into the original equation gives $\sqrt{8-4} = 2$, which is a true statement since $\sqrt{4} = 2$. So $x = 8$ is the solution to this equation.
3. If we subtract 5 from both sides first, we get $\sqrt{x} = -3$. But we know $\sqrt{x} \geq 0$, so there is no solution. If we try to square both sides, then $x = 9$ seems like a possible solution. But when we substitute this into the original equation, we get $\sqrt{9} + 5 = 2$, which isn't true.

PROBLEM 1

Solve each of the following equations. Compare the approaches you use in these 2 problems. Make sure to check your solutions.

1. $\sqrt{x+2} = 3$
2. $\sqrt{x-5} + 4 = 10$

Once we have mastered the basic techniques for solving radical equations, we can solve more complicated equations.

EXAMPLE 2

Solve the equation $x = \sqrt{3x+4}$.

SOLUTION In this equation, the radical is already isolated, so we begin by squaring both sides of the equation to get $x^2 = 3x + 4$. We can then rearrange the terms of this equation to get $x^2 - 3x - 4 = 0$. This factors into $(x-4)(x+1) = 0$, so we have $x = 4$ or $x = -1$. We will now substitute both of these values into the original equation to determine whether they are solutions. Substituting $x = 4$ into the original equation gives:

$$4 = \sqrt{3 \cdot 4 + 4} \Rightarrow 4 = \sqrt{16}$$

which is true. Substituting $x = -1$ into the original equation gives:

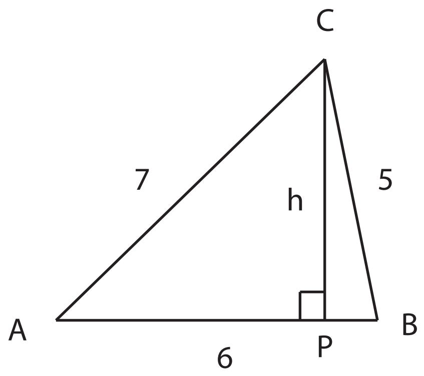
$$-1 = \sqrt{3 \cdot -1 + 4} \Rightarrow -1 = \sqrt{1}$$

which is false. So $x = 4$ is a solution, but $x = -1$ is not. This means that $x = 4$ is the only solution of the equation $x = \sqrt{3x+4}$.

Occasionally, we encounter situations that lead to even more complicated radical equations.

EXAMPLE 3

In triangle ΔABC, $AB = 6$, $BC = 5$ and $CA = 7$. What is the height of the triangle, measured from vertex C to side $\overline{AB}$?



SOLUTION Let h be the height of ΔABC, and let P be the point at which the perpendicular from C intersects side $\overline{AB}$. We know that ΔAPC is a right triangle, so the length of $\overline{AP}$ is given by:

$$(AP)^2 + h^2 = 7^2$$

by the Pythagorean Theorem. So $AP = \sqrt{49 - h^2}$. Similarly, since ΔBPC is a right triangle, we have $BP = \sqrt{25 - h^2}$. Since $AP + BP = 6$, we can then say that $\sqrt{49 - h^2} + \sqrt{25 - h^2} = 6$. We will now try to solve this equation for h.

This time, our equation has 2 radicals, so if we isolate one of them, the other one will not be isolated. Still, we will try isolating one of them:

$$\sqrt{49 - h^2} = 6 - \sqrt{25 - h^2}.$$

Now we square both sides of the equation. We must be careful here, since the right side of the equation is a binomial.

$$49 - h^2 = (6 - \sqrt{25 - h^2})^2$$

$$= 36 - 12\sqrt{25 - h^2} + (25 - h^2)$$
$$= 61 - h^2 - 12\sqrt{25 - h^2}$$

We can now add h^2 to both sides of the equation to get $49 = 61 - 12\sqrt{25 - h^2}$, and then subtract 61 from both sides to get $-12 = -12\sqrt{25 - h^2}$. Dividing both sides by -12 yields:

$$\sqrt{25 - h^2} = 1.$$

We can then square both sides of the equation again to get $25 - h^2 = 1$. So $h^2 = 24$, and thus $h = \pm 2\sqrt{6}$. We know that h must be a positive number, so we have $h = 2\sqrt{6}$. Substituting this value into the original equation gives:

$$\sqrt{49 - (2\sqrt{6})^2} + \sqrt{25 - (2\sqrt{6})^2} = 6$$

which is true since $\sqrt{49 - (2\sqrt{6})^2} + \sqrt{25 - (2\sqrt{6})^2} = \sqrt{25} + \sqrt{1} = 6$. So the height of the triangle is $2\sqrt{6}$. We can also use this to determine the area of the triangle:

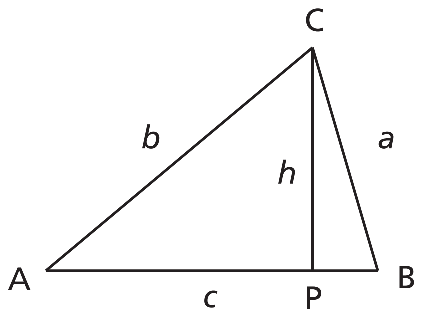
$$\text{Area } = \frac{1}{2}h \cdot AB = \frac{1}{2} \cdot 2\sqrt{6} \cdot 6 = 6\sqrt{6}.$$

Note that we were able to find the area of the triangle given only the lengths of the 3 sides of the triangle! There is a formula that allows us to do this for any triangle; we will explore this formula in Exercise 5.

As we saw in this example, when we encounter an equation involving more than one radical, it can be useful to square both sides of the equation, even if doing this does not eliminate all the radicals in the equation. In many cases, doing this will at least reduce the number of radicals in the equation; we can then rearrange the equation and square again in order to eliminate all of the radicals.

EXERCISES

1. Solve and check your answers.
   a. $\sqrt{x+5} = x - 1$
   b. $2\sqrt{2x+5} = x$
   c. $3\sqrt{x-7} = \sqrt{4x+17}$
   d. $\sqrt{\frac{3x-7}{2}} = \sqrt{x+1}$
   e. $\sqrt{15-x} = x + 5$
   f. $\sqrt{x^2+11x+4} = \sqrt{x+15}$
   g. $\sqrt{x^2+7x-2} = x - 6$
   h. $\sqrt{\frac{1}{x}} = \frac{2}{x}$
2. Solve and check your answers.
   a. $\sqrt{5x+15} = 7 - x$
   b. $\sqrt{5x+15} = x - 7$
3. Billy is using a formula from a physics book that tells him that the time t needed for a falling object to be at height h is given by:
   $$t = \sqrt{\frac{2000-h}{16}}$$
   with h measured in feet and t in seconds. Find the values of t for $h = 1600, 1200, 400$, and 0 feet.
4. **Ingenuity:**
   Solve and check your answers:
   a. $\sqrt{x+2} + \sqrt{x+9} = 7.$
   b. $\sqrt{x+1} + \sqrt{2x-5} = \sqrt{5x+6}.$
5. **Investigation:**
   In Example 3, we determined the area of a triangle given only the lengths of its 3 sides. In this Investigation, we will see that this is possible for any triangle, and we will discover a beautiful geometric formula. Let ΔABC be a triangle, and let a, b and c be the lengths of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let h be the perpendicular distance from vertex C to side $\overline{AB}$.



a. In terms of a, b, c and h, compute the distances AP and BP. What must these distances add up to? Write this fact as an equation.
b. Rearrange the equation so that the 2 radicals are on different sides, square both sides of the equation, and simplify. How many radicals remain in the equation after you do this?
c. Eliminate the remaining radical(s) in the equation. You should end up with a square of a rational expression on one side of the equation. Leave this square in factored form for now; do not compute the square.
d. Isolate h^2 on one side of the equation. The other side of the equation should be a difference of squares; factor this difference of squares.
e. You should now have an equation stating that h^2 is equal to a product of 2 factors, where each factor is a rational expression. Simplify these rational expressions by getting a common denominator for each.
f. Now, factor these rational expressions as much as possible. (*Hint:* The numerator of each expression should be a difference of squares.)
g. Now solve for h. You will end up with a square root on the other side of the equation.
h. Now that you know the height of the triangle, find the area in terms of a, b and c.
i. Let s be the *semiperimeter* of the triangle; that is, $s = \frac{a+b+c}{2}$. Show that the expression you obtained in part (h) is equal to $\sqrt{s(s-a)(s-b)(s-c)}$. This elegant expression for the area of a triangle is called *Heron's formula*.

SECTION 11.4 CHAPTER REVIEW

Key Terms

- cube root
- domain
- extraneous solutions
- nonnegative number
- radical
- radical equation
- radical function
- radicand
- range
- square root function

Rules of Exponents and Roots

Product of Powers Rule:
$x^m \cdot x^n = x^{m+n}$

Distributive Property of Exponents:
$y^n \cdot z^n = (y \cdot z)^n$

Power of a Power Rule:
$(z^a)^b = z^{ab}$

Quotient of Powers Rule:
$\frac{x^n}{x^m} = x^{n-m}$

Zero Power:
If $x \neq 0$, $x^0 = 1$

Negative Power Rule:
If $x \neq 0$, $x^{-1} = \frac{1}{x}$

One Half Power:
If $x \neq 0$, $\sqrt{x} = x^{\frac{1}{2}}$

Inverse Root Rule:
If $x > 0$, $\sqrt{x^2} = x$

Product of Roots:
If $x > 0$ and $y > 0$,
$\sqrt{xy} = \sqrt{x}\sqrt{y}$

Quotient of Roots:
If $x > 0$ and $y > 0$
$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Practice Problems

1. A right triangle has legs of 4 and 12. Find the length of the hypotenuse of the triangle. Calculate the perimeter and area of the triangle.
2. Let $g(x) = -4 + \sqrt{x}$.
   a. What is the domain of g?
   b. What is the range of g?
   c. Make table of at least 5 points on the graph of g and plot the points.
   d. Let $h(x) = g(x-5)$. Without graphing, predict what the graph of h would look like. What is the domain of h?
3. Suppose we have a right triangle with legs of lengths 2 and $(x+3)$ units.
   a. Make a sketch of the triangle.
   b. Write an expression for the length of the hypotenuse in terms of x, call it $h(x)$.
   c. Identify the domain and range of $h(x)$. Plot $h(x)$.
4. Simplify the following. Write you answer using a radical. Note we assume that all variable are nonnegative.
   a. $\sqrt{96}$
   b. $\sqrt{m^5n^3p^4}$
   c. $\sqrt{72q^5t}$
   d. $\sqrt[3]{54w^4x^2}$
5. Simplify the following write your answer using a fractional exponent. Note we assume that all expressions under the root are nonnegative.
   a. $\sqrt[3]{250w^7}$
   b. $\sqrt{2x^2+12x+18}$
   c. $\sqrt{12(x-3)^3}$
   d. $\sqrt{12x^4y^2-8x^4y^2}$
6. Solve the following equations. Check your answers.
   a. $\sqrt{f+5} = \sqrt{3f-7}$
   b. $\sqrt{g^2+3g+5} = g+2$
   c. $\sqrt{z-3} = z+2$
7. Solve the following equations. Check your answers.
   a. $\sqrt{x+2}\sqrt{3x-4} = 5$
   b. $\sqrt{x-1}\sqrt{2x+3} = \sqrt{10x+15}$

RATIONAL EXPRESSIONS

12

SECTION 12.1 OPERATIONS WITH RATIONAL EXPRESSIONS

We began our study of arithmetic with counting numbers, $1, 2, 3, \ldots$. We then extended these to include negative integers and zero. The next extension was to study fractions, $\frac{a}{b}$, where a and b are integers. We then introduced variables, and made expressions such as $(x + 2)$ or $(x^2 - 2x + 4)$. In this section, we will look at fractions where the denominator is an expression with variables. We call these kind of expressions *rational expressions*. Often the numerator will be an expression with variables as well. So a rational number is a ratio of integers and a rational expression is a ratio of algebraic expressions.

EXAMPLE 1

Suppose that Mary can complete a job in x hours. Write a rational expression for the part of the job that she can complete in 3 hours.

SOLUTION In x hours, Mary will do one job. In 1 hour, Mary will do $\frac{1}{x}$ of a job. In 3 hours, Mary will do $\frac{3}{x}$ of a job. Does this make sense?

What if it takes Mary 6 hours to complete a job? This means that $x = 6$, and the amount of job Mary completes will simplify to $\frac{3}{6}$ job $= \frac{1}{2}$ job in 3 hours.

Equivalent Rational Expressions
In Section 6.2, we used the quotient of powers rule,

$$\frac{x^n}{x^m} = x^{n-m},$$

to simplify rational expressions. The process is the same you use to simplify fractions.

EXPLORATION 1

1. Find the prime factorization of each of the following pairs of numbers:
   a. 6 and 70
   b. 36 and 30
   c. 24 and 72
2. Simplify the following fractions:
   a. $\frac{6}{70}$
   b. $\frac{30}{36}$
   c. $\frac{24}{72}$
3. Simplify the following rational expressions:
   a. $\frac{ab}{acd}$
   b. $\frac{abc}{a^2b^2}$
   c. $\frac{a^3b}{a^3b^2}$

Using the factoring techniques from Chapter 7, we can simplify more complicated rational expressions.

EXAMPLE 2

Consider the rational expression:

$$\frac{x^2 - 5x + 6}{x^2 - 4x + 4}.$$

1. Factor $x^2 - 5x + 6$ and $x^2 - 4x + 4$.
2. What are permissible values of x?
3. Simplify $\frac{x^2-5x+6}{x^2-4x+4}$.

SOLUTION

1. $x^2 - 5x + 6 = (x - 3)(x - 2)$ and $x^2 - 4x + 4 = (x - 2)^2$.
2. We usually allow the variable x to take on any value. But, when working with rational expressions, we must be careful not to divide by 0. So we can not let x equal any value which makes the denominator 0. The factored form is helpful in figuring out which values we must avoid. In this case, the denominator is $x^2 - 4x + 4 = (x - 2)^2$, so $x \neq 2$.
3. We use the factored form to see how to simplify the expression:

$$\frac{x^2 - 5x + 6}{x^2 - 4x + 4} = \frac{(x - 3)(x - 2)}{(x - 2)^2} = \frac{(x - 3)}{(x - 2)}.$$

PROBLEM 1

Consider the rational expressions:

- $\frac{x^2-2x-15}{x^2-2x-8}$
- $\frac{x^2-9}{x^2-x-6}$

- $\frac{3x^2-12x+12}{x^3-4x^2+4x}$

For each rational expression above:

1. Determine which values of x make the denominator 0.
2. Simplify the expression.

In the next example, we see that sometimes we can simplify complicated rational expressions.

EXAMPLE 3

Consider the rational expression:

$$\frac{x(x+5)+x^2-4x-11-(x+2)(x-1)}{(x+7)(x-2)-2(x+2)-(x-3)(x+4)}.$$

1. Determine which values of x make the denominator 0.
2. Simplify the expression.

SOLUTION One method to deal with such a complicated expression is to use the distributive property and combine like terms to find a equivalent expression with no parentheses.

Numerator

$$\begin{aligned} &x(x+5)+x^2-4x-11-(x+2)(x-1)\\ &\quad = x^2+5x+x^2-4x+3-(x^2+x-2)\text{[Dist. Prop.]}\\ &\quad = x^2+5x+x^2-4x-11-x^2-x+2\text{ [Dist. Prop.]}\\ &\quad = x^2-9\text{ [Comb. Like Terms]} \end{aligned}$$

Denominator

$$\begin{aligned} &(x+\ 7\ )(x-2)-2(x+2)-(x-3)(x+4)\\ &\quad = x^2+5x-14-2x-4-(x^2+x-12)\text{[Dist. Prop.]}\\ &\quad = x^2+5x-14-2x-4-x^2-x+12\text{ [Dist. Prop.]}\\ &\quad = 2x-6 \end{aligned}$$

Now the expression is easier to work with.

1. The denominator will be 0 when $2x-6=0$. So $x \neq 3$.
2. We can also factor and simplify.

$$\frac{x(x+5)+x^2-4x-11-(x+2)(x-1)}{(x+7)(x-2)-2(x+2)-(x-3)(x+4)}$$

$$= \frac{x^2 - 9}{2x - 6} = \frac{(x-3)(x+3)}{2(x-3)} = \frac{x+3}{2}$$

In Example 3, we saw how a complicated expression can be simplified. However, you need to be very careful that at each step you use the rules of algebra to ensure that the result is an equivalent expression. In the excitement of working with expressions we often desire a simpler equivalent expression and it is easy to say expressions are equivalent when they really are not. Let's explore some common errors to watch out for.

EXPLORATION 2

For each of the following pairs of expressions determine if they are equivalent or not. Explain your answer.

1. $\frac{x^2-2x}{x}$ and $x - 2$
2. $25x - (3 - 10x)$ and $15x - 3$
3. $\frac{x}{x+2}$ and $1 + \frac{x}{2}$
4. $\frac{x^2+6x+9}{x-3}$ and $-(x + 3)$
5. $(x - 2)^2 - x$ and $x^2 - 5x + 4$
6. $(2x + 3)^2$ and $4x^2 + 9$
7. $\frac{x+2}{x}$ and 2
8. $\frac{x^2-25}{x+5}$ and $x - 5$

Adding Rational Expressions

Rational expressions can be added and subtracted just like integers. The key is to make sure that the rational expressions you are combining have the same denominator. If this is the case, then you can combine fractions as usual, since

$$\frac{a}{x} + \frac{b}{x} = \frac{a+b}{x}.$$

EXAMPLE 4

Combine the rational expressions $\frac{4}{x} + \frac{3}{x}$.

SOLUTION Since both denominators are the same, you can add the fractions by combining the numerators.

$$\frac{4}{x} + \frac{3}{x} = \frac{7}{x}$$

This is exactly how you would combine fractions if the denominator were a number, say $x = 10$.

$$\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$$

Next we explore some trickier problems that occur when you combine rational expressions with different denominators.

EXPLORATION 3

1. Find the following sums. Carefully write out each step.
   a. $\frac{1}{3} + \frac{1}{2}$
   b. $\frac{2}{3} + \frac{1}{4}$
   c. $\frac{5}{4} + \frac{2}{3}$
2. How did you find the common denominator in part a? Use the same patterns to find the sums of the following rational expressions.
   a. $\frac{1}{x} + \frac{1}{y}$
   b. $\frac{2}{x} + \frac{1}{y}$
   c. $\frac{5}{x} + \frac{2}{y}$

In Exploration 3 we found the common denominator by taking the product of the original denominators.

EXAMPLE 5

Combine the rational expressions $\frac{1}{x} - \frac{1}{y}$.

SOLUTION The first step is to find equivalent fractions with the same denominators. In this example, you can use the denominator xy.

$$\frac{1}{x} = \frac{y}{y} \cdot \frac{1}{x} = \frac{y}{xy}$$

and

$$\frac{1}{y} = \frac{x}{x} \cdot \frac{1}{y} = \frac{x}{xy}$$

Subtracting yields,

$$\frac{y}{xy} - \frac{x}{xy} = \frac{y - x}{xy}$$

EXAMPLE 6

Combine the rational expressions $\frac{1}{x} + \frac{1}{2}$.

SOLUTION The first step is to find a common denominator. We need a denominator that is multiple of both 2 and x, and as above you can use the product, namely $2x$.

$$\frac{1}{x} = \frac{2}{2} \cdot \frac{1}{x} = \frac{2}{2x}$$

and

$$\frac{1}{2} = \frac{x}{x} \cdot \frac{1}{2} = \frac{x}{2x}$$

Adding yields,

$$\frac{2}{2x} + \frac{x}{2x} = \frac{2 + x}{2x}$$

When working with more complicated denominators we often find equivalent fractions using the least common denominator. The *least common denominator* is the least common multiple (LCM) of the original denominators. For example, if we want to add $\frac{1}{6} + \frac{3}{4}$, we start by finding the least common multiple of 6 and 4. This is 12. So we find equivalent fractions with a common denominator of 12.

$$\begin{aligned} \frac{1}{6} + \frac{3}{4} &= \frac{1}{6} \cdot \frac{2}{2} + \frac{3}{4} \cdot \frac{3}{3} \\ &= \frac{2}{12} + \frac{9}{12} \\ &= \frac{11}{12} \end{aligned}$$

One way you may have learned to find the least common multiple is to use the prime factorization. Since:

$$6 = 2 \cdot 3 \qquad \text{and} \qquad 4 = 2^2,$$

we can see that the least common multiple must contain two 2's and one 3. In other words the LCM should be $2^2 \cdot 3 = 12$. If we compare the prime factorization of 6, 4 and 12, we can see how we form the equivalent fractions above. For example, since $6 = 2 \cdot 3$ and $12 = 2^2 \cdot 3$, we need to multiply $\frac{1}{6}$ by $\frac{2}{2}$. In the next exploration we will see that working with rational expressions is very similar to the method using prime factorization.

EXPLORATION 4

1. For each pair of numbers, write out the prime factorization.
   a. 6 and 10
   b. 12 and 15
   c. 8 and 9
2. Find the least common multiple in prime factored form of each pair above.
3. Find the following sums. Carefully write out each step.

a. $\frac{1}{6} + \frac{3}{10}$
b. $\frac{5}{12} + \frac{7}{15}$
c. $\frac{3}{8} + \frac{2}{9}$

4. Find the least common multiple for each of the following:
   a. xy and yz
   b. x^2y and yz
   c. x^2 and y^2.
5. Find the sums of the following rational expressions:
   a. $\frac{1}{xy} + \frac{3}{yz}$
   b. $\frac{5}{x^2y} + \frac{7}{yz}$
   c. $\frac{3}{x^2} + \frac{2}{y^2}$

EXAMPLE 7

Find $\frac{2}{a^2bc^2} - \frac{3}{abc^3}$.

SOLUTION We first find the least common multiple of the denominators: a^2bc^2 and abc^3. We see that our multiple must contain two a's, one b and three c's. So the LCM is a^2bc^3. Now we write equivalent fractions with common denominator, a^2bc^3:

$$\frac{2}{a^2bc^2} = \frac{c}{c} \cdot \frac{2}{a^2bc^2} = \frac{2c}{a^2bc^3},$$

and

$$\frac{3}{abc^3} = \frac{a}{a} \cdot \frac{3}{abc^3} = \frac{3a}{a^2bc^3}.$$

Subtracting yields

$$\frac{2c}{a^2bc^3} - \frac{3a}{a^2bc^3} = \frac{2c - 3a}{a^2bc^3}.$$

EXERCISES

1. The percentage of a saline solution is given by the number of grams of salt in the solution divided by the total weight of the solution times 100 percent.
   a. How much salt is in 50 grams of 2% saline solution?
   b. What is the percentage of the solution if we dissolve 5 grams of salt in 35 grams of water?
   c. Write a rational expression for the percentage of a saline solution in which 5 grams of salt is dissolved in x grams of water.

2. There are ten more boys than girls in a class. Let $x =$ the number of girls.
   a. Write a rational expression for the ratio of boys to girls.
   b. Write a rational expression for the percentage of girls in the class.

3. Find the following sums:
   a. $\frac{2}{x} + \frac{3}{x}$
   b. $\frac{4}{a^2} + \frac{5}{a^2}$
   c. $\frac{x}{2} + \frac{3x}{2}$
   d. $\frac{3}{2x} + \frac{5}{2x}$

4. Find the following differences:
   a. $\frac{5}{x} - \frac{2}{x}$
   b. $\frac{3}{xy^2} - \frac{2}{xy^2}$
   c. $\frac{1}{x} - \frac{6}{x}$
   d. $\frac{3x}{7} - \frac{x^2}{7}$

5. For each of the following pairs of numbers, find the prime factorization. Use the prime factorization to find the least common multiple in factored form.
   a. 60 and 24
   b. 100 and 56
   c. 210 and 36

6. Compute the following:
   a. $\frac{7}{60} + \frac{11}{24}$
   b. $\frac{9}{100} + \frac{15}{56}$
   c. $\frac{11}{210} - \frac{1}{36}$
7. Combine the following rational expressions:
   a. $\frac{3}{xy^2} + \frac{2}{x^2y}$
   b. $\frac{1}{xy} - \frac{1}{x^2y^2}$
   c. $\frac{3x}{14} + \frac{5x}{6}$
   d. $\frac{2d}{a^2bc} + \frac{3}{ab^3c^2}$
   e. $\frac{2}{(x-1)(x-2)} + \frac{5}{(x-1)^2}$
8. **Investigation:**
   Sometimes rational expressions involve polynomials in the denominator. It may be convenient to factor in order to find the common denominator.
   a. Factor each of the following pairs of polynomials:
      i. $x^2 + 2x + 1$ and $x^2 - 1$
      ii. $x^2 - 3x - 4$ and $x^2 + 5x + 4$
      iii. $x^2 + x - 6$ and $x^2 - 2x$
   b. Combine the following rational expressions:
      i. $\frac{1}{x^2+2x+1} + \frac{1}{x^2-1}$
      ii. $\frac{3}{x^2-3x-4} + \frac{5}{x^2+5x+4}$
      iii. $\frac{x}{x^2+x-6} + \frac{3}{x^2-2x}$
9. For each rational expression determine which values of x make its denominator 0 and then simplify the expression.
   a. $\frac{5x-35}{x^2-x-42}$
   b. $\frac{x^2+7x-18}{x^2-81}$

c. $\frac{x^3+5x^2+4x}{x^2+4x}$

d. $\frac{3x^2+7x+2}{3x^2+5x-2}$

e. $\frac{y^2-4x^2y^2}{3y-9xy+6x^2y}$

f. $\frac{x^4-16}{x^3-2x^2+4x-8}$

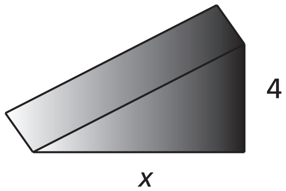
g. $\frac{(3x+1)^2+x(x-2)+9(x+1)-13}{(1+3x)(1-3x)-x(x+10)+8-3(x+2)}$

10. **Ingenuity:**
Three quarters of fish in a pond are females and one quarter are males. We add 10 males and females to the pond.
   a. Write a rational expression for ratio of female fish to male fish.
   b. Is the ratio of female fish to male fish still 3 to 1? If not, what the ratio larger or smaller than what we started with?

11. **Ingenuity:**
Suppose a, b, c and d are positive integers such that $\frac{a}{b} < \frac{c}{d}$. Show that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

SECTION 12.2 DIRECT AND INVERSE VARIATION

When several variables are related by an equation, we often characterize this relationship as either direct or inverse, especially in science applications. In this section we examine what these terms mean.



EXPLORATION 1

Consider the triangular ramp in the figure above. The steepness of the ramp is given by $s(x) = \frac{4}{x}$ where x is length of the base in feet. The area of the triangle on the side is given by $A(x) = \frac{1}{2} \cdot 4 \cdot x = 2x$.

1. Make a table of values for x, $s(x)$, $A(x)$.
2. Use the values from the table to graph $s(x)$ and $A(x)$ on the same coordinate grid.
3. Describe the graph of $A(x)$.
4. What happens to the area of the triangle when we double the length of the base? What if we triple it?
5. Describe the graph of $s(x)$.
6. What happens to the steepness of the ramp when we double the length of the base? What if we triple it?

In Exploration 1, A is the area of the triangle, then the relation between A and x is given by the equation $A = 2x$. The bigger x gets, the bigger A gets. If we double x, then we double A. Similarly, if we triple x, then we triple A. Recall, that we studied this kind of proportional relationship in Section 3.3. We say that x and y are proportional or that A varies directly as x.

DIRECTLY VARIES
If you have two variables, y and x, then we say that y *varies directly* as x if $y = kx$ for some constant k. k is called the *constant of proportionality* (or constant of variation).

The phrase "varies directly" really means "equals a constant times". In Exploration 1, A varies directly as x, and the constant of proportionality is 2. Note that the graph of $A(x)$ is a line and the constant of proportionality is the slope of the line.

EXAMPLE 1

The area of a circle varies directly as the square of the radius.

1. Express this relationship with an equation.
2. What is the constant of proportionality?
3. What is the radius of a circle with area equal to $10cm^2$.

SOLUTION

1. The area of a circle, A, is given by the equation $A = \pi r^2$.
2. The constant of proportionality is our old friend π.
3. If $A = 10$, then $10 = \pi r^2$. Solving for r gives $r = \sqrt{\frac{10}{\pi}}$.

EXAMPLE 2

Suppose a car travels at a rate of 50 miles per hour. Then the distance D the car travels varies directly as the time t the car travels. Find an equation that relates D and t. What is the constant of proportionality?

SOLUTION Recall the formula $D = (rate)\cdot(time)$. Since the rate is 50 miles per hour, you can write $D = 50t$. The constant of proportionality is the rate , 50 miles per hour.

Let's return to Exploration 1. If s is the steepness of the ramp, then the relation between s and x is given by the equation $s = \frac{4}{x}$. Now the bigger x gets, the smaller s gets. If we double x, then we cut s in half. Similarly, if we triple x, then we cut s in one-third. We say that s varies inversely as x. In direct variation, the variables change "directly." Inverse variation is quite similar, but the phrase "varies inversely" really means "equals a constant divided by".

INVERSELY VARIES
If you have two variables, y and x, then we say that y *varies inversely* as x if $y = \frac{k}{x}$ for some constant k. k is called the *constant of proportionality*.

EXAMPLE 3

y varies inversely as x. Find an equation that relates y and x. If $y = 20$ when $x = 2$, find the constant of proportionality. What does y equal if $x = 4$?

SOLUTION Since y varies inversely as x, then it equals a constant divided by x, $y = \frac{K}{x}$. Since $y = 20$ when $x = 2$, we substitute into this equation and obtain $20 = \frac{K}{2}$ or $K = 40$. So $y = \frac{40}{x}$. Now, if $x = 4$, then $y = \frac{40}{4} = 10$. This time, when x gets bigger, y gets smaller.

EXAMPLE 4

The pressure P of a gas in a closed system varies inversely as the volume V. Express this relationship with an equation. If $P = 100$, $V = 5$. Find the constant of proportionality.

SOLUTION $P = \frac{K}{V}$. Substituting into this equation, $100 = \frac{K}{5}$, So $K = 500$. Hence, we have the equation $P = \frac{500}{V}$.

EXPLORATION 2

For each of the following pairs, decide if the variables vary directly or inversely. Explain.

1. The circumference of a circle A and its diameter d.
2. The length L and width W of a rectangle with an area of 10 square meters.
3. The maximum weight W in pounds that can be supported by a bridge whose dimensions are 2 inch by 4 inch by L placed across a cavern.
4. The amount of money, M, raised by selling n candy bars.

PROBLEM 1

For each of the following, determine if y varies directly as x, inversely as x or neither:

1. $y = 3x$
2. $y = \frac{x}{3}$
3. $y = \frac{10}{x}$
4. $y = \frac{1}{3x}$
5. $y = 2x + 4$
6. $y = \frac{1}{x} - 2$
7. y is the temperature in Fahrenheit, x is the temperature in Celsius.

In many problems, you will have several variables and a combination of both direct and inverse variation.

EXAMPLE 5

The Force F between two objects varies directly as the product of the masses and inversely as the square of the distance between the objects. This is known as the "universal law of gravitational attraction". Express this law as an equation.

SOLUTION Let the masses of the objects be denoted by the variables M_1 and M_2. Let $d =$ the distance between the objects. Then:

$$F = \frac{K \cdot M_1 \cdot M_2}{d^2.}$$

In this problem: K is the constant of proportionality. You only need one constant since it takes into account all of the different variables. You multiply by M_1 and M_2, since F varies directly as these variables. You divide by d^2 since F varies inversely as the square of the distance. The constant K is very special and is called the gravitational constant.

When one variable varies directly with two variables, this is also called a joint variation. Joint variation is just saying that there are two variations that are both direct. But as you have seen above, you can have different combinations of direct and inverse variation all in one problem.

EXAMPLE 6

The temperature T varies jointly with the pressure P and Volume V. This is known as the universal gas law. Write an equation that relates T, P and V.

SOLUTION

$$T = K \cdot P \cdot V.$$

In direct and inverse variation problems, the first step is to find an equation that expresses the variation as either direct or inverse. Then you can use the other information in the problem to find the constant of proportionality. In the exercises below, you will explore different combinations of direct and inverse variation.

EXERCISES

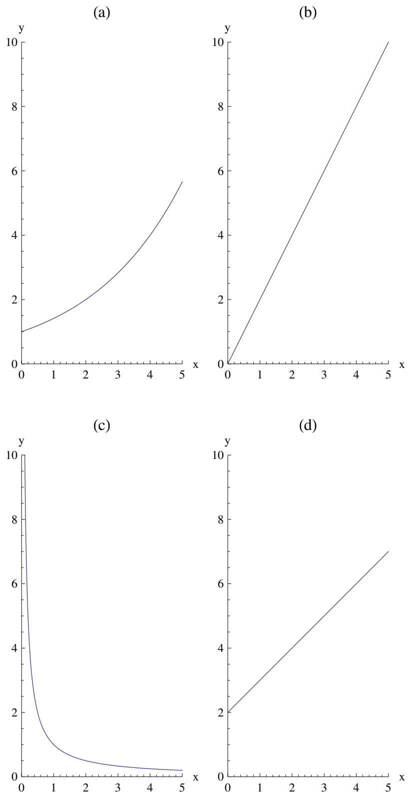
1. For each of the following determine if y varies directly as x, inversely as x or neither.
   a. $y = \frac{1}{2}x$
   b. $y = \frac{x}{9}$
   c. $y = \frac{\pi}{x}$
   d. $y = \frac{1}{2x}$
   e. $y = 8x - 4$
   f. $y = \frac{1}{x} + 7$
   g. y is the distance between Dallas and Houston in kilometers, x is the distance in miles.
2. If y varies directly as x, what will the graph of y vs. x look like?

   For Exercises 3 through 13, express each variation in the form of an equation, find the value of the constant k and solve for the value of the specified variable.

3. If x varies directly with y and $x = 12\pi$ when $y = 6$, what is the value of x when $y = 10$?
4. If x varies inversely with y and $x = 15$ when $y = 7$, what is the value of x when $y = 35$?
5. If x varies jointly with y and z^2 and $x = 300$ when $y = 4$ and $z = 5$, what is the value of z when $x = 600$ and $y = 2$?
6. Let r^2 vary inversely with h. If $r = 5$ when $h = 24$, what is the value of r when $h = 12$?
7. Let t vary jointly with b and w and inversely with r. If $t = 3$ when $b = 20$, $h = 30$ and $r = 200$, what is the value of b when $t = 4$, $w = 50$ and $r = 400$?
8. If x varies inversely with both y and z and $x = 50$ when $y = 4$ and

$z = 5$, what is the value of x when $y = z = 10$?

9. Let A vary jointly with h and b. If $A = 100$ when $b = 50$ and $h = 4$, what is the value of A when $b = 20$ and $h = 10$?
10. Let S vary directly with the square of r. If $S = 100\pi$ when $r = 5$, what is the value of r when $S = \pi$?
11. Let A vary directly with square of s. If $A = 9\sqrt{3}$ when $s = 6$, what is the value of A when $s = 10$?
12. Let h vary directly with V and inversely with the square of s. If $h = 30$ when $V = 640$ and $s = 8$, what is the value of V when $s = 4$ and $h = 60$?
13. Using the relationship of the previous problem, solve for V in terms of s, h and k. Based on your result, state how V varies in terms of h and s.
14. For each graph in the figure below determine if y varies directly as x, inversely as x or neither.



SECTION 12.3 RATIONAL EQUATIONS

EXPLORATION 1

One of the jobs of the scientists at the Fish and Wildlife department is to estimate the number of fish of each type in a lake. It isn't possible to drain the lake or to count every fish in the lake one by one, so how do you think they estimate the total number? The most common method they use is called capture/recapture. In this method, scientists capture some fish from the lake. Then they tag and release them. The tagged fish are allowed to mix with the rest of the fish. Then they capture a second set of fish and look at the number of tagged and untagged fish in the second set. Suppose in the first set the scientists capture 100 fish, tag them and release them. Let N be the unknown total number of fish in the lake.

1. Write a rational expression for the ratio of tagged fish to the total number fish in the lake.
2. The scientists then capture a second set. In the second set they capture 20 tagged fish and 80 untagged fish. What is the ratio of tagged fish to total fish caught in the second set?
3. Use the answers above to estimate N. Explain your method.

As we saw in Exploration 1, you will encounter equations that contain rational expressions. To solve these equations you will use the same properties of equality we have used so far:

1. Addition: add equal amounts to both sides of the equation.
2. Subtraction: subtract equal amounts from both sides of the equation.
3. Multiplication: multiply both sides of the equation by the same (non-zero) number.
4. Division: divide both sides of the equation by the same (non-zero) number.

Each of these operations produces an equivalent equation, meaning that any solution to the new equation will be a solution to the original equation, and any solution to the original equation will be a solution to the new equation. When you have an equation that contains a rational expression, the multiplication property is particularly important. If you multiply both sides of the equation by the denominator, the resulting equivalent equation will no longer have any rational expressions. Now you can use the techniques you have already learned to solve this new equation. There is one complication. In order to use the multiplication property, we must multiply be a non-zero number. With rational equations, the denominator is an expression involving variables. So it is important to check the solution set with the original equation and remove any solutions which make the denominators 0.

EXAMPLE 1

Solve the equation $\frac{10}{x} = 2$ for the unknown x.

SOLUTION By inspection, you could observe that the answer is $x = 5$. But let's try to get this answer using our operations. We begin by multiplying both sides of the equation by x. This gives us the equivalent equation:

$$10 = 2x.$$

Now divide both sides by 2, and we get:

$$\frac{10}{2} = \frac{2x}{2}$$
$$5 = x$$

In more complicated problems, it is important to check that the solution works in the original equation. For rational equations, it is important to make sure the solution does not make the denominator 0. In this case, it is easy to check that $\frac{10}{5} = 2$.

In fact, if you ever simplify the equation to $\frac{a}{b} = \frac{c}{d}$, then you can multiply both sides of the equation by bd. This will give the equivalent equation $ad = bc$. Again, the key is that $\frac{a}{b} \cdot bd = ad$ and $\frac{c}{d} \cdot bd = bc$. Do you see why this works?

EXAMPLE 2

Solve the equation $\frac{20}{(x-5)} = 2$ for x.

SOLUTION We begin by multiplying both sides of the equation by the denominator $(x - 5)$. This gives us the equivalent equation:

$$20 = 2(x - 5)$$

which simplifies to $20 = 2x - 10$. Next we add 10 to both sides of the equation to obtain $30 = 2x$. Finally we divide by 2, to obtain $x = 15$. Again we should check our answer:

$$\frac{20}{(15 - 5)} = \frac{20}{10} = 2$$

EXPLORATION 2

1. Two thirds of a class are girls. There are 10 more girls than boys. How many boys and girls are there?
2. The ratio of boys to girls in the 8th grade is 7 to 5. There are 26 more boys than girls. Let x be the number of girls.
   a. Write a rational expression involving x for the ratio of boys to girls.
   b. Using the information given, write an equation involving x. Then solve the equation to find x.

In Explorations 1 and 2, we saw that equations involving rational expressions often arise in problems of ratios and proportions. One method you may have learned for solving proportions is “cross multiplication”. Why does this method work? Suppose a, b, c and d are numbers with

$b \neq 0$ and $d \neq 0$, then a general proportion problem will have the form:

$$\frac{a}{b} = \frac{c}{d}$$

Using the multiplication property of equality, we can multiply both sides of the equation by bd and obtain an equivalent equation:

$$\begin{aligned} \frac{a}{b} \cdot bd &= \frac{c}{d} \cdot bd. \\ ad &= bc \end{aligned}$$

In other words, we find that the product (ad) of the numerator of left hand side (a) and the denominator of the right hand side (d) is equal to the product (bc) of the denominator of the left hand side (b) and the numerator of the right hand side (c).

EXAMPLE 3

Solve the equation $\frac{x}{x-1} = \frac{4}{x}$. Check your answer.

SOLUTION We begin by clearing the denominators, by multiplying both sides by the product $x(x-1)$.

$$\begin{aligned} \frac{x}{x-1} \cdot (x(x-1)) &= \frac{4}{x} \cdot (x(x-1)) \\ x^2 &= 4(x-1) \\ x^2 &= 4x - 4 \\ x^2 - 4x + 4 &= 0 \end{aligned}$$

Now we use factoring to solve the quadratic equation.

$$x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0$$

So the solution is $x = 2$. We check our answer by substituting $x = 2$ into the original equation.

$$\begin{aligned}\frac{2}{2-1} &= \frac{4}{2} \\ 2 &= 2\end{aligned}$$

So, it checks out.

EXAMPLE 4

Solve

$$\frac{x}{x^2-5x+6} = \frac{x+14}{x^2+2x-8}.$$

Check your answer.

SOLUTION Let's try to clear the denominators by multiplying both sides of the equation by $(x^2-5x+6)(x^2+2x-8)$. We get:

$$\begin{aligned}x(x^2+2x-8) &= (x+14)(x^2-5x+6) \\ x^3+3x^2-8x &= x^3-5x^2+6x+14x^2-70x+84\end{aligned}$$

Now use the properties of equality to find to get:

$$\begin{aligned}7x^2-56x+84 &= 0 \\ 7(x^2-8x+12) &= 0 \\ 7(x-6)(x-2) &= 0\end{aligned}$$

So our solutions appear to be $x = 6$ and $x = 2$. Let's check each solution with the original rational equation. Substituting $x = 6$ we get:

$$\frac{6}{6^2-5\cdot 6+6} = \frac{6+14}{6^2+2\cdot 6-8}$$

$$\frac{6}{36-30+6}=\frac{20}{36+12-8}$$
$$\frac{6}{12}=\frac{20}{40}$$
$$\frac{1}{2}=\frac{1}{2}$$

So it checks!

But when we substitute $x = 2$ we get:

$$\frac{2}{2^2-5\cdot 2+6}=\frac{2+14}{2^2+2\cdot 2-8}$$
$$\frac{2}{4-10+6}=\frac{16}{4+4-8}$$
$$\frac{2}{0}=\frac{16}{0}$$

which has 0 in the denominators. So $x = 2$ is not a solution to the original equation. What went wrong? In the first step when we used the multiplicative property of equality, we multiplied by $(x^2 - 5x + 6)(x^2 + 2x - 8)$. However, when $x = 2$, $(x^2 - 5x + 6)(x^2 + 2x - 8) = 0$. The multiplicative property only works for non-zero numbers. So the original equation is not equivalent to the final quadratic equation we used to get the two solutions.

We can also solve equations involving sums and differences of rational expressions.

EXAMPLE 5

Solve the equation $\frac{1}{x} + \frac{1}{3} = 1$.

SOLUTION We begin by isolating the term with x on one side of the equation. To do this, we subtract $\frac{1}{3}$ from both sides of the equation.

$$\begin{aligned}\frac{1}{x} &= 1 - \frac{1}{3}\\ \frac{1}{x} &= \frac{2}{3}\end{aligned}$$

Next we multiply both sides of the equation by $3x$. (You can do this in two steps, first multiply by x, and then multiply by 3.)

$$\begin{aligned}\frac{1}{x} \cdot 3x &= \frac{2}{3} \cdot 3x\\ 3 &= 2x\\ \frac{3}{2} &= x\end{aligned}$$

In the final step, we divided by 2, to get the answer $x = \frac{3}{2}$. Let's check that this satisfies the original equation:

$$\frac{1}{\frac{3}{2}} + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1.$$

Note, in this problem you simplified the complex fraction $\frac{1}{\frac{a}{b}} = \frac{b}{a}$. To do this, you multiplied both the numerator and denominator by b. Dividing by a fraction $\frac{a}{b}$ is the same as multiplying by the reciprocal $\frac{b}{a}$. This is a useful property that you will use often.

EXAMPLE 6

Solve the equation $\frac{4}{x} - \frac{3}{x^2} = 1$.

SOLUTION In Example 5, we began by isolating the term with x. However, in this case, there are two terms which involve x, so this will not work. Instead, we will use the method of clearing the denominators, which means we will multiply both sides by the least common denominator. Our denominators are x and x^2, so the LCM is x^2.

$$(\frac{4}{x} - \frac{3}{x^2})x^2 = 1 \cdot x^2$$
$$\frac{4}{x} \cdot x^2 - \frac{3}{x^2} \cdot x^2 = x^2$$
$$4x - 3 = x^2$$
$$0 = x^2 - 4x + 3$$

Now we use factoring to solve the quadratic equation .

$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

So the solution is $x = 3$ and $x = 1$. We check our answer by substituting into the original equation.

$$\frac{4}{3} - \frac{3}{9} = \frac{12}{9} - \frac{3}{9} = \frac{9}{9} = 1$$

and

$$\frac{4}{1} - \frac{3}{1} = 4 - 3 = 1$$

So both solutions are valid.

EXERCISES

1. Solve. Check your answer.

   a. $\frac{3}{x} = 6$
   b. $\frac{x}{4} = \frac{5}{2}$
   c. $\frac{100}{x^2} = 4$
   d. $\frac{12}{2x} = \frac{14}{3x}$
   e. $\frac{3}{x-2} = 5$
   f. $\frac{10}{x+4} = 2x$

2. Solve. Check your answer.

   a. $\frac{10}{x(x-2)} + \frac{4}{x} = \frac{5}{x-2}$
   b. $\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2-6x+8}$
   c. $\frac{x-4}{4} + \frac{x}{3} = 6$
   d. $\frac{3}{2x} - \frac{2x}{x+1} = -2$
   e. $\frac{k+1}{3} - \frac{k}{5} = 3$

3. A boat can travel 8 mph in still water. In a river with an unknown current it can travel 30 miles downstream and then turn around and return to its starting point in 8 hours. What is the rate of the current?

4. Amanda plans to drive 200 miles to visit a friend and estimates when she will arrive based on her average speed. However, she encounters construction along the way and her anticipated average speed is reduced by 10 miles per hour. As a result, she arrives an hour later than her estimate. What was her anticipated average speed?

SECTION 12.4 CHAPTER REVIEW

Key Terms

constant of proportionality
direct variation
inverse variation
joint variation
rational equation
rational expression

Formulas

Direct Variation
$y = kx$

Inverse Variation
$y = \frac{k}{x}$

Practice Problems

1. Simplify the following sums and differences.
   a. $\frac{5}{a} + \frac{3}{b}$
   b. $\frac{9}{xy^2} + \frac{2}{x^2y}$
   c. $\frac{4}{2a} - \frac{5}{3b}$
   d. $\frac{7}{b^2} - \frac{6}{b}$
   e. $\frac{1}{(x+1)^2} + \frac{1}{x^2+3x+2}$
   f. $\frac{1}{x^2-2x-15} - \frac{1}{x^2+7x+12}$
2. Determine if y varies directly or inverse as x for the following equations.
   a. $y = 0.5x$
   b. $y = \frac{6}{x} + 2$
   c. $3x - 2y = 0$
   d. $y = \frac{x}{\pi}$
   e. $y = \frac{3}{2x}$
   f. $5x - 2y = 2$
3. If x varies inversely with y and $x = 4$ when $y = \frac{1}{2}$, what is the value of x when $y = 3$? What is the constant of proportionality?
4. If x varies directly with y and $x = 5$ when $y = 16$, what is the value of x when $y = 35.2$? What is the constant of proportionality?
5. Solve the following equations.
   a. $\frac{15}{(x+4)} = -6$
   b. $\frac{x}{2} = \frac{3x-4}{x}$
   c. $\frac{1}{5} + \frac{2}{x} = 4$

6. Jayden plans to drive 1800 miles to Minnesota to visit his parent. He wants to estimate how long it will take him based on an average speed. However, he finds that the speed limit has increased, so his average speed was 5 miles per hour faster than he predicted. As a result, he arrives 2 hours earlier than his estimate. What was his original predicted average speed?

PERSONAL FINANCE 13

SECTION 13.1 INTEREST

Let's begin by looking at simple interest, and then see how banks extend this idea to everyday business. First, what does it mean to invest a principal amount of money P in a bank at a simple interest rate r? The bank's customers are letting the bank use their money for a period of time, and in return the bank is willing to return their original amount at the end of the period plus extra money called Interest. The amount of extra money depends on the interest rate that the bank pays and the period of time.

So if Mrs. Hamlisch puts \$200 into a savings account at the beginning of the year at 12% interest rate, at the end of the year her account will have \$200 plus the interest that she earned. This interest is 12% of \$200, or, from the simple interest formula $I = Prt$,

$$I = (0.12)200 = 24.$$

So at the end of the year she will have \$200 + \$24 = \$224 in her bank account.

EXAMPLE 1

Mr. Lablanc invests \$500 at a simple interest rate of 6%. How much interest will he earn in one month? In six months? In one year?

SOLUTION The yearly interest he will earn is 6% of his original principal if he put his money in the bank for one year. However, in one month, he will only earn $\frac{1}{12}$ of this amount. The interest earned in one month is $I = (500)(0.06)(1/12) = \$2.50$. Similarly, in 6 months, the interest he will earn is $I = (500)(0.06)(\frac{6}{12}) = \$15$. In one year the interest he will earn is $I = (500)(0.06)(1) = \$30$. Compare the three amounts of interest to see if they are reasonable.

Each case above uses the Simple Interest Formula:

SIMPLE INTEREST FORMULA
If a principal amount P is invested at an interest rate r for t years, then the *simple interest* earned will be $I = Prt$.

We can use the simple interest formula to find a formula for the amount of money A that will be in a simple interest account after t years. The amount A is the original principal P plus the interest I earned over the period of time t. So:

$$A = P + I = P + Prt.$$

To simplify the formula factor P from the right-hand expression to obtain

$$A = P + Prt = P(1 + rt).$$

PROBLEM 1

Compute the total amount of money Mr. Lablanc will have in his account from Example 1. Do this in two ways:

1. Add the amount of interest earned to the initial amount. This is the total amount in the account after each time period.
2. Check your work by using the Simple Interest Formula for amount.

Most accounts, however, set more than one interest period a year. This is called compounding. Financial institutions divide the year into a certain number of periods and add simple interest to an account after each period.

EXAMPLE 2

If you invest a principal P at an interest rate r compounded monthly, how much interest will P earn in one period?

SOLUTION Each period is one month long, so the length of time for one period is $\frac{1}{12}$ of a year. The interest earned for one period will be $I = P \times r \times (\frac{1}{12})$. This is the same as we found in Example 1.

In general, if there are m periods in a year, the length of time for each period will be $(\frac{1}{m})$ of a year. The interest earned in one period will be $I = P\frac{r}{m}$.

The periodic interest rate, then, is $\frac{r}{m}$. We can I the periodic interest amount and $i = \frac{r}{m}$. So the interest earned in one period Is $I = Pi$. That means the amount of money in an interest-earning account at the end of a period is $P + Pi$. This looks just like the simple interest formula except the interest rate r is replaced by the periodic interest rate $i = \frac{r}{m}$.

If an account earns interest compounded every six months, the periodic interest rate per each six-month period is $i = \frac{12\%}{2} = 6\%$. If the account earns interest compounded quarterly, or four times a year, the periodic interest rate is $i = \frac{12\%}{4} = 3\%$. Many accounts earn interest each month, so $i = \frac{r}{12}$.

EXAMPLE 3

Suppose we deposit \$100 at 12% per year compounded monthly. How much will be in the account after three months? Find a formula for the amount in the account at the end of t months.

SOLUTION Let's make a list of the amount of money in your account at the end of each of the first 3 months. Since one month is $\frac{1}{12}$ of a year, the interest rate for one month is $i = \frac{.12}{12} = .01$. The amount in the account after one month would be $100 + (.01)(100) = 100 + 1 = 101$.

Let $A(t) =$ amount in the account at the end of t months. So,

$$\begin{aligned} A(0) &= 100 = P = \text{ amt. in account at beginning of first month} \\ A(1) &= P + Pr \\ &= P(1+i) \\ &= P(1.01) = \text{ amt. in account at end of first month} \end{aligned}$$

Notice that we factor the P out of the sum producing the factor $(1 + i)$. Computing the next two months, we get

$$\begin{aligned} A(2) = A(1) + A(1)i &= A(1)(1+i) \\ &= [P(1+i)](1+i) \\ &= P(1+i)^2 \\ &= P(1.01)^2 \end{aligned}$$

and similarly,

$$A(3) = A(2) + A(2)r = A(2)(1+i)$$

$$= [P(1+i)^2](1+i)$$
$$= P(1+i)^3$$
$$= P(1.01)^3$$

Using this pattern, find the values for $A(4), A(5), A(12)$ and $A(t)$. Fill in the following table.

month	$A(t)$
0	\$100
1	\$101
2	\$102.01
3	\$103.03
4	\$104.06
5	
12	
t	

Of course this pattern works with different interest rates. The key idea is that for each period that passes, the amount at the end of the period is equal to the amount at the beginning of the same period multiplied by $(1+i)$, leading to the

COMPOUND INTEREST FORMULA
If an initial principal P is invested at an interest rate r compounded m times per year, then the amount in an account after n periods is $A(n) = P(1+i)^n$, where $i = \frac{r}{m}$ is the interest earned each period.

EXAMPLE 4

Sally invests \$1000 at 12% simple interest for three years. Ann invests \$1000 at a rate of 12% compounded monthly for three years.

1. How much money will Sally have after three years?
2. How much money will Ann have?
3. Are you surprised by the results in any way?
4. If the time invested were ten years, by how much would the amounts differ?

SOLUTION

1. The amount that Sally will have is $A = 1000(1 + 0.12 * 3) = \$1360$.
2. First, find the periodic interest rate: $i = \frac{0.12}{12} = 0.01$. Second, find the number of periods in three years. Because $m = 12$, there are 12 periods per year. In three years, there are 36 periods. Finally, use the compound interest formula to find Ann's amount, using a calculator:

$$A = 1000(1 + 0.12/12)^{36} = \$1430.77.$$

3. Ann will have $1430.77 - 1360 = \$70.77$ more than Sally after 3 years.
4. Performing these calculations for ten years, Sally will have $A = 1000(1 + .12 \times 10) = \$2200$. Ann will have $A = 1000(1 + .01)^{120} = \$3300.39$.

So after ten years, Ann will have $1100.39 more than Sally.

PROBLEM 2

Suppose we deposit $1000 at 12% per year compounded monthly. How much will be in the account after 6 months? Find a formula for the amount in the account at the end of t months. Graph the function using your graphing calculator. Using the graph, estimate how many months it will take before you have $1100.

EXERCISES

1. Sue invests \$500 in the bank at a simple interest rate of 12%. How much interest will she earn after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?
2. Chris invests \$100 in the Simple Bank of America at a simple interest rate of 8%. How much will be in his account after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?
   e. 5 years?
3. When interest is compounded monthly, how many periods are there in
   a. 3 months?
   b. 1 year?
   c. 5 years?
4. Chris discovers he could invest Texas Compound Bank. In that bank he can invest \$100 in at an interest rate of 8% compounded **monthly**. How much will be in his account after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?
   e. 5 years?
   f. Compare with your answer from Exercise 2. Should he switch banks?
5. Jackie and Jennifer are sisters. On January 1, 1950, Jackie put \$1000 in her safe and forgot about it. Her sister Jennifer only had \$100 at that time. But she put her money into an account at the bank which paid 5% compounded yearly and forgot about it. On January 1,2000, they decided to take out their money and throw a party. Who had more money to spend on the party? How much more?

6. A bank offers three types of accounts. In the Bronze account, you earn 12% annual interest compounded monthly. In the Silver account, you earn 12.2% compounded twice a year. Finally, in the Gold account you earn 12.4% compounded once a year. Which is the better deal? If you deposit $100 how much will you have at the end of the year?
7. How much money will Mr. Garza need to deposit into an account earning 12% per year (1% per month) compounded monthly in order that he have $500 at the end of 3 years?
8. John deposited $200 in the bank at a interest rate of 9% compounded monthly. How much will be in the account after 6 months?
9. Victoria invests $100 at a simple interest rate of 8%. Penelope invests $100 at a rate of 8% compounded monthly. How much will each girl have:
   a. After 6 months?
   b. After 1 year?
   c. After 10 years?

   Compare the amounts and summarize what you found.
10. Using the simple interest formula $A = P(1 + rt)$,
    a. Given A, r, t , solve for the principal P.
    b. Given A, P, r, solve for the time t. Using this information, find out how long will it take Sam to have $200 in his account, after he invests $100 at a simple interest rate of 8%.
11. Solve the compound interest formula $A = P(1 + i)^n$ for P. What does the new formula tell you? Make up a problem that you could use your new formula to solve.
12. Use your calculator to graph the following two functions. Then explain what information it gives.: $A = 100(1 + 0.08t)$. $A = 100(1 + 0.08/12)^12t$. What difference do you notice in the two graphs? Which grows faster?
13. Compare the difference in simple interest from money invested at 8% and interest compounded monthly for money invested at 8% after five years.

14. Explain to a fifth grader the difference between simple interest and compound interest.

15. **Investigation:**
How long will it take money to double at a compound interest rate of 12% compounded monthly? At 8%? Research and explain the Banker's Rule of 72.

16. **Ingenuity:**
Compute each of the products from a-d. Then speculate what the products are for parts e-f.
a. $(1 + x + x^2)(1 - x)$
b. $(1 + x + x^2 + x^3)(1 - x)$
c. $(1 + x + x^2 + x^3 + x^4)(1 - x)$
d. $(1 + x + x^2 + x^3 + \cdots + x^8)(1 - x)$
e. $(1 + x + x^2 + \cdots + x^{19} + x^{20})(1 - x)$
f. $(1 + x + x^2 + \cdots + x^{(n-1)} + x^n)(1 - x)$

SECTION 13.2 COST OF CREDIT

What happens if instead of depositing money in the bank, we borrow money from the bank. For example, if we use a credit card. Do you think the interest rate will be the same, smaller, or larger? Now A will be the amount we owe. How can we show that we owe the bank instead of the bank owing us?

Usually a traditional loan uses the compound interest formula. As with the simple interest formula, the principal P is the amount of the loan , and the amount A is the total amount to be repaid . Here is an example that uses the compound interest formula to understand loans.

EXAMPLE 1

You use a credit card to purchase a \$100 MP3 player. The credit card company charges 24% per year compounded monthly. How much do you owe after t months if you don't pay the credit card company any money? How much do you owe at the end of the year?

SOLUTION You used the card to purchase a \$100 MP3 player so the initial value is \$100. To indicate that you owe the money, write $P = -100$. The annual rate is 24% or $i = 2\%$ per month. So the amount after t months is

$$A = -100(1 + .02)^t = -100 \cdot 1.02^t$$

At the end of the year, the amount is

$$A = -100 \cdot 1.02^{12} = -126.8242$$

Hence you owe \$126.82 to the credit card company.

EXPLORATION 1

Many Americans have money in savings accounts that pay interest and at the same time owe money to credit card companies. Suppose we put $1000 dollars in the bank and the bank pays 12% compounded monthly. At the same time we use the credit card to purchase $1000 of clothes. The credit card company charges 24% compounded monthly. How much do we have in the bank after t months? How much do we owe to the credit card company after t months? Does this make sense to you? What function could we graph to investigate what happens? Graph the function and see what happens at the end of the year.

When you borrow money to purchase a very expensive item like a car or a house, the lender (usually a bank) asks you to pay something each month. After each payment is made, that much less is owed the bank. In this way, you slowly pay off your loan. The formulas in this situation are complicated, but we can compute a few steps to explore how this works. Then we can use an online calculator to analyze a more realistic situation.

EXPLORATION 2

Imagine you borrow $10000 to purchase a car. The bank says it will charge 10 % annual interest rate.

1. Suppose you pay the bank $500 every 6 months.
   a. How long do you think it will take to repay the loan?
   b. Since you are paying each six months, the bank recomputes the interest and the money owed each six months. The money owed is called the *principal* of the loan. How much interest do you owe for the first 6 months? Adding the initial principal, $10000, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment? What do you notice?
   c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding

the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?

d. At this rate, how long will it take to pay off the loan?

2. Now suppose you pay the bank $1000 every 6 months. You may use the table below to organize your calculations.
   a. Now how long do you think it will take to repay the loan?
   b. How much interest do you owe for the first 6 months? Adding the initial principal, $10000, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment?
   c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?
   d. Compare the first payment and the second payment. How much did the principal go down after the first payment? How much did it go down after the second payment? Why are these different?

Principal Amt. at Beginning	Interest Owed for Period	Amt. at End of Period	Payment
10000			1000

If you continue the calculations in 2, you will see that it takes over 7 years to pay off the loan if you pay $1000 every six months. In the middle of 8th year, you will have to make a final payment of $210.71.

It is also possible to use an on-line calculator to make these computations. For example, use a search engine to find a loan calculator. What is the web address?

EXPLORATION 3

Use an online calculator to investigate how car loans work. Typically, when you buy a car you negotiate with the car dealer or bank the terms of the loan. Two important features are the annual interest rate and total length of the loan. For a one year loan, you must pay back the whole loan in one year (12 monthly payments). For a 6 year loan, you pay the loan back over a longer period (72 monthly payments). As you saw in Exploration 2, the amount you still owe on the loan and the interest on the loan are calculated after each payment.

Set the original amount of the loan to $P = \$10000$. Use the calculator to fill in the table.

1. Set the interest rate to 5%. Fill in the following table:

Length of Loan	Monthly Payment	Total Interest
1		
2		
3		
4		
5		
6		

   a. Using the data in the table, plot the Length of the Loan versus the Monthly Payment.
   b. What happens to the Monthly Payment as the length of the loan increases? Why does this make sense?
   c. Based on the graph, which type of loan would you prefer and why?
   d. On a separate coordinate plane, now plot the Length of the Loan versus the Total Interest.
   e. What happens to the Total Interest as the length of the loan increases? Why does this make sense?
   f. Now which loan do you prefer?

2. Set the interest rate to 10%. Fill in the following table:

Length of Loan	Monthly Payment	Total Interest
1		
2		
3		
4		
5		
6		

a. Add the data from this table to plot of the Length of the Loan versus the Monthly Payment above. If possible, use a different symbol to tell the two data sets apart.

b. Add the data from this table to plot of the Length of the Loan versus the Total Interest above. If possible, use a different symbol to tell the two data sets apart.

c. Now which loan would you prefer?

3. Interest rates on credit cards tend be very high. Predict what the graphs would look like if the interest rate was 20%.

PROBLEM 1

John has a yearly income of $45,000. He is able to pay all of his bills on time each month, and has $425 after regular expenses. John is considering buying a new car. Assuming that John qualifies for a loan at 3% for 6 years, how expensive a car can he afford? John has three cars he is considering:

Car 1: Costs $18,000 with other monthly expenses for insurance, gas and repairs of $120 per month.

Car 2: Costs $24,000 with other monthly expenses for insurance, gas and repairs of $150 per month.

Car 3: Costs $30,000 with other monthly expenses for insurance, gas and repairs of $180 per month.

Which of these three options can John afford to buy? Would it be financially wise to spend his entire $425 for the car? Explain.

Scientific and business calculators typically have options to compute monthly payments as well. In this example. we will show how one calculator can be used to compare loan options.

EXAMPLE 2

Janice has a credit card bill of $2500 that she needs to pay off. There are several options she is considering:

1. Repay the amount over a period of two years at an interest rate of 6% compounded monthly. If she chooses this method, what will her monthly payment be, and what will the total cost of the loan be?
2. Repay the amount over a period of four years at an interest rate of 6% compounded monthly. What will her monthly payment be, and What will the total cost of the loan be?
3. Repay the amount over a period of six years, at an interest rate of 6% compounded monthly. What will her monthly payment be, and what will the total cost of the loan be?
   In each of these options, Janice has the same interest rate, but a different length of time to make her payments. However, Janice needs to make sure that she makes her payments on time, or her interest rate will be increased to 18% compounded monthly.
4. How much is Janice's monthly payment if she repays the credit card bill of $2500 over a two-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?
5. How much is Janice's monthly payment if she repays the credit card bill of $2500 over a four-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?
6. How much is Janice's monthly payment if she repays the credit card bill of $2500 over a six-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?

SOLUTION The TI-83 Plus has special application to solve problems about loans and investments. To use the calculator

- Press *APPS* key. Choose *1: Finance*.
- Choose *1: TVM Solver*.
- A screen will appear where you enter the information. The following table explains each number you must enter.

Variable	Interpretation
N	total number of payments
$I\%$	annual interest rate (% not decimal)
PV	present value or principal
PMT	payment
FV	future value, worth at end
P/Y	number of payments per year
C/Y	number of times compounded each year
PMT	payment at of end month or beginning

When you are borrowing money, the present value is the amount you borrow. You get this money right away or at the present. You slowly pay off the loan until you owe no more money. So the future value is 0. Since we don't know the monthly payment, we leave that blank. When you borrow money, your first payment is at the end of the first month, so we choose the END for the final option on the screen. To find the monthly payment for the Janice's first loan option, we enter

Variable on TI-83/84	Value
N	24
$I\%$	6
PV	2500
PMT	
FV	0
P/Y	12
C/Y	12
PMT:	**END**

To have the calculator compute the monthly payment, press *ALPHA* and then press *SOLVE*. . The calculator should fill in the PMT cell. In this case, we get -110.80. The number is negative because you pay that amount and the amount owed goes down.

The TVM solver does not report the total amount paid nor the interest paid. But these can be easily computed. To find the total amount paid multiply the monthly payment by the number of months. To find the interest paid, subtract the amount of the loan from the total amount paid.

The table below shows the difference between an interest rate of 6% and of 18% compounded monthly.

Amount	Months	Rate	Payment	Total	Interest Paid
2500	24	0.06	$110.80	$2,659.20	$159.20
2500	48	0.06	$58.71	$2,818.08	$318.08
2500	72	0.06	$41.43	$2,982.96	$482.96
2500	24	0.18	$134.81	$3,235.44	$735.44
2500	48	0.18	$73.44	$3,525.12	$1,025.12
2500	72	0.18	$57.02	$4,105.44	$1,605.44

Different Payment Methods Notice the significant difference that the higher interest rate makes both to monthly payments as well as the interest paid over the life of the loan. Also, the amount of interest increases with the time it takes to repay the loan. If you borrow at a high interest and pay the loan off over a long period of time, the amount of interest paid can be very large. In Example 2, when the interest rate was 18% and the loan was over 6 years, the interest paid was $1605. This is 64.2% the original price of the car. However, some consumers cannot afford the monthly payment necessary to repay quickly, so they choose a longer term for the loan.

The benefits and costs of financial responsibility Credit Bureaus are organizations that collect information about individuals' borrowing and bill-paying habits. The credit bureau then uses this information to assign a credit score to everyone who wants to borrow money. Banks use this credit score to determine what interest rate they should charge a particular borrower. The lower the score, the higher the interest rate you will be charged. Hence, it is important to have a good credit history to qualify for a lower interest rate so borrowing is not so costly. In fact, with a good credit history, consumers may qualify for higher loan amounts than usual. Factors that influence a credit history include the timely payment of all bills are, a stable and adequate source of income, and the absence of bankruptcy history. Bankruptcy is a legal term used when a person cannot repay his debts.

It is financially beneficial to pay all credit card bills on time. It is absolutely financially necessary to make the minimum required payment. Not meeting the required payment usually results in a very high interest rate. The table above demonstrated that a higher interest rate results in much higher monthly payments, and costs much more money over the life of the loan.

Credit Cards: The Real Story

Credit cards are a very convenient method of payment. But it is important to recognize some key facts:

- Eventually always must pay the bill.
- Interest rates on credit cards tend to be higher than other types of credit or loans.
- Typically, if you pay the entire balance of the credit card bill at the end of each month, you are not charged any interest.
- If you don't pay the entire bill at the end of the month, the amount you owe can increase rapidly.

Let's explore a typical situation.

EXPLORATION 4

A credit card account charges no interest on purchases made during a month, if you pay the entire balance at the end of the month. If you don't pay the entire bill at the end of the month, you are required to pay a minimum payment that is equal to 2% of the balance and you are charged 15% annual interest, compounded monthly on the unpaid charges. Make a table that identifies the advantages and disadvantages of paying the balance on a credit card monthly versus paying the minimum required. Use examples and calculations to support your claims.

One type of loan is called an easy-access loan. Many banks assess a penalty if a customer overdraws his account or charges more than his credit limit. To protect against these accidents, some people take out an "easy access" loan, or line of credit. In this case, when the bank account is overdrawn or the credit card is charge too much, the bank will not charge an overdraft penalty. Instead, the bank will loan the money through as an easy-access loan. This loan is like a regular loan and will need to be repaid.

Typically, easy-access loans are for much shorter duration than a regular loan. Usually these will be repaid almost immediately, but nonetheless the customer is charged interest. In Problem 2 below, you will calculate the cost of an easy access loan with different interest rates and for different time periods. Again, use a calculator to find the monthly payment.

PROBLEM 2

Sarah took out an easy-access loan of $750. The bank has two rates for easy-access loans. For customer with good credit, the rate is 6%. However, if a customer's credit is not so good, the rate is 12%. And if any payments are missed, the rate is 18%. There are different options for the length of time to repay the easy-access loan. The options for payment periods are one month, three months, six months, or one year. Fill out the table below to determine the monthly cost of repaying an easy-access loan, and the total cost of the loan.

Amount	Months	Rate	Payment	Total	Interest
750	1	0.06			
750	3	0.06			
750	6	0.06			
750	12	0.06			
750	1	0.12			
750	3	0.12			
750	6	0.12			
750	12	0.12			
750	1	0.18			
750	3	0.18			
750	6	0.18			
750	12	0.18			

EXERCISES

Use an online loan calculator when necessary.

1. Explain the difference between a credit card loan and an easy-access loan.
2. Change the principal in Example 2 to $5000 and compute the costs.
3. Lisa wants to buy a new car that costs $18,500. There is an 8.25% sales tax, a charged based on a given percentage of the selling price, so that her total cost will be the cost of the car plus sales tax. She needs to save up for a 10% down payment of the total cost.
   a. How much is her down payment?
   b. How much will Lisa need to borrow?
4. Sue wants to purchase a new car and that costs $20,000, including taxes. She will make a down payment of $2000 and will borrow the remaining $18,000. The interest rate she qualifies for depends on her credit score. The possible interest rates are 2% compounded monthly, 6% compounded monthly, and 12% compounded monthly. If she uses a three year loan, how will the interest rate affect her monthly payment?
5. Sue also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sue only qualifies for the 6% interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time for repaying the loan affect the amount of interest she will pay over the life of the loan?
6. Sarah has a credit card bill of $3500 that she cannot pay all at once. So she decides to repay it over time. The possible rates she will have to pay are 6%, 12%, and 18% compounded monthly. Calculate the total cost of repaying her credit card bill with each of these rates over a three-year period.
7. Sarah also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sarah qualifies for the 6% interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time to repay the

credit card bill affect the amount of interest she will pay over the life of the loan?

8. Janet has overdraft protection on her checking account and accidentally overdrafted her account by $500. She decided to take out an easy-access loan for $500. There are several possible interest rates she might be charged: 5%, 10%, or 15%. Use a calculator to calculate the total cost of repaying the loan with each of the above rates over a two-year period. Over a four-year period. Which costs more in interest: 5% over 4 years or 10% over 2 years? Explain.

9. Below are two payment options for buying a $250,000 house.
    a. Obtain a 6% loan for 30 years.
    b. Obtain a 4% loan for 15 years.

    Explain the advantages and disadvantages of each option.

10. Nancy is getting ready to buy a car but is not sure how expensive a car she can afford. With her credit score, she qualifies for a 6% loan compounded monthly for three years, or a 12% loan compounded monthly for six years. What are the advantages and disadvantages of the two different payment methods? Explain. Why might the type of payment method Nancy chooses determine how expensive a car she can afford?

11. Sandra and Bill have a combined yearly income of $115,000. They are able to pay all of their bills on time each month, and have $1250 after expenses. Their rent is $800 per month. Sandra and Bill are considering buying a new house. Assuming they qualify for a loan of 3% for 30 years, how expensive a house could they afford? There are three houses they are considering-one costs $100,000, one costs $150,000, and one costs $200,000. Which of these three options can they afford? Are there other house expenses they need to consider besides the monthly payment? Which of the three choices is financially responsible? What are some of the possible costs that could result from not being able to make their monthly payments?

12. Identify some of the benefits of financial responsibility. What does it mean to be financially responsible?

13. Identify some of the costs of financial irresponsibility. How can these costs rise and cause future problems?

SECTION 13.3 PLANNING FOR THE FUTURE

Cost of College One of the most important decisions you will make is your career. Once you decide, it may take many years to prepare for your chosen career. Many times preparing for a career involves getting a college education. Perhaps one of the best reasons to go to a two-year or four-year college is that this will give you more career options. There have been many studies about whether college graduates earn more money than their counterparts. While there might be some debate about how much more money a they will earn, there is widespread agreement that college graduates have more options than their contemporaries.

Anyone interested in a career in Science, Technology, Engineering, or mathematics (STEM) will almost certainly need a college degree, and quite possibly even more advanced training like a masters or PhD. Many other higher-paying careers require a college education, or an associate two-year degree from a community or junior college. One advantage of going to a junior college is that this will let you try it out, and it provides a convenient stopping point after two years if you decide you don't want to continue. And if you do decide to continue, then you may transfer to a 4-year college and receive credit for your first two years at the junior college. However, there can be problems making this transfer as well since some of your courses taken at the junior college might not match the requirements of the degree program at the 4-year college.

As reported by cbsnews.com the highest-paying bachelor's degrees for new graduates in 2012 were

- Electrical engineering $52,307
- Chemical engineering $51,823
- Mechanical engineering $51,625
- Computer engineering $50,375
- Computer programming $48,714
- Industrial engineering $48,566
- Computer science $47,561

Table 13.1 The average cost of college: 2012-13

	Public 2-year (in-state)	Public 4-year (in state)	Private 4-year
Tuition & fees	$3,131	$8,655	$29,056
Room, board, books, etc.	$12,453	$13,606	$14,233
Total cost	$15,584	$22,261	$43,289
Net price (after scholarships, grants, aid)	$4,350	$5,750	$15,680

- Civil and environmental engineering $45,621

Notice that almost all of these are in engineering, and they all require a strong mathematics background.

Table 13.1 shows the average cost of colleges from 2012. On average, for a public school, an education at a two-year college will cost $15,584 and at a four-year state college will cost $22,261 per year. The net price reflects the fact that most students obtain scholarships, grants, and financial aid to help pay for college. The net cost is what you and your family need to pay on average, assuming that you can obtain scholarships and other financial aid to cover the rest of the cost.

EXPLORATION 1

Work in groups. Have each group member explore one two-year college and one public four-year college. Compare their costs, including the family contribution at each. Discuss in your group how the costs vary from college to college. Is there anything that surprises you?

Saving for College It's not too early to make a savings plan to save the money needed for at least the first year of college. Because the amount of scholarships and financial aid available is unknown, you should plan to save $22,261, an amount necessary to attend the first year of some four-year public colleges. This assumes that the costs of college will not rise over the next few years, which is probably wishful thinking.

The savings plan will involve you (or your parents) making monthly payments into a savings account. It is hard to find a savings account that pays a high interest rate, but let's assume that you can find an account that pays an interest rate of 3% compounded monthly.

EXAMPLE 1

How much must be invested each month into a savings account that pays 3% interest compounded monthly to have $22,261 after four years? How much to have $22,261 after eight years? After twelve years? What action do your answers suggest?

SOLUTION To solve this problem we will use the TVM solver on the calculator. In this case, you want to invest enough money monthly in an savings account, so that at the end you have enough money for college. Since we start with no money, the present value is 0, but the desired future value is $22,261. The process doesn't start until we invest our first payment, so the payments occur at the beginning of each month. To see how much you must invest monthly over 4 years, enter the following into the calculator:

Variable on TI-83/84	Value
N	48
$I\%$	2
PV	0
PMT	
FV	22261
P/Y	12
C/Y	12
PMT:	**BEGIN**

To have the calculator compute the monthly payment, press *ALPHA* and then press *SOLVE*. The calculator should fill in the PMT cell. In this case, we get −435.99. The number is negative because you pay that amount each month. If we change the number of years we have to invest and repeat the calculation, we can make a table:

Years	Monthly Payment ($)
4	435.99
8	204.95
12	128.30

Clearly, the longer you wait to save for college the more money you will need to invest each month. So it is a good idea to start early.

EXPLORATION 2

Unfortunately, the cost of going to college does not stay the same over time. Due to inflation, the costs increase each year. Suppose that the cost of college increases by 3% per year and it costs $22261 this year.

1. How much will college cost in 4 years?
2. How much more will you have to save each month in order to pay for college in four years? How does this compare to Example 1.
3. What if you plan to attend college 8 years from now?

To offset the cost of saving, plan to apply for scholarships and financial aid. In this way, your family contribution on average could be reduced to the net price in the table above, less than one-third of the total cost.

Father Off in the Future Just as small amounts of money invested monthly amount to significant savings for college, the same is true of a retirement plan. Investing or paying into an account at regular intervals, establishes an annuity or tax-free savings account.

EXAMPLE 2

The Ortizes are newly-weds and have figured that they need to save $50,000 to make the down payment on their dream house. To do this, they plan to make monthly deposits into an account that pays an interest rate of r = 6% compounded monthly. They can afford to save $600 dollars each month. How long must they wait until they have money they need?

SOLUTION We can use TVM solver to determine this. In this case we know the monthly payment, but do not know how many months are needed. The future value is $50000 and the present value is $0. The payment is −600 because the Ortizes will pay this each month. Enter the following into the calculator:

Variable on TI-83/84	Value
N	
$I\%$	6
PV	0
PMT	-600
FV	50000
P/Y	12
C/Y	12
PMT:	**BEGIN**

Leave the N = cell blank and press *APLHA* and then *SOLVE*. The calculator should fill in $N = 69.54$. So the Ortizes need to save for 70 months or almost 6 years.

It is also important to start saving for retirement early as well. Advances in medicine has increased the number of years that Americans are living after retirement.

EXPLORATION 3

Planning for retirement is a complicated issue. There are even retirement specialists who help others make decisions about their retirement. However, there is a lot of useful information available on the web. Use the internet to explore the following questions.

1. How much money do experts suggest you need to save before you retire?

2. How much money should you invest each year in order to achieve this goal?
3. What is the effect of waiting 10 years before you start saving for retirement? What about waiting 20 years?
4. There are many ways to save or invest for retirement. Describe at least three of these. How do they compare?

EXERCISES

1. Find the tuition and fees of two in-state junior colleges. Estimate the total cost of attending each. Find the commuting and residential cost for each.
2. Find the tuition and fees of one public in-state four-year college, one out-of-state public college, and one private four-year college. Estimate the total cost of attending each. Include travel costs, and assume that you will be living away from home.
3. Investigate three colleges that have programs you might be interested in attending. How much does each cost to attend per year?
4. Devise a savings plan to make a college education possible. Explain how small amounts of money invested each month could enable a future student to save enough to help pay for college.
5. Discuss retirement plans with your parents. Are they able to save money each month for retirement?

SECTION 13.4 CHAPTER REVIEW

Key Terms

Compound Interest	Minimum Payment
Credit	Monthly Payment
Credit Card	Principal
Credit Score	Room and Board
Deposit	Scholarship
Down Payment	Simple Interest
Easy Access Loan	Terms of a Loan
Interest Rate	Tuition

Formulas

Simple Interest: $I = Prt$

Compound Interest: $A(n) = P(1 + \frac{r}{m})^n$

Practice Problems

1. Angelina invests $900 in the bank at a simple interest rate of 3%. How much interest will she earn after
   a. 1 month?
   b. 1 year?
   c. 2 years?
2. Valerie deposited $1200 into a bank account and left the account alone. The account collects 4% interest, compounded quarterly.
   a. How much will be in the account after 5 years? After 10 years? After 20 years?
   b. Write an equation to model the amount of money in the account.
3. Suppose that you use a new credit card to help furnish your home. After all the purchases, the total balance on your card is $12,457 (Wow!). The credit card charges 18.8% interest compounded monthly. If your were not to make a payment for 6 months, what would your balance be?

4. As a vehicle ages, its value decreases or depreciates. The function that models this is $V(t) = I(.082)^t$, where I is the original cost of the vehicle when it was new, t is the time in years from the date of purchase and $V(t)$ is the value after t years. Patricia's SUV cost $32,000 brand new.
   a. Find the value of Patricia's vehicle 3 years after she purchased it.
   b. Instead of buying the car, Patricia could have leased the car. She would have paid $400 per month and then would have returned the car at the end of 3 years. Would this have been a better deal? Explain.
5. Susana Ormany is financial advisor on television. One day she says, its better to have compounding work for by investing, than against you by borrowing. Explain what she means using two examples.
6. Lalo and Lazaro are brothers. Lalo is very financially responsible and always pays his bills on time. Lazaro, on the other hand, often pays late. The brothers go to a bank separately to ask for a loan. Which brother is likely to have to pay the higher interest rate? What will the effect of this be on the monthly payments, each one has to pay?

STATISTICS 14

SECTION 14.1 MEASURING CENTER

In this chapter we will be applying some of the algebra skills you learned in previous chapters to statistics. Statistics is the study of how to use data to answer questions. When you study statistics you learn how to ask questions in a way that can be analyzed numerically, how to collect the data, how to summarize the data with numbers and graphs, and then how to use data to answer the question as honestly as possible. In this chapter, we will focus mainly on how to summarize data or *descriptive statistics*, but we will also briefly touch on the other parts of statistics. First we begin with some notation.

DATA SET
Suppose n is a positive integer. A *data set* is a collection of observations: $x_1, x_2, \ldots, x_n$. The *sample size* n is the number of observations.

EXAMPLE 1

Anna recorded the number of hours she slept each night last week:

$8, 6, 6, 11, 8, 9, 8$. So in this case, $n = 7$, and $x_1 = 8$, $x_2 = 6$, $x_3 = 6$, $x_4 = 11$, $x_5 = 8$, $x_6 = 9$, and $x_7 = 8$.

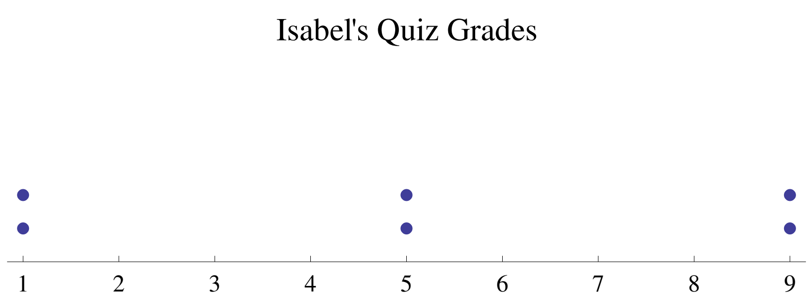
There are two kinds of variables that can occur in set of data: *numerical* and *categorical*. For categorical variables, each data value represents a label or category. For numerical variables, each data value represents a number. If you survey your class and ask each person what is their favorite subject in school, that is a categorical variable. If you ask, how many brothers and sisters they have, that is a numerical variable. Be careful, sometimes statisticians who collect categorical data will use numbers as the labels. For example, write 0 for male and 1 for female. This is still a categorical variable, because the data value represents a category.

Let us begin with simplest way to display numerical data called a *line plot*. To make a line plot you simply draw a number line and place a dot for each data point. If you have two or more points with the same value, you stack the points.

EXAMPLE 2

Ms. Garcia gave a 10 point quiz every week this six week period. Isabel quiz grades were: $1, 1, 5, 5, 9, 9$. Make a line plot of her grades.

SOLUTION



Measuring the Center of a Data Set

When we summarize statistics, our goal is to find a few numbers called *statistics* which describe the entire data set. This becomes more important as we collect larger and larger data sets. Often, we are content to condense very large sets of data down to two numbers: the center and the spread.

Although, the "center" is not a well defined concept, statisticians use this term to mean a number which reflects where most of the data is. Depending on the situation, we may use different formulas to find the center of a data set.

EXPLORATION 1

Here are the quiz grades for four students in Ms. Garcia's class. Isabel:$1, 1, 5, 5, 9, 9$, Maya: $5, 5, 5, 5, 5, 5$, Aden: $1, 1, 1, 9, 9, 9$, Sofia: $1, 1, 7, 7, 7, 7$. Using four number lines each labeled 0 through 10, make line plots for each student's grades. For each set of a data find a number you think describes the center. How did you decide on these numbers?

Mean

In Exploration 1 many of you may have used the mean or average of the data to describe the center.

MEAN

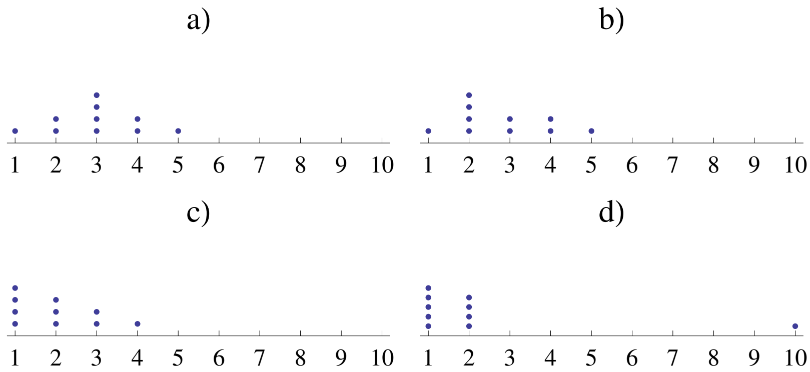
The *mean*, often denoted $\bar{x}$, is a measure of center. To find the mean of a given data set, find the sum of all the data points and divide by the number of data points.

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

The mean is the measure of center that we prefer for most cases. But the mean is not always in the center of the data, sometimes not even close.

EXPLORATION 2

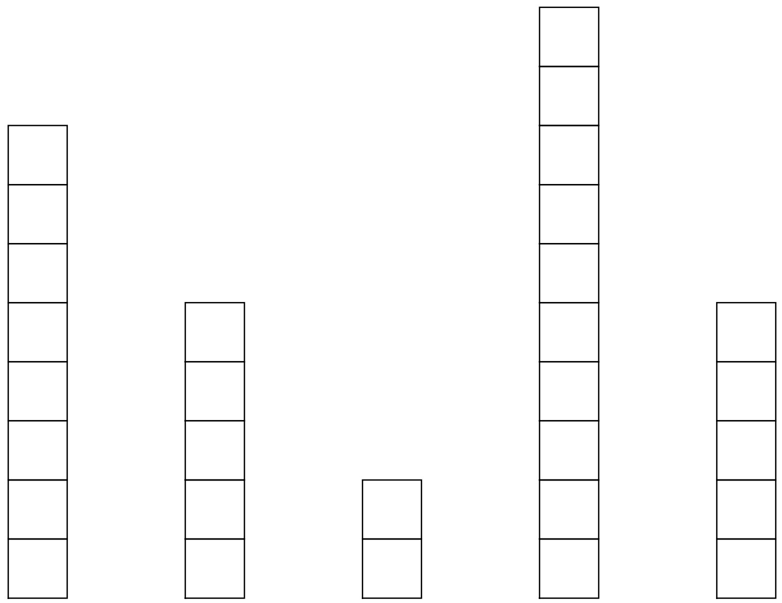
The figure below shows the line plot for four data sets. Find the mean of each data set and mark it on the number line using an $\bar{x}$. Is the mean always in the "center" of the data? For what kind of data sets do you think the mean would be a good measure of center? For what kind of data sets would the mean not be a good measure of center? What could we use instead of the mean?



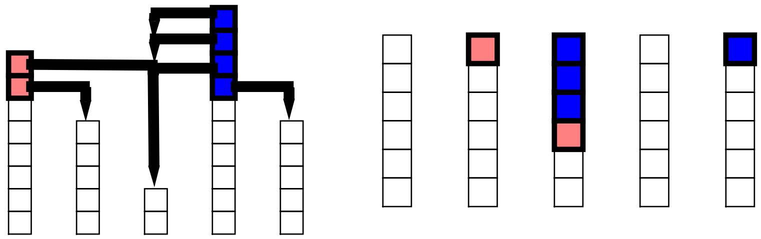
The basic formula for the mean given in the definition is pretty simple. However, there are other ways to think about how to compute the mean and to interpret what it means for a set of data. One perspective is called *leveling off*.

EXAMPLE 3

A kindergartner used legos to represent numbers, see the figure below. Which numbers are represented? Can you find the mean of the numbers without using the formula from the definition?



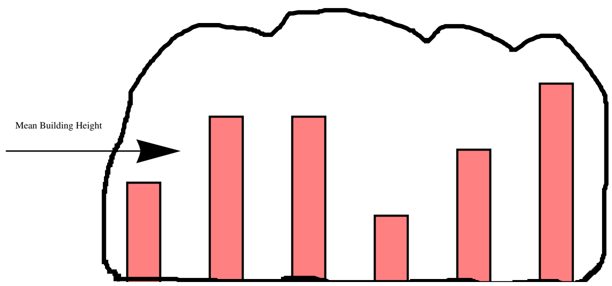
SOLUTION The numbers represented are $8, 5, 2, 10, 5$. If we don't want to use the formula to find the mean we can move some Lego blocks from the tall towers to the lower towers, until each tower has the same number of blocks. This way we replace the original towers with 5 towers of the same height, but which altogether have the same total number of blocks.



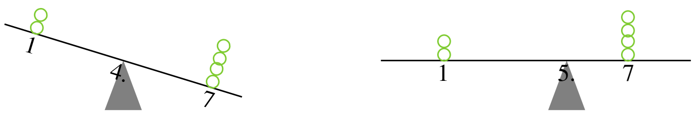
The leveling off method gives an important perspective of what the mean does. First, we can easily see that the mean must be greater than or equal to the smallest data value and less than or equal to largest data value. Second, the mean can be thought of as a "fair share" where the total is divided equally among each of the parts.

PROBLEM 1

While visiting a demolition site, you happened to look at a cityscape through a hole in the wall. You can see six buildings, and you know a seventh building is located to the right of the six in view. A construction worker gives you the average height of the buildings. See the figure below. How can you use the given information to determine the height of remaining building?



Another way to interpret how the mean is related to the data set is to imagine that the line plot iss a see-saw. Each data point represents a weight on the see saw. In this situation we can think of the mean as point that exactly balances the see saw. See figure below. Engineers and physicists use this property to figure out how to find the *balance point* of physical shapes.



Median

Let's return to Exploration 1. When asked to find a number that represents the center of the data set, some of you may have chosen the median.

MEDIAN
The *median* is a measure of center. The median is a number M such that at least 50% of the data is greater than or equal to M and at least 50% of the data is less than or equal to M.

To find the median we first determine the number of data points in the data set n. Then:

1. Order the data from smallest to largest.
2. If n is odd, M is the middle ordered number.
3. If n is even, there is no middle number. In this case, M is the average of the middle pair of numbers.

EXPLORATION 3

Reexamine the data in the figure from Exploration 2. Find the median of each data set and mark it on the number line using an M. Is the median always in the "center" of the data? For what kind of data sets do you think the median would be a better measure of center than the mean?

PROBLEM 2

Think about playing the lottery. If our data set includes the winnings for each person who played the lottery last week. What do you think the median would be? Which would be larger the median or the mean?

EXPLORATION 4

The mean and median do not have be one of the original numbers in the data set. Create two data sets where neither the mean nor the median are members of the data set. Explain how you came up with the data set.

Next let's explore how to compute the mean and median when the data has already been summarized.

EXAMPLE 4

Each student was given a small box of raisins. Students then counted the number of raisins in their box. The teacher summarized the results in the following table:

Number of Raisins	*Frequency*
28	6
29	3
30	4
31	3
32	1
34	2
35	6
36	1
38	3
40	1

Find the mean and median number of raisins per box.

SOLUTION It is important to realize that some boxes had the same number of raisins. For example, 6 different students counted 28 raisins in their box. First let's find the total number of students by summing the frequency column.

$$n = 6 + 3 + 4 + 3 + 1 + 2 + 6 + 1 + 3 + 1 = 30$$

To find the mean we must sum all 30 values, this includes counting 28 six times. The easiest way to do this is to include a third column in the table.

Number of Raisins	*Frequency*	*Number × Frequency*
28	6	168
29	3	87
30	4	120
31	3	93
32	1	32
34	2	68
35	6	210
36	1	36
38	3	114
40	1	40
Sum	*30*	*968*

So the mean is $\bar{x} = \frac{968}{30} = 32.266$.

Since $n = 30$ is even, we must find the middle pair of ordered values. In this case, it will be the 15^{th} and 16^{th} ordered values. One way, to find this is to include a column where will gradually sum the frequency. We call this the cumulative frequency.

Number of Raisins	*Frequency*	*Cumulative Frequency*
28	6	6
29	3	9
30	4	13
31	3	16
32	1	17
34	2	19
35	6	25
36	1	26
38	3	29
40	1	30

Looking at the table, we see that 13 students counted 30 raisins or less, while 16 counted 31 or less. This means that the 15^{th} and 16^{th} ordered values are both 31.

PROBLEM 3

Mr. Garcia's gave an exam to his two Algebra classes. The results are shown below.

Period	*Frequency*	*Average*
2nd	20	91
3rd	25	82

Use the data in the table to answer the following.

1. How many students are there in the two Algebra classes?
2. What is the sum of all the scores of all the students in the two classes?
3. What is the mean of all of Mr. Garcia's Algebra students?
4. Explain why you can't find the median of all the students in the two classes.

EXERCISES

1. For each data set make a line plot, find the mean and label the mean on the line plot.

   a. $1, 1, 1, 1, 1$
   b. $0, 1, 1, 1, 2$
   c. $-1, 0, 1, 2, 3$
   d. $-2, -1, 0, 0, 0, 1$
   e. $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$

2. Which of the following are always true about the mean?

   a. The mean is an integer.
   b. The mean is equal to one of the numbers in the data set.
   c. The mean is less than or equal to the maximum of the data set.
   d. The mean is greater than equal to the minimum of the data set.

3. Many statistics are collected on Social Media as a way to track trends and popularity. In April 4, 2012 the number of Twitter followers of the 8 finalists on American Idol were reported by www.zabasearch.com. See the results below:

Contestant	*Twitter Followers*
Colton Dixon	140895
Jessica Sanchez	96788
Phillip Phillips	124269
Skylar Laine	47665
DeAndre Brackensick	41651
Hollie Cavanagh	43135
Joshua Ledet	33908
Elise Testone	35931

Find the mean and median number of Twitter followers for the 8 contestants.

4. Tony Parker, point guard for the San Antonio Spurs, played 9 games in February 2013. The table below summarizes the number of assists he made in each of those games. Find the mean and the median number of assists.

Number of Assists	*Frequency*
3	1
7	2
8	3
11	2
12	1

5. The average teacher salary for New Mexico, Oklahoma and Texas are given below.

State	Salary
New Mexico	28,120
Oklahoma	27,684
Texas	31,874

Explain why the average salary for all three states combined will be closer to that of Texas than Oklahoma.

6. **Zeroes and Ones.** For each data set make a line plot, find the mean and label the mean on the line plot.

a. $1, 1, 1, 1, 1, 0, 0, 0, 0, 0$
b. $1, 1, 0, 0, 0, 0, 0, 0, 0, 0$
c. $1, 0, 0, 0, 0, 0, 0, 0, 0, 0$
d. $1, 1, 1, 1, 1, 1, 1, 1, 1, 0$

7. **Investigation:**

Let c be a number. For each data set make a line plot, find the mean and label the mean on the line plot. How are the numbers in each set related? What is the effect on the mean?

a. $-1, 0, 3, 4, 4$
b. $-3, -2, 1, 2, 2$
c. $99, 100, 103, 104, 104$

d. $c-1, c, c+3, c+4, c+4$

8. **Investigation:**

Suppose a is a number. For each data set make a line plot, find the mean and label on the line plot. How are the numbers in each set related? What is the effect on the mean?

a. $-1, 0, 3, 4, 4$
b. $-4, -4, -3, 0, 1$
c. $-2, 0, 6, 8, 8$
d. $-\frac{1}{2}, 0, \frac{3}{2}, 2, 2$
e. $-a, 0, 3a, 4a, 4a$

9. The median, M, is a measure of center. Let min be the minimum of the data set. That is the smallest value. And Max be the maximum or greatest value.

a. Explain why $min \leq M \leq Max$.
b. Create a data set where $n = 5$ and $min = M < Max$.
c. Create a data set where $n = 5$ and $min = M = Max$.

10. The mean, $\bar{x}$, is a measure of center. Let min be the minimum of the data set. That is the smallest value. And Max be the maximum or greatest value.

a. Explain why $min \leq \bar{x} \leq Max$. *Hint: what happens if replace values larger than $\bar{x}$ with Max?*
b. Create a data set where $n = 5$ and $min = \bar{x}$.
c. Create a data set where $n = 5$ and $\bar{x} = Max$.

SECTION 14.2 MEASURES OF SPREAD

Just like the center, the "spread" is also not a well defined concept. However, statisticians use this term to mean a number which shows how spread out the data is from the center. Again depending on the situation, we can use different formulas to find the measure the spread of a data set.

EXPLORATION 1

Here are the quiz grades for four students in Ms. Garcia's class. Isabel:$1, 1, 5, 5, 9, 9$, Maya: $5, 5, 5, 5, 5, 5$, Aden: $1, 1, 1, 9, 9, 9$, Sofia: $1, 1, 7, 7, 7, 7$. Using four number lines each labeled 0 through 10, make line plots for each student's grades.

1. For each set of a data find a number you think describes the spread. How did you decide on these numbers?
2. Do you think that the spread for Isabel's and Aden's grades is the same?
3. If you could move the dots on the number lines, how would you change Isabel's line plot so it looked like Aden's?

Range of a Data Set

Many of you may have looked at the minimum and maximum value when determining how spread out data is. This leads to simplest measure of spread.

RANGE

The *range* is a measure of spread. The range is the difference between the maximum and minimum value.

For example, for the data set $\{1, 3, 18, 12, 16\}$ the range $= 18 - 1 = 17$.

PROBLEM 1

Find the range for each data set Exploration 1.

Deviations from the Mean

One way to think about spread is to compute the *deviation* for each data point. The deviation is the difference between the data point and the mean of the data set. Let's make a table of the deviations for Isabel's quiz grades from Exploration 1. First note that the mean is $(1+1+5+5+9+9)/6 = 5$.

data point	deviation
1	$(1-5) = -4$
1	$(1-5) = -4$
5	$(5-5) = 0$
5	$(5-5) = 0$
9	$(9-5) = 4$
9	$(9-5) = 4$

Notice that the sum of the deviation column is 0.

PROBLEM 2

Make a table of the deviations for Maya's, Aden's and Sofia's grades. Find the sum of the deviations. What do you notice?

Mean Absolute Deviation

The deviation is the difference between the data point and the mean. As we saw in Problem 1, the deviation is negative if the data point is less than the mean. It is positive if the data point is greater than the mean. Further,the sum of the deviations is 0. This is true for any data set. So we can not use the sum of the deviations to measure the spread of the data.

What can we do? To deal with this issue, we can take the absolute value of the deviations. Recall that the absolute value of the difference of two number represents the distance between two points. So the absolute value of the deviation is the distance between a data point and the mean. This idea leads to a second measure of spread:

MEAN ABSOLUTE DEVIATION

The *mean absolute deviation* or MAD is a measure of spread. It is given by the formula

$$MAD = \frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + \cdots + |x_n - \bar{x}|}{n}$$

EXAMPLE 1

Make a table of the absolute deviations for Isabel's quiz grades from Exploration 1. Find the MAD for this data set.

SOLUTION Alter the table from above by taking the absolute value.

data point	absolute deviation
1	$\lvert 1-5 \rvert = 4$
1	$\lvert 1-5 \rvert = 4$
5	$\lvert 5-5 \rvert = 0$
5	$\lvert 5-5 \rvert = 0$
9	$\lvert 9-5 \rvert = 4$
9	$\lvert 9-5 \rvert = 4$

Then we take the mean of the absolute deviations:

$$MAD = \frac{4+4+0+0+4+4}{6} = \frac{16}{6} \approx 2.67$$

PROBLEM 3

Consider the quiz scores in Exploration 1.

1. Find the MAD for each data set.
2. Isabel and Aden's score have the same range, but the MAD for Aden's scores is larger than for Isabel. Explain why we might think Aden's scores are more spread out than Isabel's.

Variance and Standard Deviation

Another way to use the deviations to measure the spread of a data set is to square the deviation. Using this idea we can define two more measures of spread.

VARIANCE AND STANDARD DEVIATION

The *variance* and *standard deviation* are measures of spread. To find the variance of a given data set, find the sum of all the squared deviations and divide by the number of data points.

$$\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}$$

The standard deviation is the square root of the variance.

$$\sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

EXAMPLE 2

Find the variance and the standard deviation for Isabel's quiz grades.

SOLUTION First find the squared deviations.

data point	deviation	squared deviation
1	-4	16
1	-4	16
5	0	0
5	0	0
9	4	16
9	4	16

So the variance $= (16+16+0+0+16+16)/6 = 64/6 = 32/3 \approx 10.33$.
The standard deviation is $\sqrt{\frac{32}{3}} = 3.26$.

PROBLEM 4

Find the variance and standard deviation for Aden's grades. Using the line plots of the grades, explain why the variance is larger for Aden's grades than for Isabel's.

PROBLEM 5

Kathan is on the track the team and competes in the 100 meter sprint. His times in seconds for his last 10 races are 14, 13, 15, 14, 15, 14, 14, 15, 14, 15. Kathan's mean time is 14.3 **seconds**. What are the units of the variance? What are the units of the standard deviation? Which measure of spread has the same units as the original data and the mean?

Using a Calculator

In real data sets, n is usually large and the numbers may make calculation by hand cumbersome. This is especially true if we wish to compute the standard deviation.

EXAMPLE 3

The fat content in grams was measured on hamburgers from 7 different fast food restaurants: $19, 31, 34, 35, 39, 39, 43$. Use the calculator to find the mean and standard deviation.

SOLUTION Here is how to do it on a TI-84.

1. To Enter the data into the calculator. Use the following steps:
   - Press *STAT* key. Choose *ClrList*.
   - Enter in *ClrList L1*
   - Press *STAT* key. Choose *Edit*.
   - Place the data in *L1*.
   - Make sure you have one number per cell.
2. To find the mean and standard deviation use the following steps:
   - Press STAT.Choose *1:1-Var Stats* from the CALC menu.
   - Enter in *1-Var Stats L1*.
   - The mean is $\bar{x}$ and the standard deviation is $\sigma\ x$

Reading from the calculator we get: $\overline{x} =$ 34.2857, sd $=$ 7.22524.

PROBLEM 6

In the same study which measured the gram content in hamburgers, they found that the mean fat content for chicken sandwiches was 20.6 grams with a standard deviation of 9.8 grams. Compare to the values you computed in problem 3. Do you think chicken sandwiches are healthier? Explain.

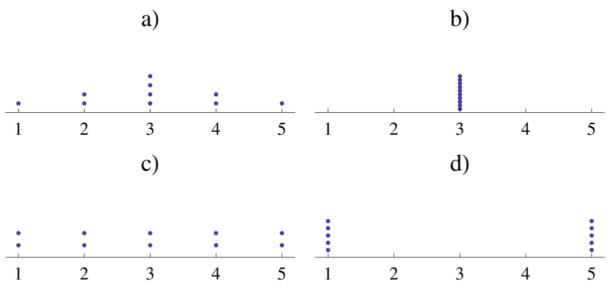
EXERCISES

1. For each data set:
    - Make a line plot
    - Find and label the mean
    - Find the Range, MAD and Standard Deviation

    a. $1, 1, 1, 1, 1$
    b. $0, 1, 1, 1, 2$
    c. $-1, 0, 1, 2, 3$
    d. $-2, -1, 0, 0, 0, 1$
    e. $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$

2. Look at the line plots of four data sets shown below. Without doing a calculation, put the data sets in order from

    a. smallest range to the largest range
    b. smallest MAD to the largest MAD
    c. smallest standard deviation to the largest standard deviation



3. **Create your own data.** For each letter below, create a data set with $n = 5$ data points which satisfies the given condition.

    a. The standard deviation is 0.
    b. The mean is 1 and the standard deviation is 0.
    c. The mean is 0 and the standard deviation is 1.
    d. The mean is 10 and three of the data points are negative.

4. Find the MAD and range for the Twitter follower data in Exercise 3

from Section 14.1.

5. **More Zeroes and Ones.** Find the variance for each data set in Exercise 6 from Section 14.1. Let $\bar{x}$ be the mean of the data set. Check that if the data is all zeroes and ones the variance equals $\bar{x} - \bar{x}^2$.

6. **NFL Running Backs.** Barry Sanders is a Hall of Fame NFL running back. He was renowned for sometimes running the full width of the field to gain only a yard on a play then, on the next, suddenly breaking through a hole for a long gain. Although he averaged 5.0 yards per carry throughout his career, Sanders holds the NFL record for the most carries for negative yardage. This is due to his common practice of running backwards to avoid a tackle in hopes of breaking out an explosive run; this, however, often led to him being brought down behind the line of scrimmage.

   a. Imagine a line plot of the yards gained on each rush by Barry Sanders. According to the description above what might the plot look like.
   b. Suppose a sports announcer speaking of another running back said, "Smith may never break it for a long gain, but he'll get you three or four yards every time." How will the MAD of Smith's yards gained each carry compare to that of Barry Sanders?

7. **Investigation:**

   a. Let c be a number. For each data set make a line plot. What do you notice about spread of the numbers? Find the MAD for each data set. What do you notice about the MAD?
      i. $-1, 0, 3, 4, 4$
      ii. $-3, -2, 1, 2, 2$
      iii. $99, 100, 103, 104, 104$
      iv. $c-1, c, c+3, c+4, c+4$
   b. Suppose a is a number. For each data set make a line plot. What do you notice about spread of the numbers? Find the MAD for each data set. What do you notice about the MAD?

i. $-1, 0, 3, 4, 4$

ii. $-4, -4, -3, 0, 1$

iii. $-2, 0, 6, 8, 8$

iv. $-\frac{1}{2}, 0, \frac{3}{2}, 2, 2$

v. $-a, 0, 3a, 4a, 4a$

SECTION 14.3 SAMPLING

In the previous two chapters you have used statistics to describe a data set. Statistics becomes most useful when you collect a sample from a population. This sample is used to get an idea of what the whole population looks like without measuring everyone or everything in the population. In this section, you will explore how to obtain samples and how this samples reflect the population they come from.

POPULATION AND SAMPLE
A *population* is set of all people or objects that you are interested in knowing about.
A *sample* is a subset of the population.

EXAMPLE 1

For each of the following identify the population and the sample.

1. A radio station wants to know what percentage of its listeners like hip-hop music. So it asks listeners to call in over the next hour and give their opinion.
2. In 2010-2011 the Substance Abuse and Mental Health Services Administration surveyed 137,913 Americans aged 12 or older and found that 4.5% reported having used pain relievers nonmedically in the past year.
3. Gallup phoned 500 adult in the U.S. and found that 55% say they experienced a lot of happiness and enjoyment the day before.

SOLUTION We need to determine who actually responded (the sample) and who those people were selected from (the population).

1. Population: all listeners, Sample: those that phoned in

2. Population: all Americans aged 12 or older, Sample: 137193 who responded
3. Population: all adults in U.S. with a phone, Sample: 500 that were interviewed.

Typically a researcher obtains a sample to find out something about the population. For example, social scientists are interested in the number of hours of television middle school children watch each day. To find this out, they take a sample of middle school children and have them record how many hours of television they are watching.

EXPLORATION 1

Consider the population of all students at your school. Work in groups of 4 or more.

1. In your group decide on a question you would like to ask the whole school.
2. Write your group's question on the board in the front of the class.

Look at all the questions your class came up with. For which question(s) do you think your class would be "representative" of the whole school? For which question do you think your class would not be "representative"?

Choose one question you think your class is representative of the whole school. Have everyone answer the question. Summarize your results.

Now we explore how we can create samples which represent the population.

EXPLORATION 2

Your teacher will give you a sheet of paper. When your teacher gives you the signal:

1. Turn the sheet over and examine the rectangles for 10 seconds. Write down your guess of the average area of all the rectangles.
2. Now select five rectangles that are representative of the rectangles on the page. Determine the area of each rectangle. Fill in the following table.

Rectangle #					
Area					

3. Compute the average area for your sample of five rectangles.
4. Record your values from Parts 1 and 3 on two number lines on the board.
5. What do you notice about the data? Did everyone get the same values?

Now we look we explore another method to select a sample of rectangles. We will use a calculator to select a *simple random sample*. In this method every sample of the same size has the same chance of being selected. First an example to show how to use the calculator.

EXAMPLE 2

Suppose there are 10 students in a club: Alejandro, Benjamin, Carlos, Diego, Estella, Frank, Gustav, Herman, Isabel, John. Use a calculator to obtain a simple random sample of 3 students.

SOLUTION We must number the students 1 through 10.

Name	Number
Alejandro	1
Benjamin	2
Carlos	3
Diego	4
Estella	5
Frank	6
Gustav	7
Herman	8
Isabel	9
John	10

Using the TI-84 we can find a random number between 1 and 10. Here's how.

- Press MATH. Tab over to the *PRB* menu.
- Select *randInt(*.
- Type in *randInt(1, 10)* and press *ENTER* to get a random number between 1 and 10.
- Record the number.
- Keep pressing *ENTER* to get more numbers.
- Stop when you have 3 distinct numbers.

Since the process is random, every student can get a unique sample. For example, when the authors of the book tried it they got the following list of numbers:

$$\{10, 3, 3, 2\}$$

Hence, our sample was { John, Carlos, Benjamin }.

EXPLORATION 3

Continue working with the sheet of rectangles.

1. Use the calculator to choose 1 rectangle at random and record its area.
2. Now use the calculator to select four more rectangles. Determine the area of each rectangle. Fill in the following table.

Rectangle #					
Area					

3. Compute the average area for your sample of five rectangles.
4. Record your values from Parts 1 and 3 on two number lines on the board.
5. Explore the line plots for Parts 1 and 3
   - Did everyone get the same values?
   - Which data is more spread out? Part 1 or 3?
   - How do the line plots compare with those from Exploration 2? Compare the centers of all the plots.
6. Below is the summary of the rectangles by area. Find the mean of the all of the rectangles.

Area	Frequency
1	16
2	2
3	6
4	16
5	8
6	6
8	8
9	5
10	7
12	9
13	1
15	1
16	10
18	5

7. Compare the mean of all the rectangles to the center of each of the line plots. Which method gives values that are closest to the true mean area of all 100 rectangles? Why do you think this happens?

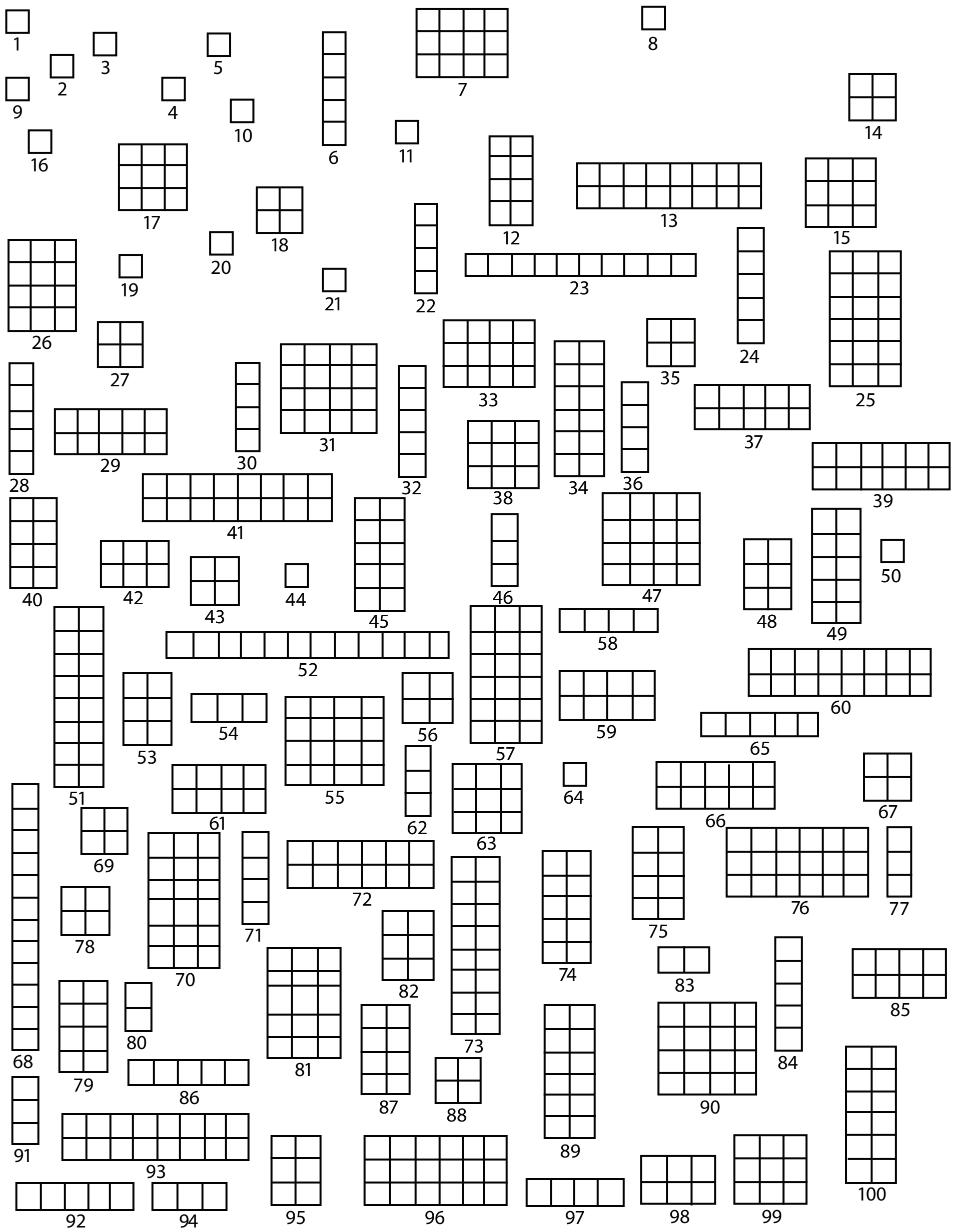
People think they can choose a sample that represents the population. But, often, the samples they choose are off target. The values tend to be too high or too low. In Exploration 3 we saw that simple random samples produce values that are on target. Samples which produce estimates which are on target are called unbiased. To do statistics it is always better to use a computer or calculator to choose the samples to ensure your samples are unbiased.

PROBLEM 1

For each of the following scenarios, determine if the sample selected is a simple random sample.

1. Students in your class are divided into girls and boys. A coin is flipped. If it's heads the girls are selected, if its tails all the boys are selected.
2. The principal has a list of all students in the school in alphabetical order. She runs down the list and selects every 10^{th} student.
3. Brandon is trying to determine what percent of students like soccer. So he goes out by the soccer field and asks the first 20 students he meets.
4. The mayor of the town wants to know if the voters favor the construction of a new park. She obtains a list of 33000 registered voters and uses a computer to select 500 of the names randomly.

100 Random Rectangles



EXERCISES

1. For each of the following determine the population and the sample:

   a. A movie executive wanted to know what effects profits. He recorded the length in minutes of 120 movies from 2012 and their profit margin.
   b. A deer hunter was interested in the average weight of white-tailed deer in a state park. He captured 50 deer in the park and weighed them.
   c. The Gallup poll conducted a telephone survey of 1180 registered voters and asked if they approve of the job the president is doing.
   d. The principle of a school is interested in the weight of student backpacks. He stands at the entrance of the school and weighs the backpacks of the first 100 students who arrive.

2. For each of the following identify if the sample is a Simple Random Sample.

   a. Teen Moda, a fashion magazine aimed at young latinas, asked all its subscribers to complete a survey online. 667 people responded.
   b. Researchers waited outside of a yogurt shop. They asked every third person who left the shop if they prefer sugar-free deserts to desserts made with sugar.
   c. The Texas Railroad Commission (despite its name) monitors the oil and gas business. To verify that oil rig operators are following regulations, inspectors plan to randomly select and inspect 100 oil rigs in Texas.

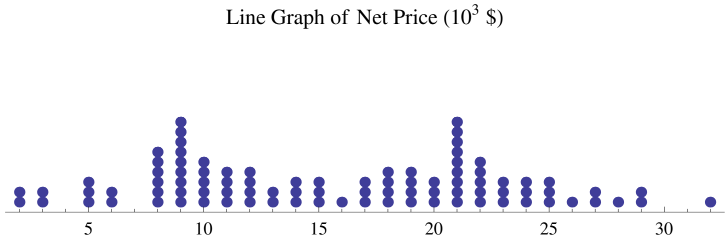
3. Use the internet to find a free applet to take simple random samples (search for "Simple Random Sample Applet"). Write down the url and then use the applet to find a simple random sample of size 5 from the numbers 1 through 100.

4. **Investigation:**

The table below shows a list of all the 4-year colleges and universities in Texas and their net price (in thousands of $) for 2010-2011 as reported by National Center for Education Statistics. Price is Average Net Price for 2010-2011 generated for full-time beginning undergraduate students who were awarded grant or scholarship aid from federal, state or local governments, or the institution. For public institutions only students paying the in-state or in-district rate are included. For institutions that charge students by program, net price is generated for the institution's largest program.

#	College	Price	#	College	Price
1	Abilene Christian Univ.	24	46	Southwestern Adventist Univ.	19
2	Angelo State Univ.	11	47	Southwestern Christian College	5
3	Argosy Univ.-Dallas	24	48	Southwestern Univ.	23
4	Arlington Baptist College	17	49	St. Mary's Univ.	18
5	Austin College	21	50	Strayer Univ.-Texas	25
6	Baptist Missionary Assoc. Theol. Sem.	3	51	Sul Ross State Univ.	8
7	Baylor Univ.	27	52	Tarleton State Univ.	15
8	Brazosport College	5	53	Texas A & M International Univ.	2
9	Brown Mackie College-San Antonio	25	54	Texas A & M Univ.-College Station	12
10	College of Biblical Studies-Houston	15	55	Texas A & M Univ.-Commerce	9
11	Concordia Univ.-Texas	19	56	Texas A & M Univ.-Corpus Christi	9
12	Dallas Christian College	10	57	Texas A & M Univ.-Galveston	11
13	DeVry Univ.-Texas	24	58	Texas A & M Univ.-Kingsville	9
14	Hallmark College of Technology/ Aeronautics	12	59	Texas A & M Univ.-Texarkana	2
15	Hardin-Simmons Univ.	19	60	Texas Christian Univ.	28
16	Houston Baptist Univ.	18	61	Texas College	13
17	Howard Payne Univ.	17	62	Texas Lutheran Univ.	17
18	Huston-Tillotson Univ.	16	63	Texas Southern Univ.	9
19	Intn'l Acad of Design and Tech-San Antonio	20	64	Texas State Univ.-San Marcos	9
20	ITT Tech. Inst.-Arlington	21	65	Texas Tech Univ.	10
21	ITT Tech. Inst.-Austin	22	66	Texas Wesleyan Univ.	14
22	ITT Tech. Inst.DeSoto	21	67	Texas Woman's Univ.	8
23	ITT Tech. Inst.-Houston North	21	68	Art Inst. of Austin	29
24	ITT Tech. Inst.-Houston West	21	69	Art Inst. of San Antonio	26
25	ITT Tech. Inst.-Richardson	20	70	Univ. of Texas at Arlington	10
26	ITT Tech. Inst.-San Antonio	21	71	Univ. of Texas at Brownsville	6
27	ITT Tech. Inst.-Webster	21	72	Univ. of Texas at Dallas	8
28	Jarvis Christian College	14	73	Univ. of Texas at El Paso	3
29	Lamar Univ.	12	74	Univ. of Texas at San Antonio	9
30	LeTourneau Univ.	21	75	Univ. of Texas of the Permian Basin	6
31	McMurry Univ.	18	76	Trinity Univ.	25
32	Messenger College	14	77	Univ. of Dallas	27
33	Midland College	5	78	Univ. of Houston	12
34	Midwestern State Univ.	8	79	Univ. of Houston-Downtown	8
35	National American Univ.-Austin	18	80	Univ. of Houston-Victoria	9
36	North American College	9	81	Univ. of Mary Hardin-Baylor	22
37	Northwood Univ.-Texas	15	82	Univ. of North Texas	10
38	Our Lady of the Lake Univ.	11	83	Univ. of Phoenix-Austin Campus	23
39	Paul Quinn College	13	84	Univ. of Phoenix-Dallas Campus	22
40	Prairie View A & M Univ.	9	85	Univ. of Phoenix-Houston Campus	22
41	Saint Edward's Univ.	20	86	Univ. of Phoenix-San Antonio Campus	23
42	Sam Houston State Univ.	8	87	West Texas A & M Univ.	10
43	Schreiner Univ.	19	88	Westwood College-Ft Worth	22
44	South Univ.- Art Inst. of Fort Worth	32	89	Westwood College-Houston S.	21
45	Southern Methodist Univ.	29	90	Wiley College	11

In this investigation, you will gather larger and larger samples to see how the sample reflects the population.



a. The line plot above shows the distribution of net price for all the colleges in Texas. The mean is 15.66, the median is 15.5, the MAD is 6.3. What are the units of each of these statistics?

b. Mark the mean and median on the line plot.

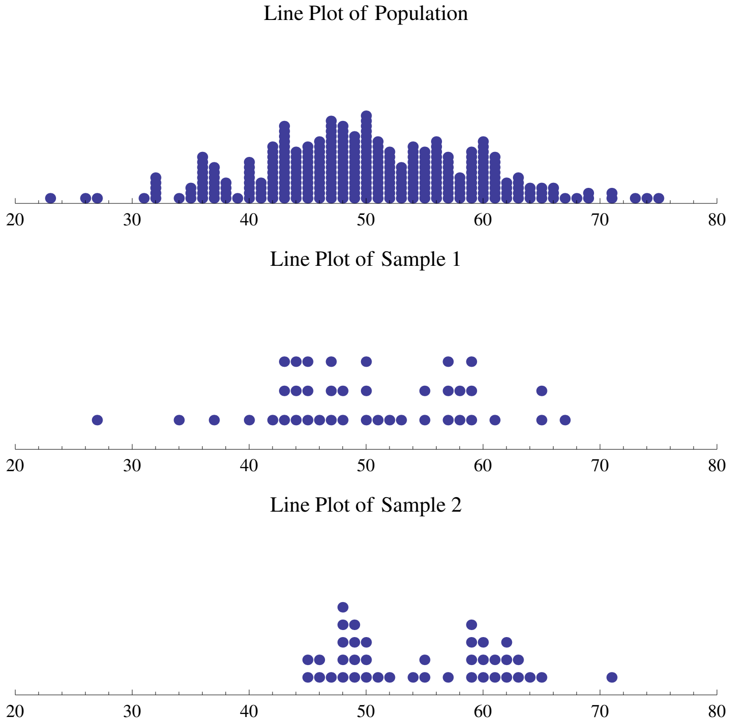
c. Use a computer or calculator to get a simple random sample of $n = 5$ colleges.
   - Make a line plot of the net price for the colleges in your sample.
   - Compute the mean and median net price for your sample. Mark them on the line plot.
   - Compute the MAD for your sample.

d. Now sample 15 more colleges. So you have a total sample of 20 colleges.
   - Make a line plot of the net price for the colleges in your sample of $n = 20$.
   - Compute the mean and median net price for your sample. Mark them on the line plot.
   - Compute the MAD for your sample.

e. Now sample 20 **more** colleges, for a total of 40 colleges.
   - Make a line plot of the net price for the colleges in your sample of $n = 20$.
   - Compute the mean and median net price for your sample. Mark them on the line plot.
   - Compute the MAD for your sample.

f. Compare the line plots for the three sample sizes to the population plot. Compare the statistics: mean, median and MAD. When does the sample begin to look like the population?

5. Below are line plots of a population and two samples from that population. Which sample is more likely to be a simple random sample? Explain.



SECTION 14.4 CHAPTER REVIEW

Key Terms

center
data set
deviation
mean
mean absolute deviation
median
population
sample
sample size
simple random sample
standard deviation
variance

Formulas

Mean: $\bar{x} = \frac{x_1+x_2+\cdots+x_n}{n}$

MAD: $MAD = \frac{|x_1-\bar{x}|+|x_2-\bar{x}|+\cdots+|x_n-\bar{x}|}{n}$

Variance: $V = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}$

Standard Deviation: $S = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}}$

Practice Problems

1. Compute the mean, MAD, variance and standard deviation for the following data sets.
   a. $4, 7, 9, 15, 10$
   b. $-1, 2, 4, 10, 5$
   c. $89, 92, 94, 100, 95$
2. To get an A for his statistics class Xingde's average exam grade must be 90 or greater. The class has five exams worth 100 points each. Xingde's first four exams grades are: $79, 88, 95, 91$. What grade must Xingde get on the fifth exam in order to get an A?

3. Without computing, match each list of numbers in the left column with its standard deviation in the right column. Explain your reasoning.

a. $1, 1, 1, 1$	i. 0
b. $1, 2, 2$	ii. .047
c. $1, 2, 3, 4, 5$	iii. .471
d. $10, 20, 20$	iv. 1.414
e. $0.1, 0.2, 0.2$	v. 2.828
f. $0, 2, 4, 6, 8$	vi. 3.399
g. $0, 0, 0, 0, 5, 6, 6, 8, 8$	vii. 4.714

MATH EXPLORATIONS

Appendix

GLOSSARY

Absolute Value: 1. A number's distance from zero. 2. For any x, is defined as follows:

$$|x| = \begin{cases} -x, \text{ if } x < 0; \\ x, \quad \text{if } x \geq 0. \end{cases}$$

Acute Angle: An angle whose measure is greater than 0 and less than 90 degrees.

Acute Triangle: A triangle in which all three angles are acute angles.

Algebraic Expression: An expression that includes one or more variables and may also include symbols indicating an operation or a relationship.

Area Model: A mathematical model based on the area of a rectangle, used to represent multiplication or to represent fractional parts of a whole.

Arithmetic Sequence: A sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if there is a number c such that for each natural number n, $a_{n+1} = a_n + c$, that is $a_{n+1} - a_n = c$.

Axis of Symmetry: The vertical line $x = -\frac{b}{2a}$ for the parabola given by $f(x) = ax^2 + bx + c$ or the vertical line $x = h$ when written $f(x) = a(x - h)^2 + k$.

Base: 1) For any number x raised to the n^{th} power, written as x^n, x is called the base of the expression. 2) In geometry any side of a triangle may be called the base.

Bankruptcy: A person or company is in **bankruptcy** when the court determines they are unable to pay their debts.

Bivariate Data: Data that has two variables.

Cartesian Plane: See: **Coordinate Plane**

Categorical Data: Data for which each observation represents a name or category.

Coefficient: In the product of a constant and a variable the constant is the numerical coefficient of the variable and is frequently referred to simply as the coefficient.

Common Factor: A factor that appears in two or more terms.

Compound Interest: Interest paid on previous interest which was added to the principal.

Congruent: Two geometric figures are **congruent** if they are the same size and shape.

Consistent system: See: **System of Equations**

Convex Polygon: A plane, closed, figure formed by three or more line segments intersecting only at end points and each interior angle being less than 180 degrees.

Coordinate(s): A number assigned to each point on the number line which shows its position or location on the line. In a coordinate plane the ordered pair, (x, y), assigned to each point of the plane showing its position in relation to the x-axis and y-axis.

Coordinate Plane: A plane that consists of a horizontal and vertical number line, intersecting at right angles at their origins. The number lines, called **axes**, divide the plane into four quadrants. The quadrants are numbered I, II, III, and IV beginning in the upper right quadrant and moving counterclockwise.

Correlation Coefficient: A number between -1 and 1 which measures the linear dependence of two variables.

Counterclockwise: A circular movement opposite to the direction of the movement of the hands of a clock.

Counting numbers: The **counting numbers** are the numbers in the following never-ending sequence: $1, 2, 3, 4, 5, 6, 7, \ldots$ We can also write this as $+1, +2, +3, +4, +5, +6, +7, \ldots$. These numbers are also called the **positive integers** or **natural numbers**.

Credit Card: A small plastic card issued by a bank or business allowing the holder to purchase goods or services on credit. A form of revolving credit.

Credit Score: A number assigned to a person that indicates to lenders their capacity to repay a loan.

Cube: The third power of a number.

Cube Root: For real number x and y, y is the cube root of x, (written $\sqrt[3]{x}$), if $y^3 = x$.

Data Set: A collection of information, frequently in the form of numbers.

Degree: The degree of a term is the sum of the exponents of the variables. The degree of a polynomial is the highest degree of any of its terms. If it contains no variables its degree is 0 if it is non-zero, and undefined if the polynomial is zero.

Dependent System: See: **System of Equations**

Dependent Variable: The variable in a function representing the elements of the range; the output values.

Dilation: A transformation that represents a change in size while maintaining the shape. The figure is enlarged or reduced by a **scale factor** k around a given point. The point is called the **center of the dilation**.

Direct Variation: For real variables x and y, y varies directly with x if $y = Kx$ or $\frac{y}{x} = K$ for a constant K, $K \neq 0$. K is called the constant of proportionality.

Discriminant: The expression $b^2 - 4ac$ that appears under the radical sign in the Quadratic Formula.

Domain: The set of input values in a function.

Down Payment: An initial payment made when something is bought on credit.

Easy Access Loan: A line of credit made available to cover overdrafting a checking account.

Elements: Members of a **set**.

Equation: A math sentence using the equal sign to state that two expressions represent the same number.

Equivalent Equations: Two equations are equivalent if they have the same solution or solution set.

Equivalent Inequalities: Two inequalities are equivalent if they have the same solution set.

Equivalent Expression: Expressions that have the same numerical value for given values of the variables.

Exponent: Suppose that n is a whole number. Then, for any number x, the n^{th} power of x, or x to the n^{th} power, is the product of n factors of the number x. This number is usually written x^n. The number x is usually called the base of the expression x^n, and n is called the **exponent**.

Exponential Function: For numbers a, k and $b \neq 0$, the function $f(t) = ab^{\frac{t}{k}}$ is called an exponential function. The number $a = f(0)$ is the initial value, b is the base and k is a constant related to growth rate or period.

Exponential Growth/Decay: Also see Exponential Function. For $a > 0$ and $b > 1$ the function denotes growth; for $a > 0$ and $0 < b < 1$ the function denotes decay.

Exponential Notation: A notation that expresses a number in terms of a base and an exponent.

Extraneous Solutions: Apparent solutions which do not satisfy the given equation; usually introduced by raising to a power or multiplying by the variable in obtaining the solution.

Extrapolation: The process of estimating beyond the original observation range.

Factor: 1) For integers a, b and c, a and b are factors of c if $c = ab$. Similarly $f(x)$ and $g(x)$ are factors of $p(x)$ if $p(x) = f(x) \cdot g(x)$. 2) Factor is also used as an instruction or command to express a given integer or polynomial as a product.

Fixed Cost: A cost that does not change with an increase or decrease in the amount of goods or services produced.

Formula: An equation showing the relationship between two or more quantities represented by variables.

Function: A **function** is a rule which assigns to each member of a set of inputs, called the **domain**, a member of a set of outputs, called the **range**.

Function Notation: $f(x)$, read "f of x", forming one side of an equation and used to indicate the value of the function when the input is x. Also see Function.

Geometric Sequence: A sequence $g_1, g_2, g_3, \ldots$ is a **geometric sequence** if there is a number b such that for each natural number n, $g_{n+1} = g_n \cdot b$, that is $\frac{g_{n+1}}{g_n} = b$.

Graph of a Function: The pictorial representation of a function by plotting all of its input-output pairs on a coordinate system.

Half-life: The time required for a substance to reduce by half.

Height: In a triangle it is the segment from a vertex perpendicular to the selected base. Also used to refer to the length of that segment.

Horizontal Axis: See: **Coordinate Plane**

Hypotenuse: The side opposite the right angle in a right triangle.

Inconsistent system: See: **System of Equations**

Independent system: See: **System of Equations**

Independent Variable: The variable in a function representing the elements of the **domain**; the input values.

Inequality: A statement that two expressions represent different values. There are various forms.

> **Strict Inequalities:** Statements such as "x is less than y", ($x < y$), and "x is greater than y", ($x > y$).
>
> **Weak inequalities:** Statements such as "x is less than or equal to y", ($x \leq y$), and "x is greater than or equal to y", ($x \geq y$).
>
> **General inequality:** The statement "x is not equal to y", ($x \neq y$).

Input Values: The values of the domain of a function.

Integers: The collection of integers is composed of the negative integers, zero and the positive integers: $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$.

Intersection of Sets: A set whose elements are all the elements that the given sets have in common.

Inverse Variation: For real variables x and y, y varies inversely with x if $yx = K$ or $y = \frac{K}{x}$ and K is a non-zero constant.

Irrational Number: A decimal number that neither repeats nor terminates. A number that can not be expressed as an integer divided by an integer.

Joint Variation: For variables x, y and z, z varies jointly with x and y if $z = Kxy$ and K is a non-zero constant.

Legs: The two sides of a right triangle that form the right angle.

Less Than, Greater Than: The statement that the number a is less than the number b, written $a < b$ means that there is a positive number x such that $b = a + x$. The number x must be $b - a$. If a is less than b, then b is greater than a, written $b > a$.

Like Terms: Algebraic terms that contain the same variables and for each variable the power is the same.

Least Squares Line: For a given bivariate data set: $(x_1, y_1), \ldots, (x_n, y_n)$, the **least squares line** is determined by choosing the slope m and y-intercept b, which makes the sum of squares:

$$(y_1 - (mx_1 + b))^2 + \cdots + (y_n - (mx_n + b))^2$$

as small as possible. This is also called the best fit line.

Linear Model for Multiplication: Skip counting on a number line.

Mean: The average of a set of data; sum of the data divided by the number of items.

Multiple: An integer or polynomial is said to be a multiple of any of its factors.

Multiplicative Identity: For each n, $n \cdot 1 = n$ and 1 is called the identity element for multiplication.

Multiplicative Inverse: The number x is called the **multiplicative**

inverse or **reciprocal** of n, $n \neq 0$, if $n \cdot x = 1$. This may also be written as $n \cdot \frac{1}{n} = 1$.

Natural Numbers: See: **Counting Numbers**

Negative 1 Power: If x is non-zero, x^{-1} is the number $\frac{1}{x}$.

Non-negative Numbers: Numbers greater than or equal to zero.

Numerical Data: Data for which each observation represents a numerical quantity or measurement.

Obtuse Angle: An angle whose measure is greater than 90 and less than 180 degrees.

Obtuse Triangle: A triangle that has one obtuse angle.

Ordered pair: A pair of numbers that represent the coordinates of a point in the coordinate plane with the first number measured along the horizontal scale and the second along the vertical scale.

Origin: The point with coordinate 0 on a number line; the point with coordinates $(0, 0)$ in the coordinate plane.

Output Values: The set of results obtained by applying a function rule to a set of input values.

Parabola: The shape of the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$. If the function is written as $f(x) = a(x - h)^2 + k$, $a \neq 0$, then the vertex is (h, k).

Parallel Lines: Two lines in a plane that never intersect.

Parent function: The simplest example of a family of functions.

Perfect Cube: An integer n that can be written in the form $n = k^3$, where k is an integer.

Perfect Square: 1) An integer n that can be written in the form $n = k^2$, where k is an integer. 2) Also a trinomial which can be written in the form $(ax \pm b)^2$.

Perpendicular: Lines are perpendicular if they intersect to form a right angle. Segments are perpendicular if the lines containing the segments are perpendicular.

Point-slope form: A form of a linear equation written as $(y - y_1) = m(x - x_1)$ where m is the slope and the line passes through the point $(x1, y1)$.

Positive Integers: See: **Counting Numbers**

Power: See: **Exponent**

Polynomial: A polynomial is an algebraic expression obtained by adding, subtracting and /or multiplying real numbers and variables.

Population In statistics, the set of values or objects the sample is taken from.

Principal: The amount borrowed or still owed on loan.

Proportional Relationship: Two variables have a proportional relationship if their ratio is always the same. We represent this relationship with the equation $\frac{y}{x} = m$ or $y = mx$, where x and y are the variables and m is the constant ratio.

Quadrant: See: **Coordinate Plane**

Quadratic Equation: An equation with a second degree term as its highest degree term.

Quadratic Expression: A polynomial containing a second degree term as its highest degree term.

Radical: An indicated root of a number or polynomial denoted by $\sqrt[n]{\ }$ so that $r = \sqrt[n]{x}$ implies $r^n = x$. The index n is generally omitted when $n = 2$ and we write $\sqrt{x}$ to mean the square root of x.

Radical Equation: An equation in which the variable appears in the radicand.

Radical Function: A function which is the square root of a variable expression. More generally, a function of the n^{th} root of a variable expression.

Radicand: The number or expression that appears in the radical sign; the number or expression whose root is to be found.

Range: See: **Function**

Rate: 1. A rate is a division comparison between two quantities with different units of measure. Also see **Unit Rate**. 2. The amount of interest charged on an annual basis.

Rational Equation: An equation involving one or more rational expressions.

Rational Expression: An expression in the form $\frac{a}{b}$ with a and b being polynomial expressions, b of at least degree one and $b \neq 0$.

Rational Number: A number that can be written as $\frac{a}{b}$ where a is an integer and b is a natural number.

Reciprocal: See: **Multiplicative Inverse**

Reflection: A transformation that represents a flipping motion. Points are flipped across the **line of reflection**. This creates the mirror image of the points on the other side of the line.

Representative: A sample is **representative** of the population if it exhibits the same characteristics as the population. Simple random samples are typically representative.

Right Angle: An angle formed by the intersection perpendicular lines; an angle with a measure of 90 deg.

Right Triangle: A triangle that has a right angle.

Room and Board: A sum of money charged by school, college or university for lodging and food.

Roots of a Quadratic: The solutions of $ax^2 + bx + c = 0$. The same values are zeros of the quadratic expression or the function $f(x) = ax^2 + bx + c$. They are also the x-intercepts of the intersection of the graph of $f(x)$ with the x-axis.

Sample: A subset of the population.

Scale Factor: For the parabola $f(x) = a(x - h)^2 + k$, a is the scale factor.

Scatter plot: A graph in which the values of two variables are plotted along two axes. The pattern of the points can reveal any correlation that is present.

Scholarship: A payment made to support a student's education, awarded on the basis of academic or other achievement.

Scientific Notation: Base ten numbers written in the form $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.

Sequence: A list of terms ordered by the natural numbers. The outputs of a function whose domain is the natural numbers.

Set: A collection of objects or elements.

Set Notation: A symbolic description of the elements of a set. "A is the set of all x's such that x is an element of the natural numbers with x greater than 2 and less than 11" would be written $A = \{x|x \text{ is a natural number }, 2 < x < 11\}$.

Similar: Two figures are **similar** if they have the same shape.

Simple Interest: Interest paid a single time on a principal invested or borrowed. Computed using $I = Prt$.

Simple Random Sample: A sample chosen using a process so that each sample of the same size has the same chance of being selected.

Simplifying: Combining like terms of a polynomial by carrying out the indicated additions or subtractions.

Simultaneously: See: **System of Equations in two Variables**

Slope of a Line: If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope of the line is the ratio $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ provided $x_2 \neq x_1$

Slope-intercept form: The equation $y = mx + b$ is the slope-intercept form of a line, where m is the slope and b is the y-intercept of the line.

Solution of an Equation: A solution to an equation with variable x is a number that, when substituted for x, makes the two sides of the equation equal. If the equation has more than one solution, then the collection of solutions is called the **solution set**.

Solution of an Inequality: The values that may be substituted for the variable in an inequality to form a true statement.

Solution of a System of Linear Equations: See **System of Linear**

Equations.

Square Root: For non-negative x and y, y is the square root of x if $y^2 = x$. For x a real number, $\sqrt{x^2} = |x|$ because the square root symbol denotes the non-negative root.

Square Root Function: See **Radical Function**.

Standard Form: A form of a linear equation written as $Ax+By = C$.

Subset: Set B is a subset of set A if every element of set B is also an element of set A.

System of Linear Equations: Two equations that both impose conditions on the variables. An ordered pair is a solution of the system if and only if it is a solution of each of the given equations. Systems of equations may be classified as follows:

1. A system with one or more solutions is called **Consistent**.
2. A system with no solution is called **Inconsistent**.
3. A Consistent system with exactly one solution is called **Independent**.
4. A Consistent system with more than one solution is called **Dependent**.

System of inequalities in two variables: Two inequalities that both impose conditions on the variables. If the inequalities form an "and" statement the solution is all ordered pairs that satisfy both inequalities. If the inequalities form an "or" statement the solution is any ordered pair that satisfies either inequality.

Term: 1. Each member of a sequence. 2. Each expression in a polynomial separated by addition or subtraction signs.

Translation: A transformation that represents a sliding motion. Each point is moved the same distance in the same direction. For a parabola, a horizontal or vertical shift in the position of the parent function.

Tuition: A sum of money charged for teaching or instruction by a school, college, or university.

Unit Rate: A ratio of two unlike quantities that has a denominator of 1 unit.

Variable: A letter or symbol that represents an unknown quantity.

Variable Cost: A cost that increases with an increase in the amount of goods or services produced.

Vertex: 1. The common endpoint of two rays forming an angle. 2. The highest or lowest point of the graph of a parabola.

Vertex form of a Parabola: A quadratic function written as $f(x) = a(x-h)^2+k$ is in **vertex form**. The **vertex** is (h, k) and the coefficient a is the scale factor.

Vertical Axis: See: **Coordinate Plane**

Whole Numbers: The whole numbers are the numbers in the following never-ending sequence: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$

x-**axis:** The horizontal axis of a coordinate plane.

x-**intercept:** The point of intersection of a graph with the x-axis.

y-**axis:** The vertical axis of a coordinate plane.

y-**intercept:** The point of intersection of a graph with the y-axis.

Zero power: For any number x, $x \neq 0$, $x^0 = 1$.

Zeros of a Quadratic: See **Roots of a Quadratic**.

GLOSARIO

Absolute Value: Valor Absoluto : La distancia de un número a cero. Por cualquier valor x, es definido como:

$$|x| = \begin{cases} -x, \text{ if } x < 0; \\ x, \quad \text{if } x \geq 0. \end{cases}$$

Area Model: Modelo Basado en la Area: Un Modelo matemático basado en la área de un rectángulo, usado para representar dos números multiplicados ya sean enteros o quebrados (fracciones.)

Base: Base: Por cualquier valor x, elevado a la potencia de n, escrito como x^n, x es llamado la *base* de la expresión.

Cartesian Plane: Plano Cartesiano: ver **Coordinate Plane**

Coefficient: Coeficiente: En el producto de un constante y un variable, el constante es el factor numérico del término, y es referido como el coeficiente.

Compound Interest: Interés Compuesto: Interés pagado sobre interés previamente ganado mas el principal.

Consistent System: Sistema Consistente: ver **Sistema de Ecuaciones.**

Convex Polygon: Polígono Convexo: Una figura cerrada en un plano formada por tres o mas segmentos lineales unidos solamente en los extremos y cada ángulo interior con medida de menos de 180 grados.

Coordinate(s): Coordenado(s): Un número asignando a un punto en la recta numérica. En el plano coordenado, el par ordenado (x, y) sitúa cada punto y lo sitúa en respecto a los ejes de x, y y.

Coordinate Plane: Plano Coordenado: Un plano consistiendo de una recta numérica horizontal y una recta numérica vertical que se intersectan en el origen formando ángulos rectos. Las rectas, llamadas *ejes*, dividen el plano en cuatro cuadrantes. Estos son numerados I, II, III, IV comenzando con el cuadrante en la esquina derecha de arriba y siguiendo contra-reloj.

Counterclockwise: Contra-reloj: un movimiento circular opuesto a la dirección de movimiento de las manecillas del reloj.

Counting Numbers: Numeros de Conteo: Números usados para contar. Llamados números naturales o enteros, son en secuencia, 1, 2, 3, 4, 5, 6, 7...

Cube: Elevado al Cubo: elevar un número a la tercera potencia.

Data Set: Conjunto de Datos: Una colección de información frecuentemente el forma de números.

Degree: Grado: El grado del término es la suma de las potencias de los variables. El grado de un polinomio es la potencia más grande de sus integrantes. Si contiene solo números, el valor es cero, y si es cero, el grado es indefinido.

Dependent System: Sistema Dependiente: ver **System of Equations.**

Dependent Variable: Variable Dependiente: El variable en una función que representa los elementos del rango; los datos de salida.

Discriminant: Discriminante: la expresión $b^2 - 4ac$ que aparece bajo la raíz cuadrada en la formula cuadrática.

Domain: Dominio: El conjunto de primeros números de una función.

Elements: Elementos: miembros de un conjunto.

Equation: Ecuación: Una proposición matemática con igualdad señalando que dos expresiones son iguales.

Equivalent Equations: Ecuaciones Equivalentes: Dos ecuaciones son iguales si tienen la misma solución o conjunto de soluciones.

Equivalent Inequalities: Desigualdades Equivalentes: Dos desigualdades son iguales si tienen el mismo conjunto de soluciones.

Exponent: Exponente (Potencia): Por cualquier valor x, elevado a la potencia de n, escrito como x^n, x es llamado la *base* de la expresión, y^n, n es el **exponente** o **potencia**.

Exponential Function: Función Exponencial: Para los números a, k and $b > 0$, la función $f(t) = ab^{\frac{t}{k}}$ es llamada una función exponencial. El número $a = f(0)$ es el valor inicial, b es la base, y k es una constante relacionada a la tasa de crecimiento.

Exponential Notation: Notación Exponencial (Notación Potencial): Notación para expresar un número en términos de base y potencia.

Factor: Factor: 1. Un entero que divide un número dado exactamente. 2. Definición de verbo. 3. Un polinomio que divide el polinomio dado exactamente.

Function: Función: Una función es una regla que asigna a cada primer número de un conjunto, llamado **dominio**, un número de salida, llamado el **rango**. La regla no permite que los primeros números se repitan.

Graph of a Function: Grafica de una Función: Una representación pictórica de una función graficando pares ordenados en el sistema coordinado.

Horizontal Axis: Eje Horizontal: ver **Coordinate Plane.**

Hypotenuse: Hipotenusa: El lado opuesto al ángulo recto en un triangulo recto.

Inconsistent System: Sistema Inconsistente: ver **System of Equations.**

Independent System: Sistema Independiente: ver **System of Equations.**

Independent Variable: Variable Independiente: El variable en una función que representa los elementos del dominio; los primeros datos.

Inequality: Desigualdad: Es un enunciado que dos expresiones representan diferentes valores. Hay varias formas de desigualdad.
Strict Inequalities: Desigualdades Estrictas: Enunciado como "*x es menor que y*", (x, y), y "*x es mayor que y*", ($x > y$).

Weak Inequalities: Desigualdades Débiles: Enunciado como "*x es menos que o igual a y*", ($x = y$), y "*x es mas que o igual a y*", ($x = y$).

General Inequality: Desigualdad General: Enunciado como "*x no es igual a y*", ($x \neq y$).

Input Values: Primeros Valores: Los valores del **dominio** de una función.

Integers: Enteros: La colección de enteros es compuesta de números negativos, cero y los números positivos:. . . , -4, -3, -2, -1, 0, 1, 2, 3, 4,. . .

Intersection of Sets: Traslape de Conjuntos: Un conjunto con elementos cuales son los elementos comunes de los conjuntos dados.

Irrational Number: Numero Irracional: Un decimal que no repite ni termina. Un número que no se puede representar como un entero dividido por otro entero.

Legs: Catetos: Los lados que forman el ángulo recto de un triangulo recto.

Less Than, Greater Than: Menor Que, Mayor Que: El enunciado que un número á *es menor que un número b,* escrito $a < b$ significa que hay un número positivo x de tal manera que $b = a + x$. El número x tiene que ser $b - a$. Si a es menor que b, b es mayor que a, escrito $b > a$.

Like Terms: Términos Semejantes: Términos de la misma potencia en el mismo variable o variables.

Line Plot: Grafica de Líneas: Marcando un punto en una recta numérica correspondiendo a un coordinado en la recta numérica.

Linear Model for Multiplication: Modelo Lineal para Multiplicar: Contar en grupos usando la recta numérica.

Mean: Media Aritmética: El promedio de un conjunto de datos; la suma de los datos dividida por el número de datos.

Multiplicative Identity: Identidad Multiplicativa: Por cada número n, $n \cdot 1 = n$ y 1 es llamado el elemento de identidad para la multiplicación.

Multiplicative Inverse: Inverso Multiplicativo: El número x es llamado el inverso multiplicativo o el reciproco de n, $n \neq 0$, si $n \cdot x = 1$. Se puede escribir $n \cdot \frac{1}{n} = 1$.

Natural Numbers: Números Naturales: ver **Counting Numbers**

Negative 1 Power: Potencia de Negativo 1: Si x es un número no igual a cero, x^{-1} es igual a $\frac{1}{x}$.

Ordered Pair: Par Ordenado: Un par de números que representa los coordinados de un punto en el plano coordinado con el primer número señalando la medida en el eje horizontal y el segundo señalando la medida en el eje vertical.

Origin: Origen: El punto con el coordinado 0 el la recta numérica; el punto con los coordinados (0,0) en el plano coordinado.

Output Values: Valores de Salida: El conjunto de valores obtenidos al aplicar una función a un conjunto de primeros valores.

Parallel lines: Lineas Paralelas: Dos líneas en un plano que nunca cruzan.

Parent Function: Patron de la Función: El ejemplo mas simple de una familia de funciones. La base de todas las funciones de ese tipo.

Perfect Cube: Cubo Perfecto: Un entero n que puede ser escrito en la forma $n = k^3$, y k es un entero.

Perfect Square: Cuadrado Perfecto: Un entero n que puede ser escrito en la forma $n = k^2$, y k es un entero.

Perpendicular: Perpendicular: Dos líneas o segmentos se dicen ser perpendiculares si cruzan a formar un ángulo recto.

Point-Slope Form: Forma Punto-Pendiente: Una forma de la ecuación de una recta escrita como $(y - y_1) = m(x - x_1)$ y *m* es el pendiente y la recta pasa por el punto (x_1, y_1) .

Positive Integers: Enteros Positivos: ver **Counting Numbers**

Power: Potencia: ver **Exponents**

Polynomial: Polinomio: Un polinomio es una suma y/o resta de términos.

Quadrant: Cuadrante: ver **Coordinate Plane**

Range: Rango: ver **Function**

Reciprocal: Reciproco: ver **Multiplicative Inverse**

Right Angle: Angulo Recto: un ángulo formado cuando cruzan dos líneas perpendiculares; un ángulo que mide 90 grados.

Right Triangle: Triangulo Recto: un triángulo que tiene un ángulo recto.

Sample Size: Espacio Maestral: El número de observaciones en un conjunto de datos.

Scale Factor: Factor de Escala: Si $f(x) = a(x - h)^2 + k$ el factor de escala es a.

Scientific Notation: Notación Cientifica: Números en base diez escritos en la forma *a* x 10^nen donde *n* es un entero y $1 \leq a < 10$.

Sequence: Sucesión: Un conjunto de términos puestos en orden por los números naturales. Los números de salida de una función de dominio que incluye los números naturales o enteros.

Set: Conjunto: una colección de objetos o elementos.

Set Notation: Notación de Conjunto: Un sistema que usa símbolos en ves de palabras para descibir un conjunto de valores.

Simplifying: Simplificar: Combinar términos semejantes en un polinomio sumando o restando como indicado.

Simultaneously: Simultáneamente: ver **System of Equations in Two Variables.**

Slope of a Line: Pendiente de una Recta: Si (x_1,y_1) y (x_2,y_2) son dos puntos en la recta, el pendiente de la recta es la proporción $m = \frac{sube}{corre} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ mientras x_2 ? x_1.

Slope-Intercept Form: Forma Pendiente-Intercepción: La ecuación $y = mx + b$ es la forma pendiente-intercepción de una recta, donde ḿ es el pendiente y b es la intercepción de el eje y y la recta.

Solution of an Equation: Solución de una Ecuación: La solución de una ecuación con variable x es un número que cuando substituido por x, causa que los dos lados de la ecuación sean iguales. Si la ecuación tiene mas de una solución, la colección de soluciones es llamada un **conjunto de soluciones**.

Solution of an Inequality: Solución de una Desigualdad: Los valores que pueden ser substituidos por el variable en una desigualdad para crear una certitud.

Standard Form: Forma General: Una forma de una ecuación recta escrita como $Ax + By = C$.

Subset: Subconjunto: Conjunto B es un subconjunto de Conjunto A si todos los elementos de conjunto B son parte de los elementos de Conjunto A.

System of Equations in Two Variables: Sistema de Ecuaciones con Dos Variables: Dos ecuaciones que imponen condiciones en los variables. Un par ordenado es la solución del sistema solamente si es la solución de cada una de las dos ecuaciones. Los sistemas pueden ser clasificados de la siguiente manera:

Un sistema con una o más soluciones es llamada **Consistente.**

Un sistema sin solución es llamado **Inconsistente**.

Un sistema Consistente con solo una solución es llamado **Independiente**.

Un sistema Consistente con más que una solución es llamado **Dependiente**.

System of Inequalities in Two Variables: Sistema de Desigualdades con Dos Variables: Dos desigualdades que imponen condiciones en los variables. Si las desigualdades forman una proposición "y", la solución son todos los pares ordenados que satisfacen las dos desigualdades. Si las desigualdades forman una proposición "o", la solución es cualquier par ordenado que satisface las desigualdades.

Term: Término: 1. un miembro de una sucesión. 2. Cada expresión en un polinomio separado por una suma o resta.

Translation: Traslación: Una transformación que mueve una figura sobre el plano pero no altera el tamaño ni la forma. En una parábola, un cambio vertical o horizontal en el patrón de la función.

Unit Rate: Razón Unitaria: Una razón en la cual el denominador es una unidad.

Variable: Variable: Una letra o símbolo que representa una cantidad.

Vertex: Vertice: 1. El punto común de los lados de un ángulo. 2. El punto más alto o bajo de una parábola.

Vertex Form of a Parabola: Forma de la Vértice de una Parábola: Una función cuadrática escrita como $f(x) = a(x-h)^2+k$ esta en forma de vértice. El vértice de la parábola es *(h,k)* y la a es el factor de escala.

Vertical Axis; Eje Vertical: ver **Coordinate Plane.**

Whole Numbers: Numeros Enteros: Los números enteros son los números en la suceción: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11...

x-axis: eje de x: El eje horizontal en el plano coordinado.

x-intercept: intercepción de el eje de x: El punto de intercepción de la grafica con el eje de x.

y-axis: eje de y: El eje vertical en el plano coordinado

y-intercept: intercepción de el eje de y: El punto de intercepción de la grafica con el eje de y.

Zero Power: Potencia de Cero: Por cualquier número x, $x^0 = 1$.

Summary of Important Ideas

Addition Property of Equality: If $A = B$, then $A + C = B + C$.

Addition Property of Inequality: For all numbers a, b and c, if $a < b$, then $a + c < b + c$.

Additive Identity: For any number x, $x + 0 = x$.

Additive Inverses: For any number x, there exists a number $-x$, called the additive inverse of x, such that $x + (-x) = 0$.

Associative Property of Addition: For any numbers x, y, and z, $(x + y) + z = x + (y + z)$.

Associative Property of Multiplication: For any numbers x, y, and z, $(xy)z = x(yz)$.

Commutative Property of Addition: For any numbers x and y, $x + y = y + x$.

Commutative Property of Multiplication: For any numbers A and B. $A \cdot B = B \cdot A$.

Difference of Squares: For any numbers x and y, $x^2 - y^2 = (x - y)(x + y)$.

Distance Between Two Points: For any two points on the plane, (x_1, y_1) and (x_2, y_2), the distance between the points is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Distributive Property: For any numbers x, y, and z, $x(y + z) = xy + xz$.

Division Property of Equality: If $A = B$ and C is non-zero, then $\frac{A}{C} = \frac{B}{C}$.

Division Property of Inequality: For all real numbers a and b such that $a < b$

1. If $c > 0$, the $\frac{a}{c} < \frac{b}{c}$.
2. If $c < 0$, the $\frac{a}{c} > \frac{b}{c}$.

Double Opposites Theorem: For any number x, $-(-x) = x$.

Rules of Exponents:

1. If x is a number and both m and n are natural numbers, then $(x^m)(x^n) = x^{m+n}$.
2. For each pair of numbers a and b and natural number n, $(a^n)(b^n) = (ab)^n$.
3. For each number a and natural numbers m and n, $(a^m)^n = a^{mn}$.
4. If x is not 0 and m and n are natural numbers $\frac{x^m}{x^n} = x^{m-n}$.

Linear Programming Theorem: If R is a closed, convex polygon in the coordinate plane, and f is a linear objective function in x and y, then the maximum value of f is attained at one or more vertices of R. Similarly, the minimum value of f is attained at one or more vertices of R.

Mean Absolute Deviation: $MAD = \frac{|x_1-\bar{x}|+|x_2-\bar{x}|+\cdots+|x_n-\bar{x}|}{n}$

Multiplication Property of Equality: If $A = B$, then $AC = BC$ and, for non-zero C, the equations are equivalent.

Multiplication Property of Inequality: For real numbers a and b such that $a < b$,

1. If $c > 0$, the $ac < bc$.
2. If $c < 0$, the $ac > bc$.

Multiplicative Identity: The number 1 is the multiplicative identity; that is, for any number n, $n \cdot 1 = n$.

Multiplicative Inverse: For every non-zero x there exists a number $\frac{1}{x}$, called the multiplicative inverse or reciprocal of x, such that $x \cdot \frac{1}{x} = 1$.

Product of Radicals: If $a \geq 0$ and $b \geq 0$, then $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

Pythagorean Theorem: If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

$$c^2 = a^2 + b^2.$$

Quadratic Formula: The solutions of the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are numbers and $a \neq 0$ are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Slopes of Parallel Lines Theorem: The line given by the equation $y = mx+b$ will parallel to any line with the same slope m and different y-intercept.

Slopes of Perpendicular Lines Theorem: The line given by the equation $y = mx + b$, with $m \neq 0$, will be perpendicular to any line with slope equal to $-\frac{1}{m}$

Squares of Real Numbers: If x is a real number, then $x^2 \geq 0$.

Subtraction Property of Equality: If $A = B$, then $A - C = B - C$.

Subtraction Property of Inequality: For all numbers a, b and c, if $a < b$ then $a - c < b - c$.

Transitive Property of Inequality: For all numbers a, b, and c, such that $a < b$ and $b < c$, then $a < c$.

Zero Product Property: If $xy = 0$, then $x = 0$ or $y = 0$.

MATH EXPLORATIONS

TEKS List

TEK	Page	Location
1Ai	506	Section 9.1 Exploration 1
1Ai	526	Section 9.3 Problem 2
1Aii	523	Section 9.2 Problem 1
1Aii	524	Section 9.2 Exercise 1
1Aii	352	Section 6.3 Example 6
1Aiii	321	Section 5.3 Example 4
1Aiii	325	Section 5.3 Exercise 3
1Bi	217	Section 3.6 Example 3
1Bi	221	Section 3.6 Exercise 8
1Bii	217	Section 3.6 Example 3
1Bii	221	Section 3.6 Exercise 8
1C i	72	Section 1.7 Exploration 4
1C i	80	Section 1.7 Exercise 16
1C i	68	Section 1.6 Exercise 6
1Cii	389	Section 7.2 Exploration 1
1Cii	393	Section 7.2 Exercise 1
1Cii	541	Section 10.1 Exercise 10
1Ciii	13	Section 1.2 Exploration 1
1Ciii	13	Section 1.2 Exploration 2
1Ciii	20	Section 1.2 Exercise 2
1Ciii	20	Section 1.2 Exercise 3
1Civ	512	Section 9.1 Example 1
1Civ	517	Section 9.1 Exercise 4
1Civ	678	Section 13.2 Exploration 3
1Civ	686	Section 13.2 Exercise 4
1Cv	45	Section 1.5 Exploration 1
1Cv	57	Section 1.5 Exploration 6
1Cv	60	Section 1.5 Exercise 10
1Cv	60	Section 1.5 Exercise 11
1Cv	60	Section 1.5 Exercise 12
1Cvi	57	Section 1.5 Exploration 6
1Cvi	60	Section 1.5 Exercise 10
1Cvi	352	Section 6.3 Example 6
1Cvi	362	Section 6.3 Exercise 8
1Cvii	57	Section 1.5 Exploration 6
1Cvii	60	Section 1.5 Exercise 10
1Cvii	60	Section 1.5 Exercise 11

TEK	Page	Location
1Di	46	Section 1.5 Exploration 2
1Di	58	Section 1.5 Exercise 1
1Dii	4	Section 1.1 Example 1
1Dii	11	Section 1.1 Exercise 6
1Diii	140	Section 3.1 Problem 1
1Diii	569	Section 10.3 Exercise 12
1Diii	140	Section 3.1 Problem 1
1Div	202	Section 3.5 Example 2
1Div	209	Section 3.5 Exercise 16
1Dv	293	Section 5.1 Exploration 4
1Dv	298	Section 5.1 Exercise 3
1Dvi	534	Section 10.1 Exploration 3
1Dvi	540	Section 10.1 Exercise 5
1Dvii	312	Section 5.3 Exploration 1
1Dvii	325	Section 5.3 Exercise 3
1Dviii	29	Section 1.3 Example 4
1Dviii	33	Section 1.3 Exercise 14
1Dix	550	Section 10.2 Problem 3
1Dix	552	Section 10.2 Exercise 3
1Dix	547	Section 10.2 Exploration 3
1Dx	535	Section 10.1 Exploration 4
1Dx	11	Section 1.1 Exercise 6
1Dxi	151	Section 3.2 Exploration 3
1Dxi	166	Section 3.2 Exercise 8
1Dxi	362	Section 6.3 Exercise 8
1Dxi	352	Section 6.3 Example 6
1Dxii	460	Section 8.2 Exploration 2
1Dxii	466	Section 8.2 Exercise 3
1Dxiii	57	Section 1.5 Exploration 6
1Dxiii	60	Section 1.5 Exercise 11
1Dxiii	60	Section 1.5 Exercise 11
1Dxiv	534	Section 10.1 Exploration 3
1Dxiv	541	Section 10.1 Exercise 10
1Dxv	151	Section 3.2 Exploration 3
1Dxv	152	Section 3.2 Problem 3
1Dxv	166	Section 3.2 Exercise 10
1Dxv	167	Section 3.2 Exercise 11
1Dxvi	152	Section 3.2 Exploration 4
1Dxvi	167	Section 3.2 Exercise 12
1Dxvi	140	Section 3.1 Exploration 2
1Dxvi	146	Section 3.1 Exercise 7

TEK	Page	Location
1Ei	14	Section 1.2 Problem 1
1Ei	699	Section 14.1 Exploration 1
1Ei	708	Section 14.1 Exercise 7
1Eii	13	Section 1.2 Exploration 2
1Eii	88	Section 2.1 Example 1
1Eii	108	Section 2.2 Example 2
1Eii	20	Section 1.2 Exercise 2
1Eii	60	Section 1.5 Exercise 9
1Eii	98	Section 2.1 Exercise 7
1Eii	115	Section 2.2 Exercise 5
1Eiii	120	Section 2.3 Exploration 1
1Eiii	125	Section 2.3 Exercise 9
1Eiii	68	Section 1.6 Exercise 6
1Eiv	149	Section 3.2 Exploration 1
1Eiv	164	Section 3.2 Exercise 1
1Eiv	68	Section 1.6 Exercise 6
1Ev	22	Section 1.2 Exercise 10
1Ev	24	Section 1.3 Example 1
1Ev	48	Section 1.5 Problem 3
1Ev	55	Section 1.5 Exploration 5
1Evi	160	Section 3.2 Example 3
1Evi	166	Section 3.2 Exercise 6
1Evi	22	Section 1.2 Exercise 10
1Evi	55	Section 1.5 Exploration 5
1Fi	198	Section 3.5 Exploration 1
1Fi	270	Section 4.4 Example 4
1Fi	259	Section 4.3 Exercise 3
1Fii	344	Section 6.3 Exploration 1
1Fii	354	Section 6.3 Exercise 2
1Gi	25	Section 1.3 Exploration 1
1Gi	10	Section 1.1 Exercise 5
1Gii	37	Section 1.4 Exploration 1
1Gii	41	Section 1.4 Exercise 1
1Giii	40	Section 1.4 Example 3
1Giii	41	Section 1.4 Exercise 2
1Giv	40	Section 1.4 Example 4
1Giv	167	Section 3.2 Exercise 12
1Gv	157	Section 3.2 Example 2
1Gv	166	Section 3.2 Exercise 8
1Gv	166	Section 3.2 Exercise 9
1Gvi	5	Section 1.1 Exploration 4
1Gvi	595	Section 10.6 Example 2
1Gvi	597	Section 10.6 Example 3
1Gvi	601	Section 10.6 Exercise 7
1Gvi	601	Section 10.6 Exercise 8

TEK	Page	Location
2A	4	Section 1.1 Example 1
2A	11	Section 1.1 Exercise 6
2Bi	72	Section 1.7 Exploration 4
2Bi	80	Section 1.7 Exercise 16
2Bii	544	Section 10.2 Exploration 2
2Bii	554	Section 10.2 Exercise 5
2Biii	544	Section 10.2 Exploration 2
2Biii	552	Section 10.2 Exercise 1
2Biii	554	Section 10.2 Exercise 5
2C	370	Section 6.6 Problem 1
2C	370	Section 6.6 Problem 2
2C	372	Section 6.6 Problem 4
2C	373	Section 6.6 Problem 5
2C	373	Section 6.6 Exercise 1
2Di	546	Section 10.2 Example 1
2Di	554	Section 10.2 Exercise 6
2Dii	547	Section 10.2 Exploration 3
2Dii	555	Section 10.2 Exercise 7
3A	594	Section 10.6 Exploration 4
3A	600	Section 10.6 Exercise 4
3Bi	589	Section 10.6 Exploration 1
3Bi	600	Section 10.6 Exercise 4
3Bii	589	Section 10.6 Exploration 1
3Bii	600	Section 10.6 Exercise 4
3C	589	Section 10.6 Exploration 1
3C	590	Section 10.6 Exploration 2
4A	153	Section 3.2 Example 1
4A	167	Section 3.2 Exercise 13
4B	200	Section 3.5 Problem 3
4B	209	Section 3.5 Exercise 14
4Ci	114	Section 2.2 Exploration 3
4Ci	176	Section 3.3 Exercise 5
4Cii	168	Section 3.3 Exploration 1
4Cii	175	Section 3.3 Exercise 3
4Ciii	179	Section 3.4 Exploration 1
4Ciii	175	Section 3.3 Exercise 3
4Ciii	182	Section 3.4 Example 1
4Civ	512	Section 9.1 Example 1
4Civ	517	Section 9.1 Exercise 5

TEK	Page	Location
5Ai	168	Section 3.3 Exploration 1
5Ai	175	Section 3.3 Exercise 4
5Aii	168	Section 3.3 Exploration 1
5Aii	175	Section 3.3 Exercise 3
5Aiii	168	Section 3.3 Exploration 1
5Aiii	175	Section 3.3 Exercise 4
5Bi	171	Section 3.3 Exploration 2
5Bi	209	Section 3.5 Exercise 16
5Bii	171	Section 3.3 Exploration 2
5Bii	209	Section 3.5 Exercise 16
5Biii	206	Section 3.5 Example 4
5Biii	208	Section 3.5 Exercise 9
5C	511	Section 9.1 Problem 1
5C	519	Section 9.2 Exploration 1
5C	527	Section 9.3 Problem 3
5C	528	Section 9.3 Problem 4
5D	512	Section 9.1 Example 1
5D	517	Section 9.1 Exercise 5
5E	647	Section 12.2 Example 1
5E	652	Section 12.2 Exercise 3
5Fi	171	Section 3.3 Exploration 2
5Fi	174	Section 3.3 Exercise 2
5Fii	171	Section 3.3 Exploration 2
5Fii	177	Section 3.3 Exercise 7
5Fiii	206	Section 3.5 Example 4
5Fiii	193	Section 3.4 Exercise 24
5Gi	97	Section 2.1 Problem 2
5Gi	101	Section 2.1 Exercise 15
5Gii	114	Section 2.2 Exploration 3
5Gii	116	Section 2.2 Exercise 7
5Giii	93	Section 2.1 Exploration 5
5Giii	101	Section 2.1 Exercise 15
5Giv	114	Section 2.2 Exploration 3
5Giv	116	Section 2.2 Exercise 6
5Hi	202	Section 3.5 Exploration 4
5Hi	172	Section 3.3 Exploration 3
5Hii	202	Section 3.5 Exploration 4
5Hii	176	Section 3.3 Exercise 6
5Hiii	171	Section 3.3 Exploration 2
5Hiii	177	Section 3.3 Exercise 7
5Hiv	206	Section 3.5 Example 4
5Hiv	209	Section 3.5 Exercise 17

TEK	Page	Location
5Ii	201	Section 3.5 Exploration 3
5Ii	207	Section 3.5 Exercise 5
5Ii	207	Section 3.5 Exercise 6
5Iii	198	Section 3.5 Exploration 1
5Iii	207	Section 3.5 Exercise 3
5Iiii	168	Section 3.3 Exploration 1
5Iiii	175	Section 3.3 Exercise 3
5Iiv	179	Section 3.4 Exploration 1
5Iiv	188	Section 3.4 Exercise 1
5Iiv	207	Section 3.5 Exercise 6
6A	74	Section 1.7 Problem 5
6A	80	Section 1.7 Exercise 17
6Ci	534	Section 10.1 Exploration 3
6Ci	540	Section 10.1 Exercise 5
6Cii	535	Section 10.1 Exploration 4
6Cii	541	Section 10.1 Exercise 10
7Ai	74	Section 1.7 Problem 5
7Ai	80	Section 1.7 Exercise 17
7Ai	81	Section 1.7 Exercise 20
7Aii	74	Section 1.7 Problem 5
7Aii	78	Section 1.7 Exercise 12
7Aii	81	Section 1.7 Exercise 20
7Aiii	74	Section 1.7 Problem 5
7Aiii	81	Section 1.7 Exercise 20
7Ci	536	Section 10.1 Example 2
7Ci	538	Section 10.1 Example 4
7Ci	539	Section 10.1 Exercise 1
7Ci	540	Section 10.1 Exercise 7
7Cii	539	Section 10.1 Problem 2
7Cii	540	Section 10.1 Exercise 6
7D	560	Section 10.3 Exploration 2
7D	568	Section 10.3 Exercise 6
8A	64	Section 1.6 Example 2
8A	68	Section 1.6 Exercise 5
8B	95	Section 2.1 Exploration 6
8B	99	Section 2.1 Exercise 13
8Ci	64	Section 1.6 Example 2
8Ci	66	Section 1.6 Example 3
8Ci	68	Section 1.6 Exercise 5
8Cii	66	Section 1.6 Problem 2
8Cii	69	Section 1.6 Exercise 7
8Ciii	66	Section 1.6 Example 3
8Ciii	68	Section 1.6 Exercise 5
8Civ	66	Section 1.6 Problem 2
8Civ	68	Section 1.6 Exercise 4

TEK	Page	Location
9Ai	240	Section 4.1 Exploration 4
9Ai	242	Section 4.1 Exercise 1
9Aii	240	Section 4.1 Exploration 4
9Aii	242	Section 4.1 Exercise 1
10Ai	586	Section 10.5 Exploration 3
10Ai	587	Section 10.5 Exercise 3
10Aii	576	Section 10.4 Example 2
10Aii	579	Section 10.4 Exercise 5
10Aii	579	Section 10.4 Exercise 6
10Aiii	572	Section 10.4 Example 1
10Aiii	579	Section 10.4 Exercise 5
10Aiii	579	Section 10.4 Exercise 6
10Aiv	593	Section 10.6 Exploration 3
10Aiv	600	Section 10.6 Exercise 4
10Av	581	Section 10.5 Exploration 1
10Av	582	Section 10.5 Problem 1
10Av	599	Section 10.6 Exercise 1
10Av	587	Section 10.5 Exercise 3
10Avi	576	Section 10.4 Example 2
10Avi	579	Section 10.4 Exercise 5
10Avi	579	Section 10.4 Exercise 6
10Avii	572	Section 10.4 Example 1
10Avii	579	Section 10.4 Exercise 5
10Avii	579	Section 10.4 Exercise 6
10B	589	Section 10.6 Exploration 1
10B	599	Section 10.6 Exercise 1
10Ci	572	Section 10.4 Example 1
10Ci	579	Section 10.4 Exercise 7
10Cii	576	Section 10.4 Example 2
10Cii	579	Section 10.4 Exercise 7
10Ciii	586	Section 10.5 Exploration 3
10Ciii	587	Section 10.5 Exercise 1
10Civ	586	Section 10.5 Exploration 3
10Civ	587	Section 10.5 Exercise 1
10Cv	586	Section 10.5 Exploration 3
10Cv	587	Section 10.5 Exercise 1
10Cvi	586	Section 10.5 Exploration 3
10Cvi	587	Section 10.5 Exercise 2
10Di	590	Section 10.6 Exploration 2
10Di	600	Section 10.6 Exercise 4
10Di	600	Section 10.6 Exercise 5
10Dii	590	Section 10.6 Exploration 2
10Dii	600	Section 10.6 Exercise 4
10Dii	600	Section 10.6 Exercise 5

TEK	Page	Location
11Ai	512	Section 9.1 Example 1
11Ai	515	Section 9.1 Exercise 1
11Aii	512	Section 9.1 Example 1
11Aii	515	Section 9.1 Exercise 1
11Bi	712	Section 14.2 Example 1
11Bi	717	Section 14.2 Exercise 4
11Bii	712	Section 14.2 Example 1
11Bii	717	Section 14.2 Exercise 4
11C	723	Section 14.3 Exploration 3
11C	728	Section 14.3 Exercise 3
11C	730	Section 14.3 Exercise 4
12Ai	680	Section 13.2 Example 2
12Ai	686	Section 13.2 Exercise 6
12Aii	680	Section 13.2 Example 2
12Aii	686	Section 13.2 Exercise 5
12Bi	680	Section 13.2 Example 2
12Bi	686	Section 13.2 Exercise 6
12Bii	680	Section 13.2 Example 2
12Bii	686	Section 13.2 Exercise 5
12Biii	684	Section 13.2 Problem 2
12Biii	687	Section 13.2 Exercise 8
12Biv	684	Section 13.2 Problem 2
12Biv	687	Section 13.2 Exercise 8
12Ci	690	Section 13.3 Example 1
12Ci	693	Section 13.3 Exercise 4
12Cii	691	Section 13.3 Example 2
12Cii	693	Section 13.3 Exercise 5
12Di	666	Section 13.1 Example 1
12Di	672	Section 13.1 Exercise 1
12Dii	668	Section 13.1 Example 3
12Dii	672	Section 13.1 Exercise 5
12Diii	669	Section 13.1 Example 4
12Diii	673	Section 13.1 Exercise 6

TEK	Page	Location
12Ei	683	Section 13.2 Exploration 4
12Ei	687	Section 13.2 Exercise 9
12Eii	684	Section 13.2 Problem 2
12Eii	687	Section 13.2 Exercise 9
12Eiii	684	Section 13.2 Problem 2
12Eiii	687	Section 13.2 Exercise 8
12Eiv	684	Section 13.2 Problem 2
12Eiv	687	Section 13.2 Exercise 8
12Fi	680	Section 13.2 Example 2
12Fi	687	Section 13.2 Exercise 11
12Fii	680	Section 13.2 Example 2
12Fii	687	Section 13.2 Exercise 11
12Fiii	680	Section 13.2 Example 2
12Fiii	687	Section 13.2 Exercise 12
12Gi	689	Section 13.3 Exploration 1
12Gi	693	Section 13.3 Exercise 1
12Gii	689	Section 13.3 Exploration 1
12Gii	693	Section 13.3 Exercise 2
12Gii	728	Section 14.3 Exercise 3
12Giii	690	Section 13.3 Example 1
12Giii	693	Section 13.3 Exercise 3

MATH EXPLORATIONS

Index