MATH EXPLORATIONS
PART 1

5th Edition

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Preface and Introduction

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

Learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. **A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together.** In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself.

Some basic rules for discussion within a group include:

1. Encourage everyone to participate and value each person's opinions. Listening carefully to what someone else says can help clarify a question.
2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely and never be afraid to ask questions.
3. Don't be afraid to make a mistake. In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to identify difficulties in solving a problem. Rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.
4. Always share your ideas with one another, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure

to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear there are several things to try:

1. Look for simpler cases. Looking deeply at simple cases can help you see a general pattern.
2. Ask your peers and teacher for help. Go beyond "Is this the right answer?"
3. Understand the question being asked. Understanding the question leads to mathematical progress.
4. Focus on the process of obtaining an answer, not just the answer itself; in short become problem-centered, not answer-centered. One major goal of this book is to develop an understanding of ideas that can solve more difficult problems as well.
5. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Suggestions for responding to oral questions in group and class discussion:

As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times you may want to explore first by yourself and then with others by discussing your ideas. When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.
2. In class, be recognized if you want to contribute or ask a question.
3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.

4. Finally, if you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Advice about reading and taking notes in math.

1. Reading math is a specific skill. When you read math, you need to read each word carefully. The first step is to learn the mathematical meaning of all words. Some words may be used differently in math than in everyday speech.

2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you relate what you are learning to real world situations.

3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.

4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues can help you understand and remember definitions of new terms.

Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes.

It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. Parents are encouraged to read both the book and the accompanying materials and talk to their students about what they are learning.

Possibly the most unique aspect of this book is the breadth and span of its appeal. The authors wrote this text for both a willing 4th or 5th grader and any 6th grader. Students may particularly enjoy the ingenuity and investigation problems at the end of each set of exercises which are designed to lead students to explore new concepts more deeply.

The text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 and is modeled after the Ross program at Ohio State, teaching students to "think deeply of simple things" (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with over 150 students being named semi-finalists, regional finalists, and national finalists in the prestigious Siemens Competition in Math, Science, and Engineering. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also received significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Mathworks Advisory Board.

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4–8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts, the JSMC targeted gifted students; in other districts the program was delivered to mixed groups of students. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

A concern with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The Math Explorations texts that we have written have taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills), for grades 6-8 while weaving in algebra throughout. The third volume for 8th graders allows all students to complete Algebra I. This is an integrated approach to algebra developed especially for middle school students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U. S. students on international tests.

An accompanying Teacher Edition (TE) has been written to make the textbook and its mathematical content as clear and intuitive for teachers as possible. The guide is in a three-ring binder so teachers can add or rearrange whatever they need. Every left-hand page is filled with suggestions and hints for augmenting the student text. Answers to the exercises and additional activities are also provided in the TE.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation, and Kodosky Foundation. A special thanks to the Mathworks Steering Committee, especially

Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The primary basis for the book was our JSMC curriculum, coauthored by my wife, Hiroko Warshauer, and friend and colleague, Terry McCabe. The three of us discuss every part of the book, no matter how small or insignificant it might seem. Each of us has his or her own ideas, which together I hope have made for an interesting book that will excite all young students with the joy of mathematical exploration and discovery. During the summers of 2005-2013, we have been assisted by an outstanding group of Honors Summer Math Camp alumni, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from these past summers.

Hiroko Warshauer led the team in developing this book, Math Explorations Part 1, assisted by Terry McCabe and Max Warshauer. Terry McCabe led the team in developing Math Explorations Part 2. Alex White from Texas State University provided valuable suggestions for each level of the curriculum, and took over the leadership of the effort for Math Explorations Part 3: Algebra 1.

We made numerous refinements to the curriculum in the school year 2012-2013, incorporating the 2012 revised TEKS, additional exercises, new warm-ups, and an accompanying collection of workbook handouts that provides a guide for how to teach each section. Many of these changes were inspired by and suggested by our pilot site teachers. Amy Warshauer, Alex Eusebi, and Denise Girardeau from Austin's Kealing

Middle School did a fabulous job working with our team on edits, new exercises, and the accompanying student workbook. Additional edits and proofreading was done by Michael Kellerman, who did a wonderful job making sure that the language of each section was at the appropriate grade level. Robert Perez from Brownsville developed special resources for English Language Learners, including a translation of key vocabulary into Spanish. Finally, Sam Baethge helped with putting together the glossary, proofreading, and checking and correcting the answer keys for all of the problems.

The production team was led by Namakshi P. Kant, a graduate student in mathematics education at Texas State. She did an outstanding job of laying out the book, editing, correcting problems, and in general making the book more user friendly for students and parents. Nama has a great feel for what will excite young students in mathematics, and worked tirelessly to ensure that each part of the project was done as well as we possibly could. As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help the project would not have reached its present state.

Finally, in this newest 5th edition (2016), we have continued to make edits and improvements. Genesis Dibrell, an undergraduate at Texas State, did a fabulous job of adding in these corrections, formatting the text to be consistent, and correcting any typos. Everyone at Mathworks contributed to the final product, especially Michelle Pruett, our new curriculum coordinator, and Patty Amende, who helped oversee the entire project. We could never have completed the project without the incredible help and dedication of our whole team.

Math Explorations Part 1 should work for any 6th grade student, while Math Explorations Part 2 is suitable for either an advanced 6th grade student or any 7th grade student. Finally, Math Explorations Part 3: Algebra 1 is a complete algebra course for any 8th grade student. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8 while also covering Algebra 1.

Math Explorations Part 1 was piloted by teachers and students in San Marcos, Austin, New Braunfels, and Midland. The results of these pilots have been extremely encouraging. We are seeing young 6th and 7th grade students reach (on average) 8th grade level as measured by the Orleans-Hanna test by the end of 7th grade.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with dedicated teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Michelle Pruett, Andrew Hsiau, and Patty Amende provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn't and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer

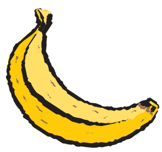
Max Warshauer
Director of Texas State Mathworks

EXPLORING INTEGERS

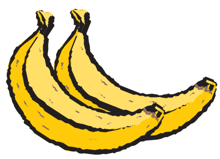
1

SECTION 1.1
BUILDING NUMBER LINES

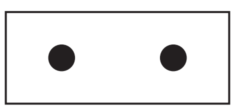
Let’s begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number one, we have in mind a picture:



Similarly, the number two describes a different quantity:

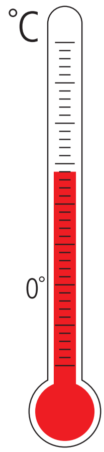


We could use a picture with dots to describe the number 2. For instance, we could draw:



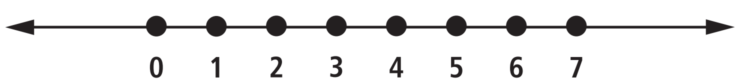
We call this way of thinking of numbers the “set model.” There are, however, other ways of representing numbers.

Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.



Next, mark off some distance to the right of the origin, and label the second point with the number 1.

Continue marking off points the same distance apart as above, and label these points with the numbers 2, 3, 4, and so on. Deciding how we label our marks is called **scaling**.



DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)

The **counting numbers** are the numbers in the following never-ending sequence

$$1, 2, 3, 4, 5, 6, 7, \ldots$$

We can also write this as

$$^{+}1, ^{+}2, ^{+}3, ^{+}4, ^{+}5, ^{+}6, ^{+}7, \ldots$$

These numbers are also called the **positive integers,** or **natural numbers**.

One interesting property of the natural numbers is that there are "infinitely many" of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0, we have a different collection of numbers that we call the **whole numbers**.

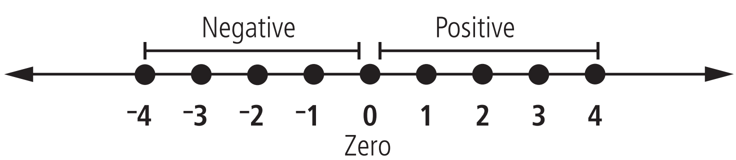
DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)

The **whole numbers** are the numbers in the following never-ending sequence:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$$

These numbers are also called the **non-negative integers**.

In order to label points to the left of the origin, we use **negative integers**: $^{-}1$, $^{-}2$, $^{-}3$,$^{-}4$, … The sign in front of the number tells us on what side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. Zero is neither positive nor negative.



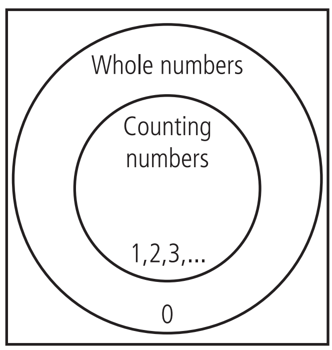
We have seen that numbers can be used in different ways. They can help us describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location can also tell us the distance the number is from the origin if we ignore the sign.

DEFINITION 1.3: INTEGERS

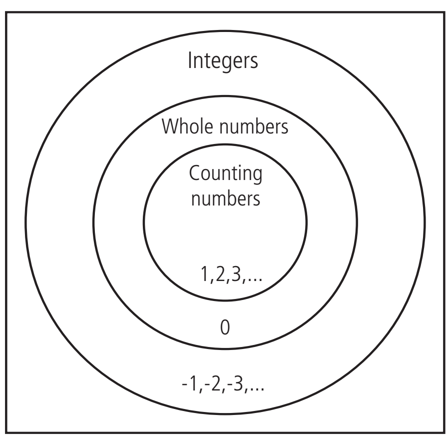
The collection of **integers** is composed of the negative integers, zero, and the positive integers:

$$\ldots, {}^-4, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, 4, \ldots$$

Notice that every counting number is a whole number, so the set of counting numbers is a subset of the set of whole numbers. One way to represent the relationship between these sets is to use a Venn Diagram. A **Venn Diagram** is simply a diagram that shows relationships between different sets in a visual way.



Notice that all of the whole numbers except 0 are inside the counting numbers. Similarly, every whole number is an integer. This is the Venn Diagram that relates the sets:



The integers that are not whole numbers are called the **negative integers**, so the integers are divided into three parts:

- The positive integers.
- Zero, which is a whole number but not a positive or negative integer.
- The negative integers.

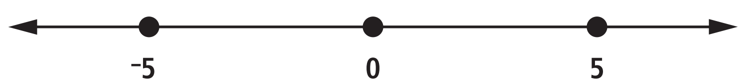
The fact that every integer is in exactly one of these three sets is called **trichotomy**, since it cuts the integers into these three parts. The prefix "tri" means three.

EXPLORATION 1: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate and label the numbers $^-10$, $^-9$,...$^-2$,$^-1$, and 1, 2, 3, …, 10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

EXERCISES

1. At the zoo, the gift shop is located at the origin of Oak Street. Its address will be labeled as 0. Going right from the origin, the seal pool is located at address 8, and the monkey habitat is at address 4. Going in the other direction from the origin, the elephant habitat is located at address $^-3$, and the lion den is at address $^-7$. Draw a number line representing Oak Street. Label each of the locations on the number line. Watch your spacing.
2. a. Copy the line below to mark off and label the integers from 0 to 5 and from 0 to $^-5$. Use a pencil to experiment with the spacing because you might need to erase.



b. Make a new number line from $^{-}10$ to 10. Use what you learned about spacing to make it accurate.

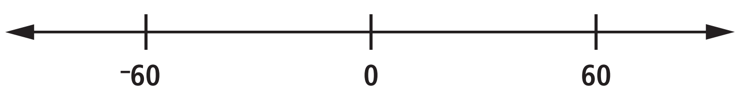
3. Write an integer for each situation below. Find the point on the number line that corresponds to the integer. (create a number line from $^{-}15$ to 15, counting by fives but leaving a mark for each integer.)

   a. Score 10 points
   b. 8° below zero
   c. A deposit of $12
   d. A gain of 7 pounds
   e. 4 ft. below sea level
   f. A debt of $11
   g. Neither positive nor negative
   h. A withdrawal of $3
   i. The opposite of $^{-}4$
   j. Put in 9 gallons

4. For each of the following integers, create a section of a number line to illustrate which integer is immediately to the left of and which integer is immediately to the right of the given integer. Be sure to label which is which.

   a. 5
   b. $^{-}3$
   c. 0
   d. $^{-}600$

5. Draw a number line like the one below to mark off the numbers with equal distances by tens from 0 to 60 and from 0 to - 60. Use a pencil to experiment with scale.



   a. Measure the distances from 0 to 30 and from 30 to 60. Are they the same?
   b. Measure the distances from 10 to 20, 30 to 40, and 40 to 50. Are they the same?
   c. Explain whether you need to rework your markings on the number line.
   d. Estimate the location of the following numbers and label each on your number line: 25, 15,$^{-}35$, $^{-}7$, $^{-}54$, 43, 18, 35, $^{-}11$, 48

6. Draw a number line so that the number ⁻600 is at the left end, and 600 is on the right end.

   a. What is the best amount to count by to ensure proper scaling?

   b. Sketch the locations of the following integers:

   25, 15, ⁻35, ⁻7, ⁻54, 43, 18, 35,⁻11, 48

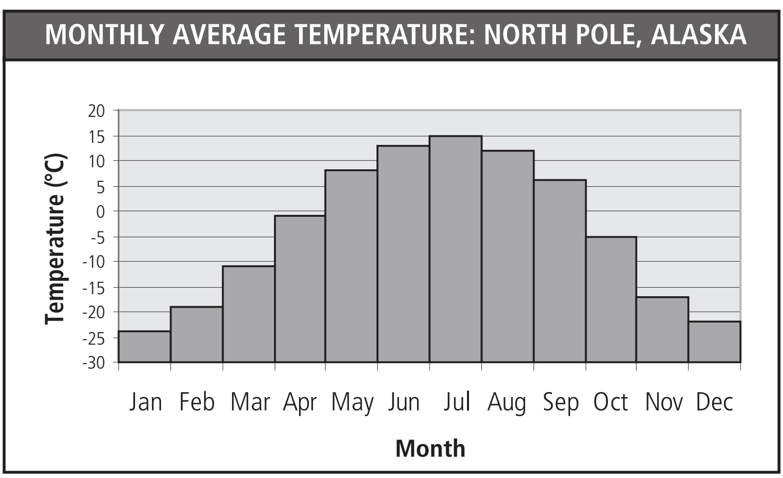
   0, 350, ⁻225, 250, ⁻450, ⁻25, ⁻599, ⁻50

7. Draw a number line. Find all the integers on your number line that are greater than 10 and less than 18. Circle the numbers you found.

8. How do you decide which number on a number line is greater? Draw 2 examples to explain your answer.

Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of North Pole, Alaska (not the actual North Pole, which is farther north). Based on the data, put the twelve months in order from coldest to warmest.



10. Nicholas visited his cousin, Marissa, in Anchorage, Alaska, where the temperature was $^{-}5$ °C. Michael visited his friend in Portland, Oregon, where the temperature was 8 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer. Remember, when we measure temperature in degrees Celsius (°C), 0 °C is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day is $^{-}7$ °C. The next day, the temperature is 5 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer.

12. One cold day, the temperature in Oslo, Norway, is $^{-}9$ °C and the temperature in Stockholm, Sweden, is $^{-}13$ °C. Which temperature is colder? How much colder?

Spiral Review:

13. Name three fractions less than $\frac{2}{3}$.

14. Write $2\frac{3}{4}$ as an improper fraction.

15. **Ingenuity:**

    a. Draw a number line, then mark and label all of the integers from 3 to 11. How many integers have you marked?

    b. Draw a number line, then mark and label all of the integers from $^{-}7$ to 5. How many integers have you marked?

    c. Suppose we drew a number line and marked all of the integers from 12 to 37. How many integers would we mark if we did this?

    d. Suppose we drew a number line and marked all of the integers from 210 to 270. How many integers would we mark if we did this?

    e. Suppose we drew a number line and marked all of the integers from $^{-}120$ to 150. How many integers would we mark if we did this?

16. **Investigation:**

    For each of the following pairs of integers, decide which integer is further to the right on the number line.

    a. 5 and 12

b. 141 and 78

c. 7 and $^{-}1$

d. $^{-}4$ and $^{-}2$

e. $^{-}7$ and $^{-}10$

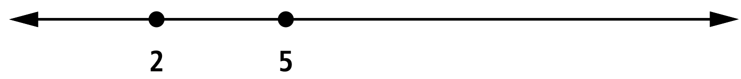
f. 9 and $^{-}55$

g. $^{-}8$ and $^{-}21$

h. $^{-}355$ and $^{-}317$

SECTION 1.2
LESS THAN AND GREATER THAN

We say that 2 is less than 5 because 2 is to the left of 5 on the number line. "Less than" means "to the left of" when comparing numbers on the number line. We use the symbol "$<$" to mean "less than." We write "2 is less than 5" as "$2 < 5$." Some people like the "less than" symbol because it keeps the numbers in the same order as they appear on the number line.



We also say that 5 is greater than 2 because 5 is to the right of 2 on the number line. "Greater than" means "to the right of" when comparing numbers on the number line. We use the symbol "$>$" to mean "greater than", so we write "5 is greater than 2" as "$5 > 2$." Just as "$=$" is the symbol for equality, "$<$" and "$>$" are symbols for **inequalities**. Inequalities are mathematical statements that relate one number as less than, less than or equal to, greater than, or greater than or equal to another number.

DEFINITION 1.4: LESS THAN AND GREATER THAN
Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are integers. We say that $\boldsymbol{x}$ is **less than** y, $\boldsymbol{x} < \boldsymbol{y}$, if $\boldsymbol{x}$ is to the left of $\boldsymbol{y}$ on the number line. We say that $\boldsymbol{x}$ is **greater than** y, $\boldsymbol{x} > \boldsymbol{y}$, if $\boldsymbol{x}$ is to the right of $\boldsymbol{y}$ on the number line.

Here x and y are called **variables**, which we will formally introduce in the next section. Variables give us a simple way to describe math objects and concepts. In this case, x and y represent two integers, and the way that we tell which is greater is to compare their positions on the number line. The two number lines below demonstrate the cases $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{x} > \boldsymbol{y}$. Can you tell which is which?

1.2

EXAMPLE 1

For each pair of integers below, locate them on a number line to determine which one is greater and which one is smaller. Express your answer as an inequality of the form $x < y$, or $x > y$ where x and y are the given integers.

a. 3 ☐ 7 c. $^{-}1$ ☐ $^{-}5$

b. $^{-}2$ ☐ 9 d. 4 ☐ $^{-}4$

SOLUTION

a. Begin by drawing a number line from $^{-}10$ to 10. Using this number line, we see that 3 is to the left of 7, so $3 < 7$.

b. We observe that $^{-}2$ is to the left of 9 on the number line, so $^{-}2 < 9$. We can also see this in a different way: We know that $^{-}2$ is to the left of 0 because $^{-}2$ is negative, and 0 is to the left of 9 because 9 is positive. Thus, $^{-}2$ must be to the left of 9, and we have $^{-}2 < 9$.

c. We notice that $^{-}5$ is to the left of $^{-}1$, so $^{-}5 < {}^{-}1$ or $^{-}1 > {}^{-}5$.

d. Because $^{-}4$ is to the left of 0, and 0 is to the left of 4, we have $^{-}4 < 4$ or alternatively, $4 > {}^{-}4$.

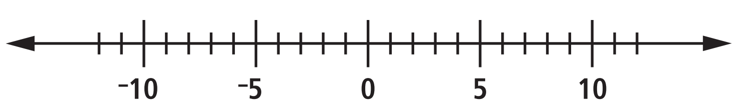
EXAMPLE 2

Use the number line to put the following integers in order from least to greatest:

2, $^{-}2$, 7, $^{-}1$, $^{-}4$, $^{-}5$, 4, 6, 3

SOLUTION

Again, we can use the number line to help us put the integers in order:



We can locate our nine given integers on the number line. You might try doing this by copying the number line above, and labeling the given numbers on your number line.

After comparing the nine numbers given, we get the following order:

$$^{-}5, ^{-}4, ^{-}2, ^{-}1, 2, 3, 4, 6, 7$$

PROBLEM 2

Put the following integers in order from least to greatest:

$$^{-}3, ^{-}1, 0, ^{-}11, 2, ^{-}6, 5$$

EXERCISES

1. Rewrite each of the following as a statement using < or >. Compare your statements to the relative locations of the two numbers on the number line. Example: $^{-}3$ is less than 8 becomes $^{-}3 < 8$.

   a. 8 is greater than 5.
   b. 3 is less than 9.
   c. $^{-}4$ is greater than $^{-}7$.
   d. $^{-}3$ is less than 2.
   e. 6 is greater than 0.
   f. $^{-}5$ is less than $^{-}3$.

2. Compare the numbers below and decide which symbol, < or >, to use between the numbers. For each, show the relationship of these numbers on a number line.

   a. $5 \square 3$
 $3 \square 5$
   b. $0 \square 2$
 $2 \square 0$
   c. $^{-}5 \square 0$
 $0 \square {}^{-}5$
   d. $^{-}4 \square {}^{-}5$
 $^{-}5 \square {}^{-}4$
   e. $^{-}6 \square 7$
 $7 \square {}^{-}6$
   f. $3 \square {}^{-}2$
 $^{-}2 \square 3$

3. Compare the numbers below and decide which symbol, < or >, to use. For each, show the relationship of these numbers on a number line.

   a. $7 \square 3$
 $^{-}7 \square {}^{-}3$
   b. $^{-}8 \square {}^{-}4$
 $8 \square 4$
   c. $5 \square 0$
 $^{-}5 \square 0$
   d. $^{-}4 \square 0$
 $4 \square 0$
   e. $^{-}2 \square 3$
 $2 \square {}^{-}3$
   f. $6 \square {}^{-}6$
 $^{-}6 \square 6$

4. Describe any patterns you see in Exercises 2 and 3. For example, you may have noticed that larger positive numbers are to the right of smaller positive numbers on a number line.

5. Compare the numbers below and decide which symbol, < or >, to use. Use your rules from Exercise 4 to help you.

a.	$6 \ \square \ 5$	c.	$^{-}7 \ \square \ 3$
	$5 \ \square \ 6$		$^{-}3 \ \square \ 7$
b.	$3 \ \square \ ^{-}1$	d.	$0 \ \square \ 4$
	$^{-}3 \ \square \ 1$		$^{-}4 \ \square \ 0$
e.	$2 \ \square \ 8$	f.	$^{-}2 \ \square \ 8$
	$^{-}2 \ \square \ ^{-}8$		$2 \ \square \ ^{-}8$

6. What are the possible values for an integer that is greater than 4 and less than 8? Circle these values on the number line.

7. Determine whether each of the following statements is true or false. Explain your answers.

   a. If an integer is greater than 6, then it is greater than $^{-}6$.

   b. If an integer is less than 4, then it is less than $^{-}4$.

8. In Green Bay, Wisconsin, the morning temperature is $^{-}5\ ^{\circ}C$. In the evening, the temperature reads $^{-}9\ ^{\circ}C$. Did the temperature rise or fall? Draw a thermometer to show how much the temperature rose or fell.

9. Mr. Canales is on a flight of stairs 34 steps above the ground. Mr. Garza has gone into the basement of the same building and is 15 steps down from ground level (let's call it the $^{-}15$th step). Who is farther from ground level? How do you know?

10. Luis has some jellybeans. He has at least 17 jellybeans and fewer than 25. How many jellybeans could he have? Give all possible answers.

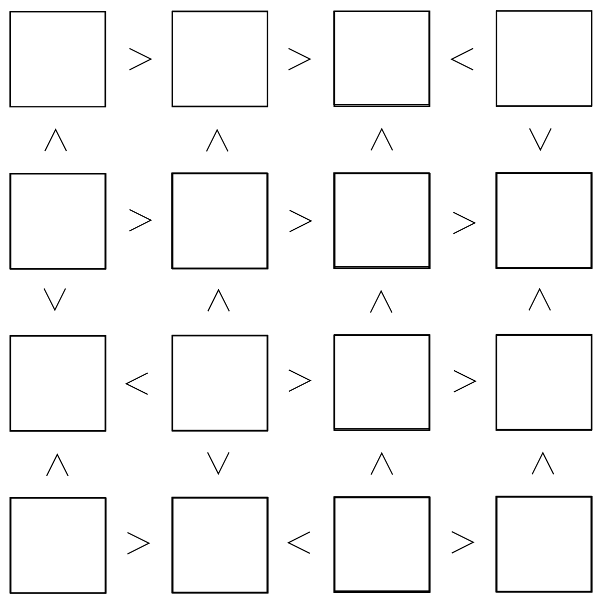
Spiral Review:

11. Jim worked 8 hours each day on Monday, Tuesday, and Wednesday last week. Sally worked 5 hours each day on Monday, Tuesday, Wednesday, and Thursday. Who worked more hours last week?

12. Jose walked from his house to school. It was a distance of 396 feet. How many yards did Jose walk from his house?

13. **Ingenuity:**

Fill in the boxes in the grid below with the numbers 1 through 4 so that each row contains all of the numbers from 1 to 4, each column contains all of the numbers from 1 to 4, and the numbers obey the inequality signs in the grid.



14. **Investigation:**

a. Ask your family for some important events and the years they occurred in your family's history, such as the year someone was born or married. Go back as far as you can, for instance the year one of your grandparents was born. Find at least 10 events and their dates.

b. Make a list of important historical events, people, or discoveries and try to find the year they occurred, lived, or appeared in history over the last three thousand years. You could also pick a time in history that you would like to visit. Find at least 10 events and their dates.

15. **Challenge Problem:**

If a, b, and c are positive integers such that $a < b < c < 10$, how many possible values are there for the three numbers?

SECTION 1.3
APPLICATIONS OF THE NUMBER LINE

When we make a number line, we place marks on it to denote different locations. We then label these marks with integers. The distance between these marks may represent more than one unit. If the numbers we want to represent on the number line are very large, we may wish to use each mark to represent ten units or perhaps even one hundred units instead of one unit. We decide the scaling by the size of the numbers we want to represent on the number line.

EXPLORATION: CONSTRUCTING A TIMELINE

In this activity, we will construct a special kind of number line called a **timeline**. Let's begin by building a timeline that goes back 100 years and forward 50 years. The first step is to draw a number line and label the origin with 0. The origin corresponds to the present year. Write the year above the line and the length of time from this year (our zero year) below the same mark. We want to label our timeline so that years that have already passed are labeled with negative numbers and years in the future are labeled with positive numbers. What scale should we use? In other words, how many years will the distance from 0 to the first mark represent? How many marks did you decide to use on your timeline? Make sure you can fit in all 150 years. Plot the special dates that you gathered from home in the Investigation from the previous section.

Next, make a timeline that goes back three thousand years and forward in time one thousand years. How long is the total span of this timeline?

CLASS EXPLORATION: EXTENDING OUR TIMELINES

Make a timeline that charts American history starting with Columbus landing in America in 1492 and continuing until the present day. Remember to think about how many years we will have to fit onto our timeline. What scale should we use?

Chart important dates on the timeline. Here are some suggestions:

- First moon landing: 1969
- Pearl Harbor: 1941
- Henry Ford builds his first car: 1893
- The beginning of the Civil War: 1860
- The end of the Civil War: 1865
- Women given the right to vote: 1920
- Declaration of Independence: 1776
- Founding of Ysleta, Texas: 1682
- Hernando Cortes leads an expedition to Mexico: 1519
- The beginning and end of interesting eras. For instance, when was the Wild West?

Next, make another timeline that charts important events in world history. Start with the completion of the first pyramid (2690 B.C.E.) and continue until the present day. How many years will we have to fit onto this timeline? What scale should we use?

Chart important or interesting dates on this timeline. Here are some suggestions:

- Trojan War: 1250 B.C.E.
- Sacking of Rome: 410 C.E.
- Marco Polo travels to China: 1300 C.E.
- Code of Hammurabi: 1790 B.C.E.
- American Revolution: 1776 C.E.
- Signing of the Magna Carta: 1215 C.E.
- Dictatorship of Caesar in Rome: 46 C.E.

EXPLORATION 2: DISTANCES ON MAIN STREET

Draw a number line, and label it Main Street.

a. Plot the following locations on Main Street: the laboratory at address 6, the zoo at address 9, the candy shop at address $^-4$ and the space observatory at address $^-7$.

b. What is the distance of each location from the post office, which is located at 0?

Using your number line for Main Street from the previous exercise, find the following distances:

a. The distance between the laboratory and the zoo.

b. The distance between the space observatory and the candy shop.

c. The distance between the zoo and the candy shop.

d. The distance between the space observatory and the laboratory.

e. The distance between the space observatory and the zoo.

f. The distance between the laboratory and the candy shop.

Discuss how you determine the distance between two locations on the number line.

EXERCISES

Use graph paper to draw a number line for each exercise.

1. In the year 540 B.C.E., Pythagoras, a Greek philosopher and mathematician, formulated a theorem still used today. The theorem is called the Pythagorean Theorem in his honor. Archimedes, another famous Greek mathematician, worked on a variety of problems. In 240 B.C.E., he developed formulas for the area and volume of a sphere. Which discovery occurred earlier in history? How did you decide?

2. The temperature in McAllen, Texas, on a hot summer day is 98 ˚F. The temperature in neighboring Mission, Texas, is 103 ˚F. Which city has the hotter temperature? How many degrees hotter is that city?

3. Cave Travis is located 325 feet below the surface of the ground and Cave Abby is located 413 feet below the surface. Which cave is farther from the surface of the ground? Explain your answer.

4. Joseph is in a cave 512 feet below sea level and directly below Aaron. Aaron is on a hill 128 feet above sea level. Who is farther from sea level? You may want to use a number line model.

5. A pilot in a helicopter hovers 463 feet above the surface of the ground. There is a person exploring in a cave 436 feet below the surface. Who is closer to the surface? Explain your choice.

6. Suppose time is measured in days, and 0 stands for today. What number would represent

   a. yesterday?
   b. tomorrow?
   c. the day after tomorrow?
   d. a week from today?
   e. a week ago?

7. On a cold day in Boston, Massachusetts, the temperature reaches a low of $^{-}4$ ˚F. The high temperature that day was 5 ˚F. What are all the other possible integer temperature readings that were reached in Boston that day?

8. What is the distance from 8 to 2? What is the distance from 2 to 8?

9. Which of the following integers is farthest from 12: 7, 9, $^{-}2$, or 23? Explain the reason for your decision by drawing a number line.

10. Find the distances between the following pairs of numbers on a number line.

    a. 7 and 2
    b. $^{-}4$ and $^{-}2$
    c. 6 and $^{-}1$
    d. $^{-}3$ and 2

Spiral Review:

11. Order the following decimals from least to greatest:

    0.5, 0.05, 5, 0.55

12. Darius ate $\frac{1}{3}$ of the pie his mother baked. Write 3 fractions equivalent to $\frac{1}{3}$.

13. **Ingenuity:**

Suppose that A, B, C, and D are points on a number line. The distances among the points A, B, C, and D are given in the following table:

	A	B	C	D
A		42	55	15
B	42		13	27
C	55	13		?
D	15	27	?	

a. If we wanted to put numbers in place of the blanks in the table, what numbers should we put?

b. Assuming that A is to the left of B, list the four points in order from left to right.

c. What is the distance between C and D?

14. **Investigation:**

Draw a number line, and mark the integers from $^{-}10$ to 10. Use your number line to answer the following questions:

a. Name two integers that are 3 units away from 0.

b. Name two integers that are 8 units away from 0.

c. Without using a number line, name two integers that are 67 units away from 0.

d. Name two integers that are 3 units away from 7.

e. Name two integers that are 4 units away from 5.

f. Without using a number line, name two integers that are 44 units away from 1960.

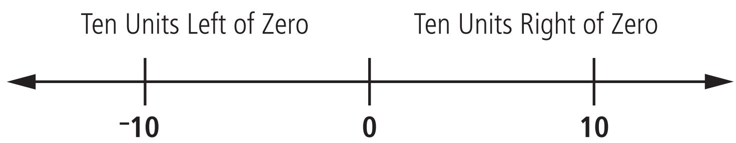
g. The integers 45 and 83 are both the same distance away from a certain integer. What is this integer?

15. **Challenge Problem:**

What day of the week was January 1, 2000?

SECTION 1.4
DISTANCE BETWEEN POINTS

We locate the numbers 10 and $^{-}10$ on the number line.



Notice that 10 and $^{-}10$ are each 10 units from 0. We have a special name for the distance of a number from 0: the **absolute value** of the number.

In mathematics, we have a special symbol to represent absolute value. For example, we write $|10|$ and read it as "absolute value of 10." We write $|^{-}10|$ and read it as "absolute value of $^{-}10$." Because 10 and $^{-}10$ are both 10 units from 0 we have the following:

The absolute value of 10 equals 10 or $|10| = 10$.

The absolute value of $^{-}10$ equals 10 or $|^{-}10| = 10$.

The absolute value of a number not only tells us its distance from the origin, it also measures the size of the number, called its **magnitude**. The positive or negative sign tells us the direction of the number relative to 0. Because 10 and $^{-}10$ are the same distance from 0, they have the same absolute value. In other words, $^{-}10 < 10$ but $|^{-}10| = |10|$.

EXPLORATION

Using the number line that you have constructed, find the distance between each pair of numbers:

a. 0 and 5	d. $^{-}1$ and $^{-}5$	g. 3 and 9
b. $^{-}0$ and $^{-}5$	e. 1 and $^{-}5$	h. $^{-}3$ and $^{-}9$
c. 1 and 5	f. $^{-}1$ and 5	i. $^{-}3$ and 9

In addition to the previous examples, you may also see $^{-}|5|$ which is read as "the negative absolute value of 5" or $^{-}|^{-}5|$ which is read as "the negative absolute value of $^{-}5$." Since $|5| = 5$ and $|^{-}5| = 5$, then we have $^{-}|5| = {}^{-}5$ and $^{-}|^{-}5| = {}^{-}5$.

EXERCISES

1. Find the absolute values of the following numbers.

   a. 0
   b. 4
   c. $^{-}4$
   d. 13
   e. $^{-}13$
   f. 42
   g. 37
   h. $^{-}58$
   i. $^{-}26$

2. Calculate the following:

   a. $|0|$
   b. $|12|$
   c. $|^{-}17|$
   d. $^{-}|17|$
   e. $^{-}|^{-}17|$
   f. $|^{-}6|$

3. What is the absolute value of the absolute value of $^{-}34$?

4. Find the distance between each number and zero:

   a. 10
   b. $^{-}10$
   c. 0

5. For each pair of numbers below, place the correct symbol < , >, or =.

   a. $|^{-}3| \;\square\; |5|$
   b. $|3| \;\square\; |5|$
   c. $|^{-}3| \;\square\; 3$
   d. $|^{-}5| \;\square\; 0$
   e. $|2| \;\square\; {}^{-}4$
   f. $3 \;\square\; |^{-}4|$
   g. $|^{-}7| \;\square\; |^{-}8|$
   h. $^{-}7 \;\square\; {}^{-}8$

6. Find the distance between 3 and 8. Did you use the number line? Can you use absolute values?

7. For each pair of integers given below, find the distance between the two integers on the number line.

   a. 3 and 5
      3 and $^{-}5$
      $^{-}3$ and 5
      $^{-}3$ and $^{-}5$
   b. 4 and 9
      $^{-}4$ and 9
      $^{-}4$ and $^{-}9$
      4 and $^{-}9$
   c. 12 and 21
      12 and $^{-}21$
      $^{-}12$ and 21
      $^{-}12$ and $^{-}21$

8. A number is distance 4 from 11. Use the number line to find this number. Is there exactly one possible number? Is there more than one? Explain using a number line.

9. Find numbers that are a distance:

   a. 2 from 5
   b. 1 from $^{-}1$
   c. 4 from 4
   d. 7 from 15
   e. 8 from 3

10. Certain school records must be kept for 4 years. If the year is 2016, identify the years of school records that must be kept.

11. **Ingenuity:**

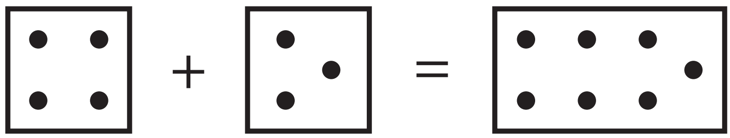
    The distance between two cities on a highway is 118 miles. If all the exits between these two cities are at least 5 miles apart, what is the largest possible number of exits between these two cities?

12. **Investigation:**

    Write a process for finding the distance between two numbers. Remember to address all possible cases: two positive numbers, two negative numbers, one of each, and at least one number equal to zero.

SECTION 1.5
ADDITION OF INTEGERS

Addition is a mathematical operation for combining integers. Visually, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:



We can also use our number line model to describe addition.

CLASS EXPLORATION: DRIVING ON THE NUMBER LINE WITH ADDITION

We can visualize adding two numbers using a car driving on the number line. We call this the Four-Step Car Model for Addition. The final location gives the sum. Let us practice how this works on a small scale. Use a number line from $^{-}15$ to 15 as your highway. You will also need a model car or something that can represent this model car.

Four-Step Car Model

Step 1: Place your car at the origin, 0, on the number line.

Step 2: If the first of the two numbers that you wish to add is positive, the car faces right, the positive direction. If the first of the two numbers is negative, the car faces left, the negative direction. Drive to the location given by the first number. Park the car.

Step 3: Next examine the second of the two numbers. If this number is positive, point the car to the right, the positive direction. If the second number is negative, point the car to the left, the negative direction.

Step 4: Because you are adding, move the car forward, the way that it is facing, the distance equal to the absolute value of the second number.

Use your car and the four-step process to compute each of the following examples. Attempt the process on your own first, and then compare your answer with the provided solution.

EXAMPLE 1

Use the Four-Step Car Model to find the sum 3 + 4, and describe how you obtain your answer using the number line.

SOLUTION

The two numbers we are adding are 3 and 4 (which we also know as $^{+}3$ and $^{+}4$).

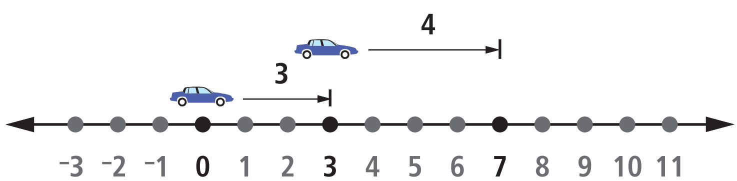
Step 1: Begin with your car at 0.

Step 2: Because the first number is positive, the car faces to the right. Drive to the location $^{+}3$. Park the car.

Step 3: Point the car to the right because the second number, $^{+}4$, is positive.

Step 4: Move the car 4 units to the right. Park the car.

You are now at location 7.



EXAMPLE 2

Find the sum $^{-}3$ + 4. How do we start the process? In which direction does your car move first and how far? Explain how you reached your solution using a car on your number line.

SOLUTION

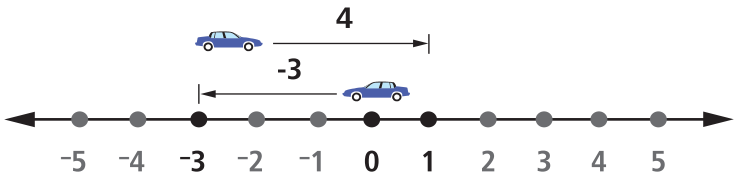
Step 1: Begin at 0.

Step 2: Because $^{-}3$ is a negative number, point the car to the left and drive 3 units. Park the car.

Step 3: Because 4, the number added to $^{-}3$, is positive, turn the car to face right.

Step 4: Move 4 units to the right, ending up at location 1. Park the car.

The result ⁻3 + 4 = 1 is demonstrated below:

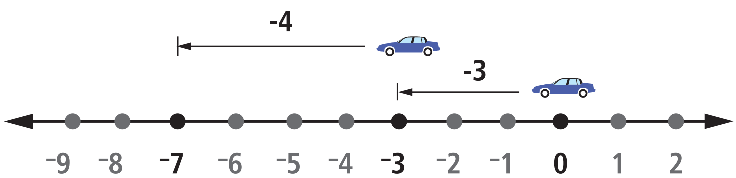


EXAMPLE 3

Use the Four-Step Car Model to find the sum ⁻3 + (⁻4), sometimes written as ⁻3 + ⁻4.

SOLUTION

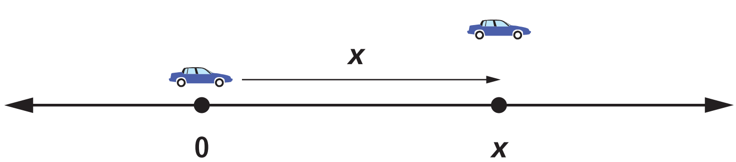
Point the car to the left and move forward 3 units. Leave the car pointing to the left because the next number is negative. Move the car forward 4 units to the location ⁻7. We have: ⁻3 + (⁻4) = ⁻7.



Remember, **when you are adding, the car always moves forward, in the direction that it is facing.** The signs of the integers tell us whether we face right, if positive, or left, if negative, before moving.

You have observed and noticed patterns in addition of integers. There are additional patterns that we can write as rules that work for any integer that we can represent by the variable x.

Think about what happens when you add 0 to a number x. You first drive $|x|$ units in the direction of the sign of x, and then you drive 0 units, remaining exactly where you were before. For example, if x is positive, then we have:



In other words, adding 0 to any number does not change its value. Because 0 has this property, we call 0 the **additive identity**.

PROPERTY 1.1: ADDITIVE IDENTITY
For any number x, $$x + 0 = 0 + x = x$$

EXAMPLE 4

For each of the numbers below, find an equivalent expression using the additive identity 0.

a. 4
c. 0
e. x
b. $^-2$
d. 9

SOLUTION

a. $4 + 0 = 4$, so $4 + 0$ and 4 are equivalent expressions

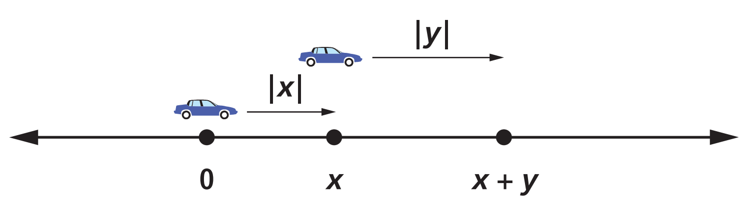
b. $^-2 + 0 = {}^-2$, so $^-2 + 0$ and $^-2$ are equivalent expressions

c. $0 + 0 = 0$, so $0 + 0$ and 0 are equivalent expressions

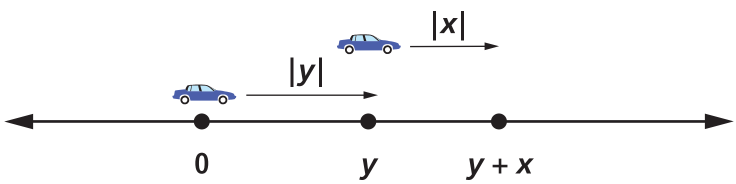
d. $0 + 9 = 9$, so $0 + 9$ and 9 are equivalent expressions

e. $x + 0 = x$ so $x + 0$ and x are equivalent expressions

Suppose that x and y represent integers. Remember, to find the sum $x + y$, we started at the point 0 on the number line, moved $|x|$ units in the direction of the sign of x, and then moved $|y|$ units in the direction of the sign of y. If x and y are positive we can model the addition as:



To find the sum $y + x$, we started at 0 and performed these two steps in the reverse order.



Reversing the order of these steps does not change the final outcome. In either case, we end up in the same place. We call this the **commutative property of addition**.

PROPERTY 1.2: COMMUTATIVE PROPERTY OF ADDITION
For any numbers x and y, $x + y = y + x$

EXAMPLE 5

Generate equivalent expressions for each side of the equalities below to show that addition is commutative.

a. $4 + 3 = 3 + 4$

b. $4 + {}^{-}6 = {}^{-}6 + 4$

c. ${}^{-}2 + {}^{-}3 = {}^{-}3 + {}^{-}2$

d. $13 + {}^{-}9 = {}^{-}9 + 13$

SOLUTION

a. $4 + 3 = 7$ $\quad 3 + 4 = 7$

b. $4 + {}^{-}6 = {}^{-}2$ $\quad {}^{-}6 + 4 = {}^{-}2$

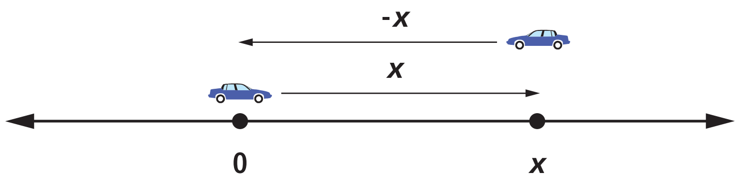
c. ${}^{-}2 + {}^{-}3 = {}^{-}5$ $\quad {}^{-}3 + {}^{-}2 = {}^{-}5$

d. $13 + {}^{-}9 = 4$ $\quad {}^{-}9 + 13 = 4$

How can we find a pair of numbers on the number line that are the same distance from zero? To do this, we need to go the same distance from 0 but in opposite directions. For example, the numbers 1 and ${}^{-}1$ are both 1 unit from 0 and $|1| = |{}^{-}1|$. Similarly, 2 and ${}^{-}2$ are the same distance from 0 and $|2| = |{}^{-}2|$. We call pairs of numbers like 2 and ${}^{-}2$ "opposites" or **additive inverses**.

What happens when you add a number to its additive inverse, for example, $x + {}^{-}x$?

Beginning at the origin, you first move a certain distance in one direction, and then move exactly the same distance in the opposite direction:



Your final position is back at the origin, 0. So, the sum of any number and its opposite is 0. We call this the **additive inverse property**:

PROPERTY 1.3: ADDITIVE INVERSE PROPERTY
For any number x, there exists a number ${}^{-}x$, called the **additive inverse** of x, such that $$x + {}^{-}x = 0$$

EXAMPLE 6

a. $2 + {}^{-}2 = 0$

b. $5 + {}^{-}5 = 0$

c. ${}^{-}9 + 9 = 0$

d. $4 + {}^{-}4 = 0$

What number is ${}^{-}({}^{-}2)$? In general, what number is ${}^{-}({}^{-}x)$?

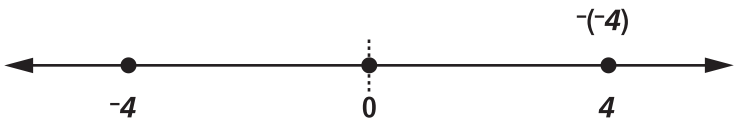
Because ${}^{-}({}^{-}x)$ is the opposite of ${}^{-}x$, we have ${}^{-}x + {}^{-}({}^{-}x) = 0$.

On the other hand, because ${}^{-}x$ is the opposite of x, we have $x + ({}^{-}x) = 0$. We can write this as ${}^{-}x + x = 0$ by the commutative property.

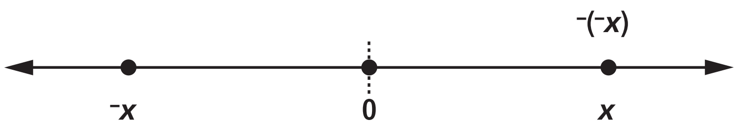
Comparing these two equations shows us that ${}^{-}({}^{-}x)$ must equal x. In short, the opposite of the opposite of x, ${}^{-}({}^{-}x)$, is the number x itself.

THEOREM 1.1: DOUBLE OPPOSITE THEOREM
For any number x, $${}^{-}({}^{-}x) = x$$

We can see this more easily in a picture. For example, if x is 4, the picture shows us that the opposite of the number 4 is $^-4$ and the opposite of $^-4$ written $^-(^-4)$ is the same as 4.



If x is $^-7$, the opposite of x is 7 and the opposite of the opposite of x is $^-7$. So, in general we have the following picture that shows us $^-(^-x) = x$:



EXAMPLE 7

Use the number line to show that the following statements are true.

a. $^-(^-10) = 10$ b. $^-(^-5) = 5$ c. $^-(^-25) = 25$ d. $^-(^-7) = 7$

Now, we look at the expressions $(x + y) + z$ and $x + (y + z)$, where x, y, and z are integers. To calculate the first expression, we first add x and y and then add z; to calculate the second expression, we first add y and z and then add x. Draw a picture using the car model for each of the expressions. Just as before, the order does not matter in determining the final value. This is called the **associative property of addition**.

PROPERTY 1.4: ASSOCIATIVE PROPERTY OF ADDITION
For any numbers x, y, and z, $$(x + y) + z = x + (y + z).$$

EXAMPLE 8

Generate equivalent expressions to each of the following using the associative property. Check each expression by evaluating the expression, first computing what is inside the parentheses.

a. $(4 + 7) + 2$ b. $^-3 + (5 + {}^-8)$

SOLUTION

a. $(4 + 7) + 2 = 4 + (7 + 2)$.

Check: $(4 + 7) + 2 = 11 + 2 = 13$

$4 + (7 + 2) = 4 + 9 = 13$

b. $(^-3 + 5) + {}^-8 = {}^-3 + (5 + {}^-8)$

Check: $^-3 + (5 + {}^-8) = {}^-3 + {}^-3 = {}^-6$

$(^-3 + 5) + {}^-8 = 2 + {}^-8 = {}^-6$

EXERCISES

1. Find the additive inverse of each number below. Generate equivalent expressions for 0 using your additive inverse and the number given. Remember that the additive inverse of $\boldsymbol{t}$ is $^-\boldsymbol{t}$ for any number $\boldsymbol{t}$.

   a. 14 b. $^-14$ c. 0 d. x e. ^-z

2. Name the property associated with each of the following:

   a. $16 + {}^-5 = {}^-5 + 16$

   b. $^-(^-1) = 1$

   c. $11 + {}^-11 = 0$

   d. $^-11 + 0 = {}^-11$

   e. $(^-2 + 4) + 1 = {}^-2 + (4 + 1)$

3. Use the indicated property and write an equivalent number expression.

   a. Use the commutative property of addition to write an equivalent number expression to $^-4 + {}^-5$

   b. Use the associative property of addition to write an equivalent number expression to $(7 + {}^-9) + 6$

For exercises 4–6, find each sum. You may use your car and number line. The sum you generate is an equivalent expression to the original.

4. a. $4 + 0$

   b. $0 + 4$

   c. $0 + {}^-6$

   d. $^-7 + 0$

5. a. $^-3 + 3$

   b. $0 + {}^-0$

   c. $4 + {}^-4$

   d. $6 + {}^-6$

6. a. $2 + 5$
   b. $^-3 + ^-5$
   c. $^-2 + ^-4$
   d. $^-1 + 5$
   e. $^-2 + 6$
   f. $^-3 + 8$
   g. $6 + ^-8$
   h. $2 + ^-4$
   i. $4 + ^-8$
   j. $5 + 2$
   k. $^-5 + ^-3$
   l. $^-4 + ^-2$
   m. $5 + ^-1$
   n. $6 + ^-2$
   o. $8 + ^-3$
   p. $^-8 + 6$
   q. $^-4 + 2$
   r. $^-8 + 4$

7. Write rules to describe any patterns you see in exercises 4–6.
   - Do you see a pattern when adding two positives?
   - adding two negatives?
   - adding a positive and a negative?
   - adding a negative and a positive?
   - Explain how your rules work using a number line.

8. For this exercise, let's pay careful attention to the order in which we add. We use parentheses to specify order. For example, $(1 + 2) + 3$ means first add 1 to 2, and then add to the result 3; $4 + (5 + 6)$ means first add 5 and 6, and then add 4 to the result. For each sum below, generate an equivalent sum using the associative property, and then calculate the result of each equivalent sum:
   a. $(2 + 3) + 4$
      $2 + (3 + 4)$
   b. $(3 + 6) + ^-5$
      $3 + (6 + ^-5)$
   c. $9 + (^-8 + 7)$
      $(9 + ^-8) + 7$
   d. $^-4 + (0 + ^-4)$
      $(^-4 + 0) + ^-4$
   e. Write a rule to describe any patterns you see in parts a.–d.

9. Predict the sign of the answer. Then find the sums. Use the number line if you need to.

   a. $7 + 6$
   b. $^{-}6 + {}^{-}4$
   c. $^{-}8 + {}^{-}3$
   d. $^{-}6 + {}^{-}6$
   e. $3 + {}^{-}6$
   f. $^{-}3 + 9$
   g. $8 + {}^{-}5$
   h. $^{-}7 + 7$

For exercises 10–16, write each problem as an addition problem and use positive and negative numbers where appropriate. Show your work on a number line.

10. a. Jeff observes that the temperature is 5˚C. If it rises 6˚C in the next three hours, what will the new temperature be?

    b. Alex observes that the temperature is $^{-}2$˚C. During the night, it falls 7˚C. What was the low temperature that night?

    c. Denise observes that the temperature is $^{-}4$˚C. If it rises 6 ˚C in the next two hours, what will the new temperature be?

    d. Carlos takes 7 steps forward then takes 5 steps back. How far is Carlos from where he started?

    e. Marissa takes 10 steps backward and then 4 steps forward. At what location does she end?

    f. Anna is looking for the hotel restaurant to meet her family for dinner. She starts at her hotel room on the 3rd floor, goes up to the 6th floor, goes down two floors and finds the restaurant. On what floor is the restaurant?

11. Jim checks the temperature and it is $^{-}8$˚C. If the temperature warms up 12˚C, what is the new temperature?

12. It was $^{-}6$ ˚C in the morning. The temperature rose 7 ˚C. What is the temperature now?

13. Carlos checks the temperature and it is 4 ˚C at 5 PM. By 10 PM, the temperature has dropped 9 ˚C. What is the new temperature?

14. Chris has $24 in his bank account. If he withdraws $30, what will his balance be?

15. The temperature in Alaska is $^{-}5$ °F on a cold winter day. If the temperature falls another 6 °F, what will the new temperature be?

16. If a football player loses 6 yards in one play, loses 2 yards in another play, and then gains 6 yards in the final play, what is the net gain or loss?

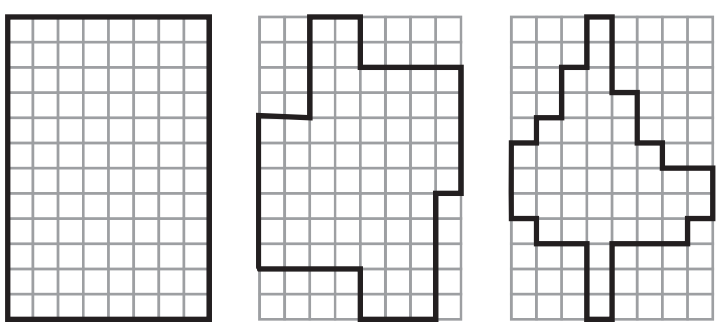
17. Eric has $250 in his bank account. Each week he earns $50 from his life guarding job and deposits it into his banking account. How much money will Eric have after 3 weeks of work?

18. Juan Carlos has $150 in his bank account. He wants to buy a new cell phone, so he makes an $89 withdrawal from his bank account. How much money will be left in his bank account after he withdraws the money?

19. Veronica has $325 in her checking account. She wants to write a check from her checking account to purchase some items. She bought a cell phone for $99, a new television for $259, and spent $25 in books for her summer reading assignment. How much money needs to be transferred from her savings account to her checking account so that she can pay for all of the items?

20. **Ingenuity:**

    In the diagrams below, assume that each of the small squares has sides of length one inch. Find the perimeter of each of the figures below. What surprising result do you notice?



21. **Investigation:**

With our car model, the car moves forward when we add. What do you think we should do when we want to subtract a number from another number? Write your best guess about how to subtract two numbers.

SECTION 1.6
SUBTRACTION OF INTEGERS

With our car model, addition involves moving the car forward in the direction indicated by the signs of the numbers we are adding. What do you think we should do when we want to subtract one number from another number? One way to model subtraction is to use the Four-Step Car Model on the number line and move the car **backward, or the opposite of forward,** for subtraction.

Try each example first, and then check your answer by comparing it to the solution given. Describe each of the steps you are using in words.

EXAMPLE 1

Use the Four-Step Car Model for subtraction to compute the difference 5 – 2, and show how to model this with a number line. Use a model car and a number line to simulate your solution. What is your final location?

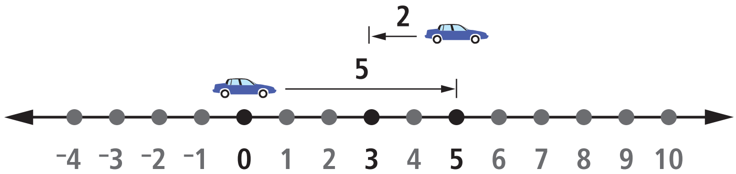
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 5 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Instead of moving forward 2 spaces, move backward 2 spaces, ending up at location 3. Remember, we move backward because we are subtracting. We can write this movement as 5 – 2 = 3.



EXAMPLE 2

Use the Four-Step Car Model to compute the difference $2 - 5$. Use a number line to show how you solved the problem.

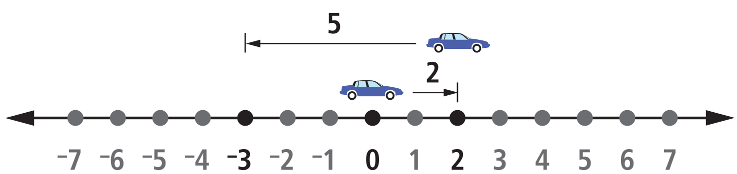
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Since we are subtracting, move backward 5 spaces, ending up at location $^{-}3$. We can write this movement as $2 - 5 = {}^{-}3$.



EXAMPLE 3

Use the Four-Step Car Model to compute the difference $^{-}7 - 3$. Use a number line to show how you solved the problem.

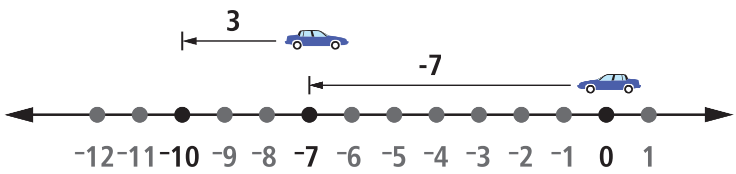
SOLUTION

Step 1: Place your car at the origin. Because the first number is negative, the car faces left.

Step 2: Move the car forward 7 units to the location given by the first number, $^{-}7$. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Now move backward 3 spaces, ending up at location $^{-}10$. Be careful! This time your car was pointing to the right, so when you back up you will move backwards to the left. We can write this movement as $^{-}7 - 3 = {}^{-}10$.



EXAMPLE 4

Use the Four-Step Car Model to compute the difference $2 - ({}^{-}5)$. Use a number line to show how you solved the problem.

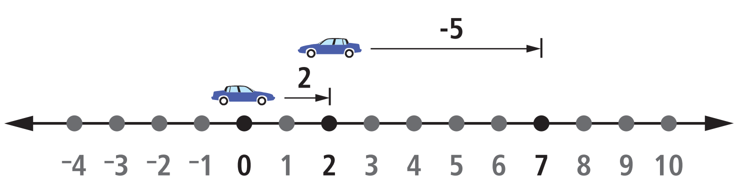
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the left because the number being subtracted is negative.

Step 4: Now move backward 5 spaces, ending up at location 7. Be careful! This time your car was pointing to the left, so when you back up you will move to the right. We can write this movement as $2 - ({}^{-}5) = 7$.



SUMMARY

In order to compute $x - y$, we use the Four-Step Car Model as follows:

Step 1: Place the car at 0, the origin. Then face the car in the direction of the sign of the first number x.

Step 2: Move the car $|x|$ units forward in the direction the car faces. Park the car.

Step 3: Next, face the car in the direction of the sign of the second number, y.

Step 4: Move the car $|y|$ units backward, the opposite direction from the way the car faces. The car is positioned on the difference $(x - y)$.

EXERCISES

Use the Four-Step Car Model with your number line to calculate each of the following exercises. Drive carefully!

1. a. $5 - 2$

   $5 + {}^-2$

   b. $6 - 3$

   $6 + {}^-3$

   c. $2 - 5$

   $2 + {}^-5$

   d. $4 - 8$

   $4 + {}^-8$

   e. ${}^-4 - 2$

   ${}^-4 + {}^-2$

   f. ${}^-7 - 4$

   ${}^-7 + {}^-4$

   g. $0 - 4$

   $0 + {}^-4$

   h. $0 - 8$

   $0 + {}^-8$

2. What patterns do you see in exercise 1? In your own words, write a rule for any patterns you observe.

3. Use the Four-Step Car Model with your number line to calculate each of the following exercises.

   a. $7 + 2$ $\quad$ $7 - ({}^-2)$

   b. $2 + 5$ $\quad$ $2 - ({}^-5)$

   c. ${}^-3 - ({}^-4)$ $\quad$ ${}^-3 + 4$

   d. ${}^-8 + 3$ $\quad$ ${}^-8 - ({}^-3)$

   e. ${}^-5 - ({}^-5)$ $\quad$ ${}^-5 + 5$

4. What patterns do you see in exercise 3? In your own words, write a rule for any patterns you observe.

5. Solve the following exercises using the Four-Step Car Model and your rules from the previous exercises.

   a. $3 - 2$

      $2 - 3$

   b. $5 - 2$

      $2 - 5$

   c. $^-3 - 8$

      $8 - (^-3)$

   d. $0 - 9$

      $9 - 0$

   e. $6 - (^-8)$

      $^-8 - 6$

   f. $5 - (^-2)$

      $^-2 - 5$

   g. $^-1 - (^-6)$

      $^-6 - (^-1)$

   h. $^-4 - (^-3)$

      $^-3 - (^-4)$

   i. What patterns do you observe in a.–h. above? Describe any patterns that you observe.

6. Calculate the following sums and differences. Use the car model as needed.

   a. $3 - 8$

      $3 + {}^-8$

      $3 - (^-8)$

   b. $7 - 4$

      $7 - (^-4)$

      $7 + {}^-4$

   c. $^-3 - 5$

      $^-3 + {}^-5$

      $^-3 - (^-5)$

   d. $^-2 - (^-6)$

      $^-2 + 6$

      $^-2 + {}^-6$

For exercises 7–11, write a subtraction expression and compute.

7. It was 8°C at 7 A.M. The temperature dropped 10°C over the next three hours. What was the temperature at 10:00 A.M.?

8. Benjamin opens a checking account at the bank in January with an initial deposit of \$50. He deposits \$20 in February, \$40 in March, \$35 in April, and \$25 in May. He needs to withdraw \$100 in June to pay for a computer camp he wants to attend this summer. What is his balance after he makes his June withdrawal?

9. The temperature in Canada is $^-4^\circ$F on a cold winter day. If the temperature falls another 3°, what will the new temperature be?

10. Whitney is going scuba diving. She will jump into the water from a deck 3 feet above the water's surface. She jumps in the water and descends 30 feet below the surface. What is Whitney's position relative to the deck?

11. Adam has a checking account. If he goes below \$50 in his checking account the bank charges his account a \$25 penalty fee. Adam has \$440 in his checking account. He withdraws \$395 from his checking account for a new surfboard. What will be the new balance in Adam's checking account? Will he have to pay the \$25 penalty fee?

12. **Ingenuity:**

    Maria has an unusual morning ritual that she performs when she goes outside to get her mail. The distance from her front door to her mailbox is 30 steps. She steps outside the front door, takes two steps forward, and then takes one step back. She then takes another two steps forward and one step back. She continues doing this until she reaches her mailbox. In all, how many steps does Maria have to take before she gets to her mailbox?

    Hint: Working this out for a 30-step trip can be quite difficult. You might want to start by seeing what happens if the mailbox is closer to the front door, perhaps 5 steps rather than 30.

13. **Investigation: Skip Counting and Scaling**

    When we initially built our number line, we used each mark to indicate one unit. The number line then corresponded to the integers 1, 2, 3, … . For larger numbers, we let the marks represent bigger lengths. So if each mark represents 5 units, then the marks correspond to the multiples of 5, and we have 5, 10, 15, 20, … using "skip counting" by 5's.

    a. Build a number line where each mark represents 3 units, skip counting by 3's. What is the 10th number to the right of 0? What is the 10^{th} number to the left of 0?

b. Build a number line where each mark represents 10 units, skip counting by 10's from $^-40$ to 40.

c. Make a table of the numbers you get when skip counting by 2's to 20, skip counting by 3's to 30, skip counting by 4's to 40, … up to skip counting by 10's to 100.

d. Now skip count by 2's, 3's, 4's, … up to 10's in the opposite direction. Make a table of the numbers you get when skip counting by 2's to $^-20$, by 3's to $^-30$, by 4's to $^-40$, etc.

e. Do you notice any patterns when you skip count? For example, the 5th number when skip counting by 3's is 15. This is the same as the 3rd number when you skip count by 5's. Does this kind of symmetry always hold?

SECTION 1.7
VARIABLES AND EXPRESSIONS

Numbers give us a way to describe different quantities. The number 5 might be 5 marbles, or it might be 5 units on a number line. When we write 5, we have in mind a definite amount or quantity. Often, however, we will want to represent an unknown quantity. To do this, we use variables. For example, Amy has some marbles, but we don't know how many she has. We could then write:

M = the number of marbles Amy has.

A **variable** is a letter or symbol that represents a quantity or number. Depending on the situation, a variable may represent an unknown quantity, or it may represent general numbers.

We use numbers, variables, and mathematical operations to form **expressions**. For example, Lisa has one more marble than Amy. We could then write "$M + 1$" to describe how many marbles Lisa has. Expressions are mathematical phrases like "$M + 1$" that we use to describe quantities mathematically. If two phrases represent the same quantity, then we say that the two expressions are equivalent. For example, "$M + 1$" and "$1 + M$" are equivalent expressions. Similarly, $3 + 6$, $6 + 3$, and 9 are **equivalent** expressions.

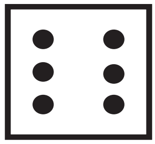
The variable M above represents the number of marbles Amy has, even though we do not know what the number is. Variables give us a convenient way to describe properties and ideas because variables can represent many different numbers.

EXAMPLE 1

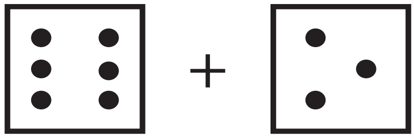
Translate "three more than six" into a mathematical expression. Represent this with a concrete set model as well as a number line. Find equivalent expressions that your models represent.

SOLUTION

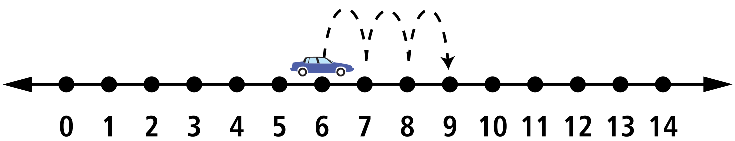
To find three more than six with a concrete set model, we begin by representing six as



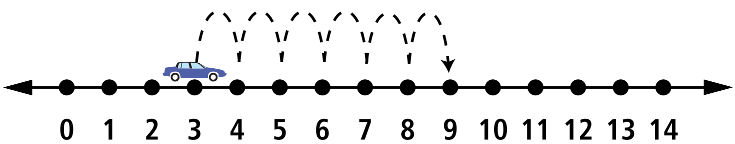
In order to find three more than 6, we must add 3. So our concrete model would be



We could represent this as 6 + 3 or equivalently as 3 + 6, or simply add and obtain 9 altogether. So the expressions 6 + 3, 3 + 6, and 9 are all equivalent, since they all have the same value. Even though these expressions are equivalent, our original expression "three more than 6" indicates that we should begin at 6, and then increase this amount by 3. So the most accurate translation would be "6 + 3." On this number line this would look like:



Use a number line to illustrate 3 + 6. Is your work similar to the representation below?



In each case, the point we end up at is 9. So each of the expressions 6 + 3, 3 + 6, and 9 are all equivalent, since they all have the same value.

EXAMPLE 2

Determine which of the expressions below are equivalent:

a. "two more than six"

b. "three less than eleven"

c. “one more than x”

Explain with concrete models, with the number line, and algebraically.

SOLUTION

With concrete models,

“two more than six”

$\Rightarrow$ •••••• + ••

$\Rightarrow$ 6 + 2

$\Rightarrow$ 8

“three less than eleven”

$\Rightarrow$ ••••••••••• – •••

$\Rightarrow$ 11 – 3

$\Rightarrow$ 8

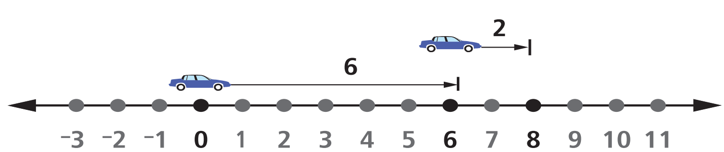
(in either case we are left with 8, so these expressions are equivalent.)

“one more than x”

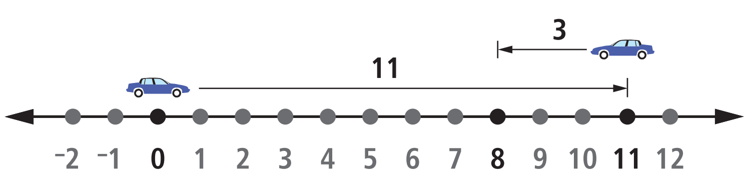
$\Rightarrow$ x + 1.

Since we do not know the value of x, this expression is not equivalent to the above.

With the number line, 6 + 2 ends up at 8



and 11 – 3 ends up at 8 as well



$x + 1$ ends up one unit to the right of x, but again, we do not know where x is. So the expressions $6 + 2$, $11 - 3$, and 8 are all equivalent. $x + 1$ is not equivalent, unless x has value 7.

PROBLEM 1

Translate "five more than two" into a mathematical expression.

EXAMPLE 3

Translate "five less than two" into a mathematical expression.

SOLUTION

"Five less than two" translates as $2 - 5$.

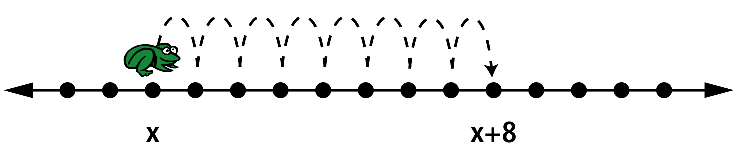
In this case, this is not the same as $5 - 2$. Do you see the difference?

EXAMPLE 4

Translate "8 more than x" into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate x.

SOLUTION

"8 more than x" can be written as $x + 8$. On the number line, we have an arbitrary point x and then go 8 units to the right as follows:



PROBLEM 2

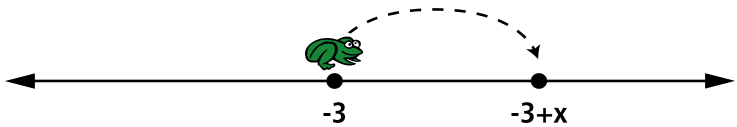
Translate "five more than x" into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate x.

EXAMPLE 5

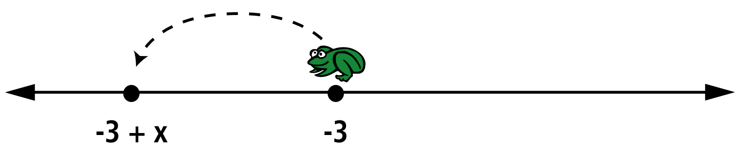
Translate "x more than $^-3$" into a mathematical expression. Illustrate this on the number line.

SOLUTION

"x more than $^-3$" can be written as $^-3 + x$. On the number line, for a positive value of x we have:



and for a negative value of x we have:



Remember that the symbol $<$ is the "less than" symbol, and the symbol $>$ is the "greater than" symbol. So, the inequality $x < 2$ says that "x is less than 2." Although we do not know what the variable x is, this inequality says that x is some number that is less than 2 and that x is to the left of 2. On the other hand, the expression "x less than 2" is written mathematically as $2 - x$.

EXERCISES

1. Translate each of the following into a mathematical expression. Determine which of these are equivalent using concrete models. Check your answer algebraically.
   a. Two more than five
   b. Five more than two
   c. Two less than five
   d. Five less than two
2. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically.
   a. Three less than eight
   b. Eight less than three
   c. Three more than eight
   d. Eight more than three

3. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically.

   a. Six more than one
   b. One more than six
   c. Six less than one
   d. One less than six

4. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically.

   a. Two more than x
   b. x less than two
   c. One less than a number three larger than x
   d. x larger than 2

5. Write the following mathematically:

   a. 7 decreased by 5
   b. 9 less than 6
   c. 11 taken away from 20
   d. 3 years younger than a 12 year old
   e. 8 inches shorter than a six-foot tall person

6. Translate each of the following into a mathematical expression:

   a. Two more than A
      Two less than A
   b. A more than two
      A less than two
   c. A more than B
      B more than A

7. Translate the following expressions or inequalities into word phrases or sentences:

   Example: $^-3 - 6$ would translate to: six less than negative three.

   a. $9 - 4$
   b. $^-3 + 6$
   c. $5 + 3$
   d. $^-4 - 3$
   e. $x + 5$
   f. $r - 6$
   g. $m < 7$
   h. $p > 0$

8. a. Using words (not mathematical symbols) explain the difference between the statement "three is less than nine" and the statement "three less than nine."

   b. Translate the two statements from part a into mathematical expressions or inequalities.

9. Translate each of the following into a mathematical expression or inequality:

   a. six less than nine
   b. eight is greater than four
   c. six is less than nine
   d. A is greater than C
   e. A is less than C
   f. A less than C

10. If B is less than 7 and greater than $^-2$, what integers could B represent?

11. Translate the inequality $^-4 < 2$ into words.

12. Translate the phrase "five greater than two" into a numerical expression.

13. Twelve is eight more than P. What integer does P represent? Show this on a number line.

14. Emily eats four cookies and has 1 cookie left. Let M = the number of cookies Emily had at the beginning. What integer does M represent?

15. Eddie and Sam like to play marbles. Eddie has x marbles. He buys 4 more marbles. Express the number of marbles Eddie has, in terms of x.

16. Eddie's friend, Miko, has y marbles. Sam's sister, Natasha, has twice as many marbles as Miko. Natasha buys four more marbles. How many marbles does Natasha have now?

Spiral Review:

17. Emma walks 210 feet to get to Central Park. Eric walks 85 yards to get to Central Park. Who has to walk further to get to Central Park? Explain.

18. Use the number line to locate the numbers $^-2$ and 1.

    a. Locate and identify two numbers greater than $^-2$.

    b. Locate and identify two numbers less than 1.

c. Locate and identify two numbers greater than $^{-}2$ and at the same time less than 1.

19. **Ingenuity:**

Choose a three-digit positive integer, and write this integer down. Then perform the following steps. After each step, write down the integer you get.

a. Multiply your integer by 3.

b. Add 2 to the integer you got in part a.

c. Multiply the result from part b. by 2.

d. Subtract 1 from the integer you got in part c.

e. Subtract your original integer from the result you got in part d.

f. Finally, multiply the result from part e. by 2.

What do you notice about your answer? (If you don't notice anything at first, it may be helpful to try this process with a different integer.) Can you explain why this happens?

20. **Investigation:**

In each of the following equations, a number has been replaced with a question mark. Determine what number needs to go in the place of the question mark so that the equation is true.

a. $5 + ? = 12$

b. $? - 4 = 2$

c. $15 - ? = 9$

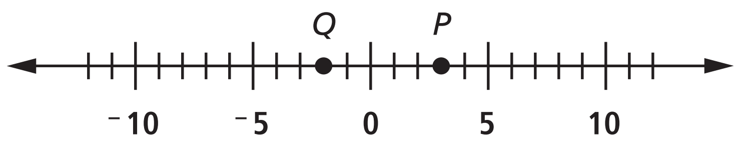
d. $3 \cdot ? = 15$

e. $4 \cdot 4 = ? - 5$

f. $2 \cdot ? - 15 = 7$

SECTION 1.8
GRAPHING ON THE COORDINATE SYSTEM

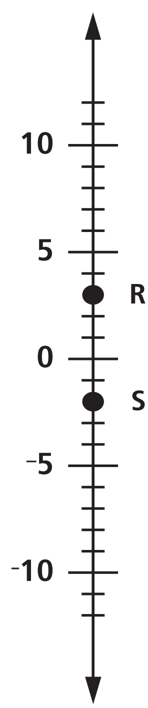
We use numbers to represent locations on number lines. To each point on the number line we associate a number, or **coordinate**, which is the location of that number.



For example, the number line above shows points ***P*** and ***Q***. To graph, or plot, a point ***P*** with coordinate 3 on the number line, we graph the point 3 units to the right of 0. Because point ***Q*** has coordinate ⁻2, we graph the point 2 units to the left of 0 on the number line.

If we draw our number line horizontally as above, then positive numbers are located to the right of 0 and negative numbers are located to the left of 0.

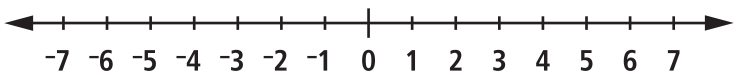
Suppose we draw our number line vertically like a thermometer. In this case, positive numbers are above zero, and negative numbers are below zero.



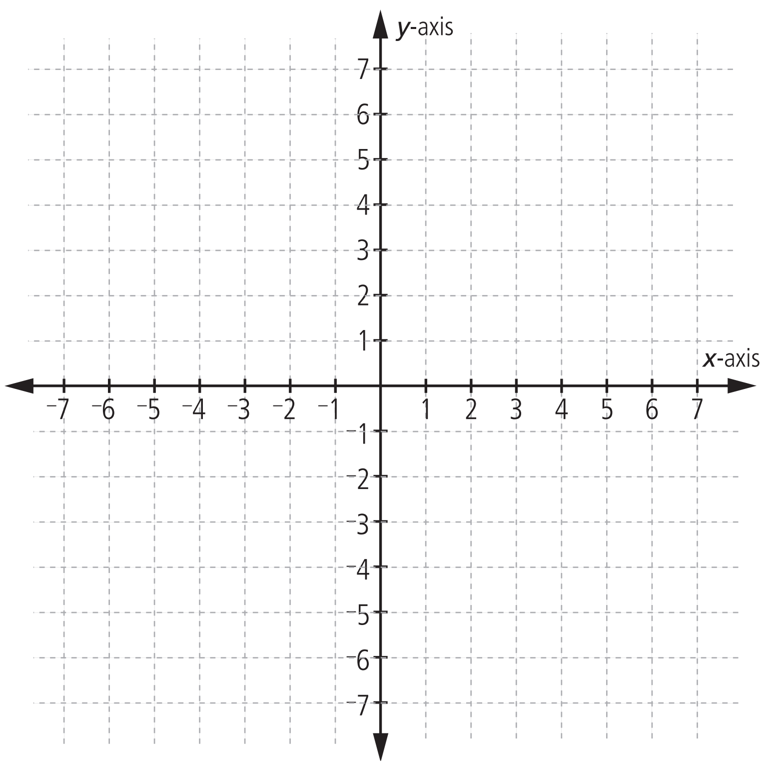
On the vertical number line, we could locate a point ***R*** with coordinate 3 by graphing the point 3 units above 0. The point ***S*** with coordinate $^{-}2$ would be located 2 units below 0. On a number line, each point corresponds to a number.

If we want to plot points on a plane, we will need to use two numbers, again called coordinates, to locate the point. A **coordinate plane** is constructed as follows:

We begin by drawing a horizontal number line and locating the zero point, which is called the **origin**:



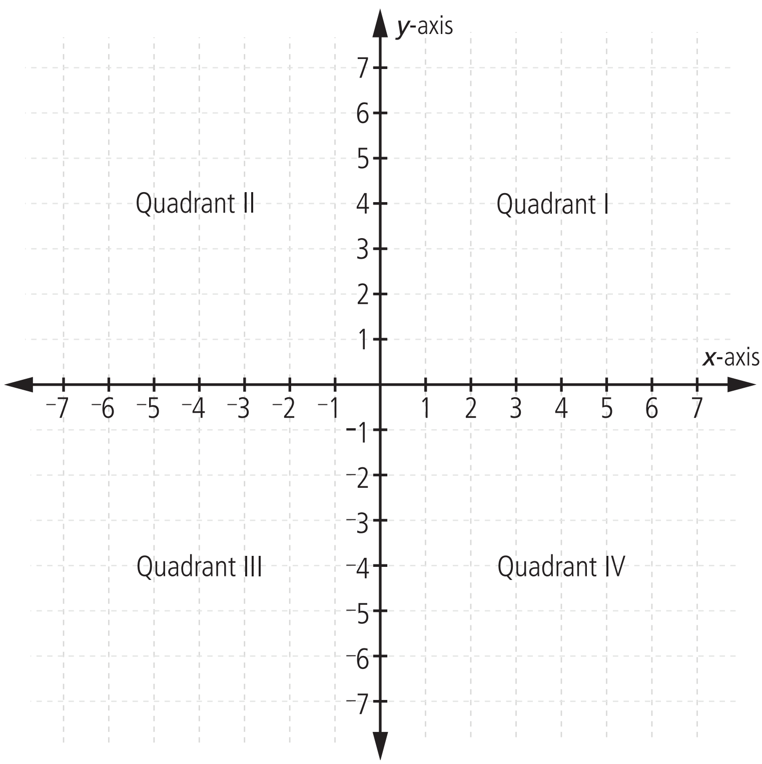
Next, draw a vertical number line through the origin of the horizontal number line so that the two zero points coincide:

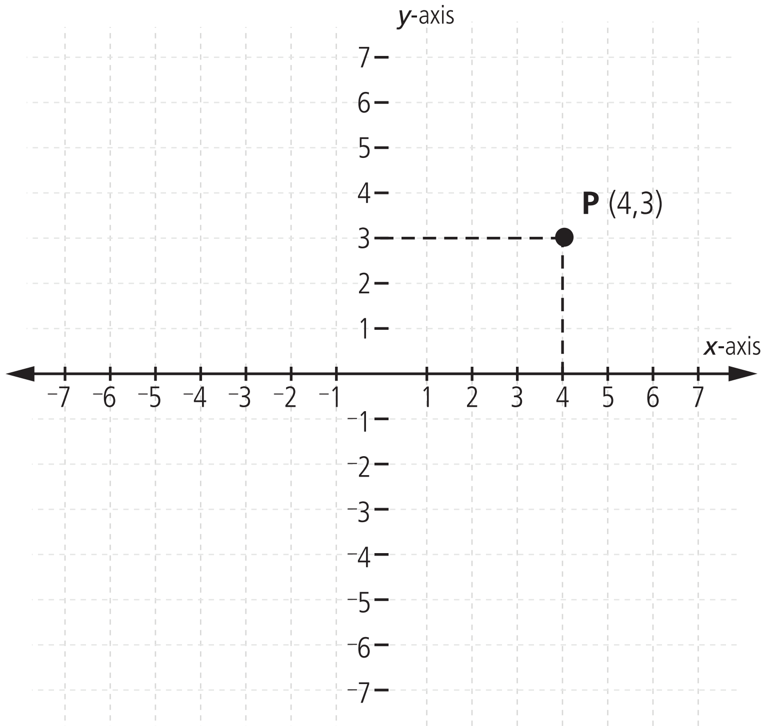


The horizontal number line is called the **horizontal axis**, or the ***x*-axis**; the vertical number line is called the **vertical axis**, or the ***y*-axis.**

Remember, the point at which the two number lines, or **axes**, meet is called the **origin** and is the zero point on both number lines.

The ***x***- and ***y***-axes divide the plane into four regions. Because there are four of them, we call each region a **quadrant**. By convention, we number the quadrants **counterclockwise**, starting with the upper-right quadrant. The axes don't belong to any quadrants but rather are their boundaries.





The coordinates for the points in the coordinate plane are always **ordered pairs** of numbers (first coordinate, second coordinate). The first coordinate is called the ***x*-coordinate**; the second is the ***y*-coordinate**.

Consider the point ***P*** on the coordinate plane above. To identify point ***P***, we begin at the origin. First we move 4 units to the right on the x-axis, then we move up 3 units. We arrive at the point ***P***. Therefore, the point ***P*** is identified as (4, 3), where 4 represents the ***x***-coordinate, and 3 represents the ***y***-coordinate.

Notice that the coordinates of the origin are (0, 0) because the origin lies at the zero point of both number lines. The points that contain integers, like (4, 3) or (5, $^{-}2$), are called **lattice points** because they fall on the cross grids, which look like a lattice.

This coordinate system with horizontal and vertical axes is called a **Cartesian coordinate system**. It is named after René Descartes, the French mathematician and philosopher who invented it.

EXERCISES

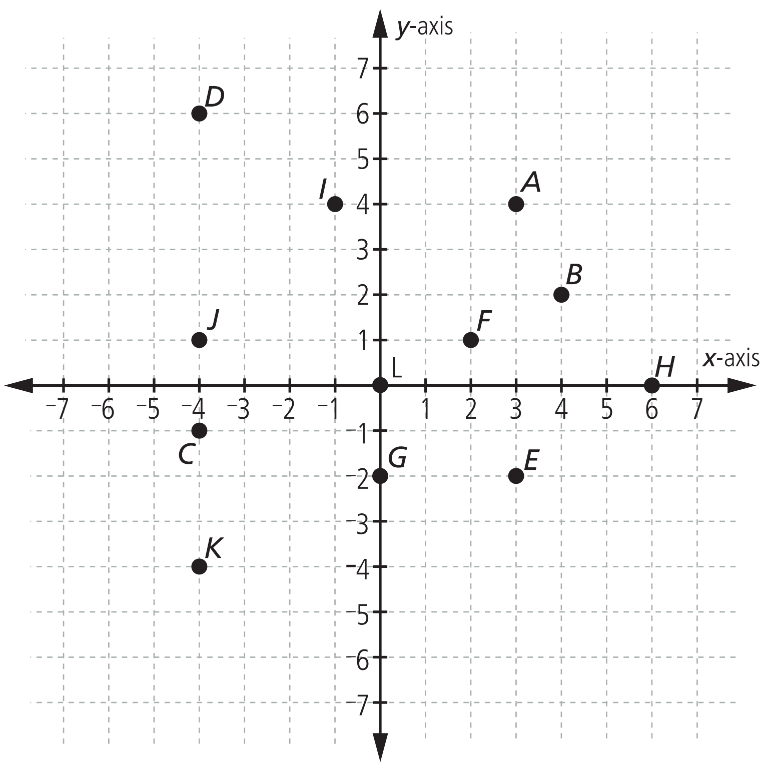
For most of these exercises, you will need to draw coordinate planes.

1. a. Draw a coordinate plane and number each axis from 5 to ⁻5.

   b. Label the following parts of the coordinate grid from part a:

x-axis	quadrant II
y-axis	quadrant III
origin	quadrant IV
quadrant I	

2. Write the coordinates for each of the points ***A*** to ***L*** shown on the coordinate plane below.



3. Create a coordinate plane on graph paper, then plot and label the following points on a coordinate plane.

M (3, 6)	*Q* ($^-4$, $^-2$)	*U* (0, 5)
N (6, 3)	*R* ($^-2$, 4)	*V* ($^-3$, 0)
O ($^-1$, 5)	*S* (2, 2)	*W* (6, 0)
P (5, $^-1$)	*T* (2, $^-2$)	*X* (0, $^-2$)

4. For each part, locate and plot 3 points on a coordinate plane that meet the following conditions:
   a. Each point has a negative *x*-coordinate and a positive *y*-coordinate.
   b. Each point has a positive *x*-coordinate and a negative *y*-coordinate.
   c. Each point has positive *x*-and *y*-coordinates.
   d. Each point has negative *x*-and *y*-coordinates.
   e. What do you notice about the points in each situation?
5. For each part, locate and plot 3 points on a coordinate plane that satisfy the following conditions:
   a. Each point has the *x*-coordinate equal to 0 and a positive *y*-coordinate.
   b. Each point has the *x*-coordinate equal to 0 and a negative *y*-coordinate.
   c. Each point has the *x*-coordinate equal to 0, but is different from part a.
   d. Each point has the *y*-coordinate equal to 0.
   e. What do you notice in each situation?
6. For each part, locate and plot 5 points on a coordinate plane that satisfy the following conditions:
   a. Each point has a y-coordinate that is double the *x*-coordinate.
   b. Each point has an *x*-coordinate that is double the *y*-coordinate.
   c. What do you notice about the points in each situation?

7. For each part, locate and plot 5 points on a coordinate plane that meet the following conditions:
   a. Each point has the y-coordinate equal to 1.
   b. Each point has the y-coordinate greater than 1.
   c. Each point has the y-coordinate less than 1.
   d. What do you notice?

8. For each part, locate and plot 5 points on a coordinate plane that meet the following conditions:
   a. Each point has the x-coordinate equal to $^-3$.
   b. Each point has the x-coordinate equal to $^-1$.
   c. Each point has x-coordinates greater than $^-3$ and less than $^-1$.
   d. What do you notice?

9. For each part, locate and plot 5 points on a coordinate plane that meet the following conditions:
   a. Each point has the same y-coordinate as the x-coordinate.
   b. Each point has the y-coordinate larger than the x-coordinate.
   c. Each point has the y-coordinate smaller than the x-coordinate.
   d. What do you notice?

Spiral Review:

10. The table below shows the time it took 4 people to row up the river. Order the rowers from fastest to slowest.

Rowing Time

Rower	Time
Robert	4 hours 23 minutes
Karen	4.5 hours
Diane	4 hours 15 minutes
David	180 minutes

11. Tamika cut 5 cucumbers for a salad. She cut each cucumber into 12 pieces. Which number sentence can be used to find ***C***, the total number of cucumber pieces Tamika cut?

a. ***C*** = 12 + 5 c. ***C*** = 12 x 5

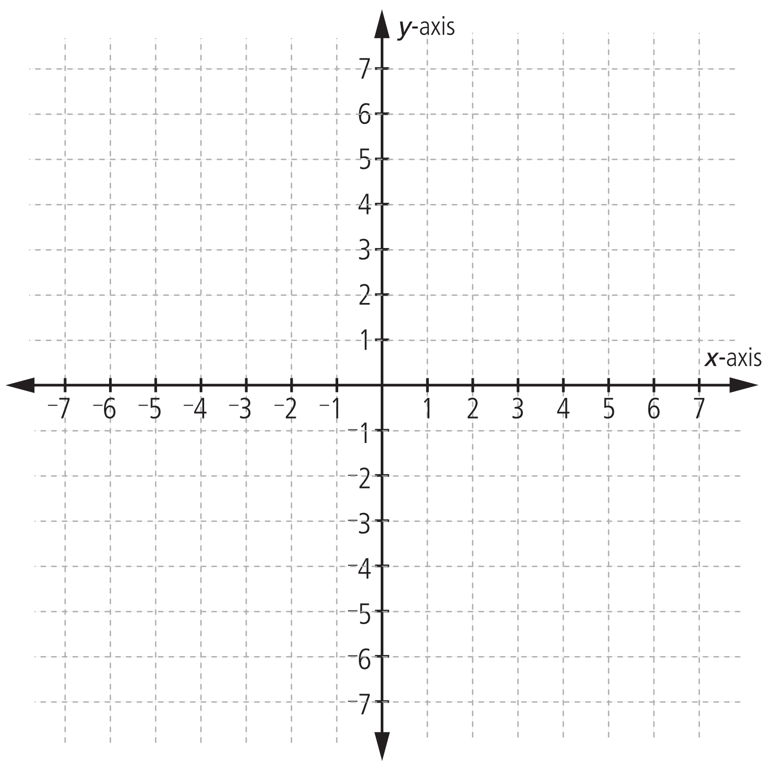
b. ***C*** = 12 – 5 d. ***C*** = 12 ÷ 5

12. **Ingenuity:**

Phyllis stands at the point (⁻2, 3) in the coordinate plane. She travels a distance of 7 units in one of the cardinal directions (up, down, left, or right), then travels 4 units in one of the cardinal directions, then travels 2 units in one of the cardinal directions, then travels 1 unit in one of the cardinal directions. After making her four moves, Phyllis is at the point (3, 6). In which direction did Phyllis travel on each move?

13. **Investigation:**

Draw a coordinate plane, and label each axis from ⁻7 to 7. Use a ruler to draw grid lines in the coordinate plane. Notice that grid lines divide the plane into 1 × 1 squares, which we call "unit squares".



a. Locate the points (1, 4), ($^-$2, 4), ($^-$2, 0), and (1, 0). Connect these four points to form a rectangle. How many unit squares lie inside this rectangle?

b. Locate the points ($^-$6, 5), (5, 5), (5, 7), and ($^-$6, 7). Connect these four points to form a rectangle. How many unit squares lie inside this rectangle?

c. Find the lengths of the sides of the rectangles you created in parts a and b. Do you see a connection between the lengths of the sides and the areas of the rectangles?

d. Locate the points (4, $^-$1), (1, $^-$4), (4, $^-$7), and (7, $^-$4). Connect these four points to form a quadrilateral. This quadrilateral happens to be a square. Suppose we want to count the number of unit squares that lie inside this square. What should we do with the unit squares that only partially lie inside the square?

e. How many unit squares lie inside the square you constructed in part d?

14. **Challenge Problem:**

Archimedes, an ant, starts at the origin in the coordinate plane. Every minute he can crawl one unit to the right or one unit up, thus increasing one of his coordinates by 1. How many different paths can Archimedes take to the point (4,3)?

SECTION 1.R
CHAPTER REVIEW

1. For each part below, draw a number line with the three given integers marked:

   a. 3, $^{-}4$, 6

   b. 20, $^{-}45$, 55

   c. $^{-}8$, 0, $^{-}3$

   d. $^{-}1214$, $^{-}1589$, $^{-}1370$

2. At 7:00 A.M., Chicago's O'Hare Airport has a temperature of $^{-}9$ °C. At 11:00 A.M. that same day, the temperature reads 4 °C. Did the temperature rise or fall? Determine by how many degrees.

3. At 4 P.M. in London, the temperature was 77 °F. At 6 A.M. the same day, the temperature was 57 °F. Did the temperature rise or fall? By how much?

4. Maggie locked all of her money inside a safe and forgot the combination. Luckily, Maggie left this note for herself: "The combination to open this safe is three positive integers. These positive integers are represented, in order, by the variables a, b, and c. c is three, $b < c$, and $a < b$." What is the combination to open the safe?

5. Homer was a Greek poet who produced several well-known epics during his lifetime. He wrote ***The Odyssey*** around 680 B.C.E. and ***The Iliad*** around 720 B.C.E. Which of these two works was written first? How many years passed between these two dates?

6. a. Which of the following numbers is farthest from 0: $^{-}2$, 3, or $^{-}5$?

   b. Which of the following numbers is closest to 14: 28, $^{-}1$, or $^{-}5$?

7. Compare the pairs of numbers below and place the appropriate symbols between them. Use < or >.

   a. 457 ☐ 81

   b. $^{-}23$ ☐ $^{-}32$

   c. 191 ☐ $^{-}3$

   d. $|^{-}11|$ ☐ 8

   e. $|^{-}23|$ ☐ $|^{-}32|$

   f. $-(^{-}5)$ ☐ $|^{-}6|$

8. Draw a number line for each problem and find the distance between the following pairs of numbers.

   Example: The distance between ⁻2 and 9 is 11.

   a. 0 and ⁻1
   b. ⁻8 and 0
   c. 0 and ⁻14
   d. 14 and 0
   e. ⁻12 and 16
   f. ⁻4 and 24
   g. ⁻20 and 10
   h. ⁻20 and ⁻40
   i. 20 and ⁻20
   j. 25 and ⁻15
   k. ⁻25 and 25
   l. 17 and 17

9. Answer the following questions using the table of information for each planet in the solar system. Justify your answers.

Studying the Planets

Planet	Distance from Sun (million km)	Average temperature (°C)
Mercury	57.9	427
Venus	108.2	464
Earth	149.6	15
Mars	227.9	⁻63
Jupiter	778.3	⁻144
Saturn	1427	⁻176
Uranus	2869.6	⁻215
Neptune	4496.6	⁻215

   a. Which planet is the hottest? Which is/are the coldest?
   b. Which planet is closest to the sun? Which is the farthest from the sun?
   c. Is there a relationship between a planet's distance from the sun and its average temperature? Explain.

10. Calculate the sums and differences. Use the car model as needed.

a. $^{-}5 + {}^{-}9$

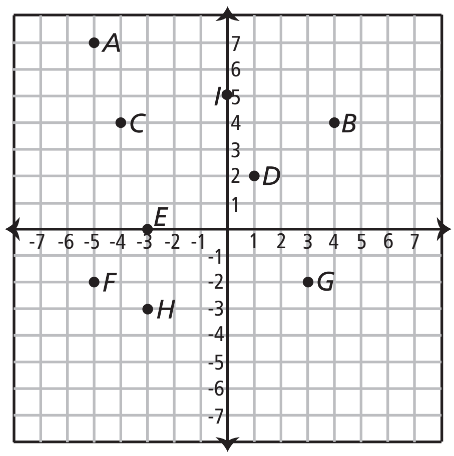
c. $^{-}9 - 5$

b. $^{-}5 - (^{-}9)$

d. $^{-}9 - (^{-}5)$

11. Ryu has 58 cents and wants to buy a toy that costs 73 cents. How much more money does he need?

12. Write the coordinates for each point shown on the coordinate plane below.



13. Draw a coordinate plane and plot the following points:

J	(1, 2)	*M*	(3, 4)	*P*	(5, 6)
K	(2, $^{-}4$)	*N*	(4, $^{-}6$)	*Q*	(7, 8)
L	($^{-}6$, $^{-}2$)	*O*	($^{-}4$, $^{-}4$)	*R*	($^{-}2$, $^{-}6$)

14. Plot the following points as specified on the coordinate plane:

a. A point on the x-axis. Identify its ordered pair.

b. A point on the y-axis. Identify its ordered pair.

c. A point in the first quadrant.

d. A point in the second quadrant.

e. A point in the third quadrant.

f. A point in the fourth quadrant.

15. Plot the following points that satisfy the stated characteristics. Identify the ordered pairs.

    a. Two points with the first coordinate 0.

    b. Two points with the second coordinate 0.

    c. Three points with the first coordinate $^-2$.

    d. Three points with the second coordinate $^-2$.

16. Plot the following points, and identify the ordered pairs that satisfy the following properties:

    a. Three points with the first coordinate positive and the second coordinate negative.

    b. Three points with the first coordinate negative and the second coordinate positive.

17. Marci is snorkeling in the San Marcos River 5 feet below the surface. She dives 3 feet deeper. How many feet below the surface is Marci now?

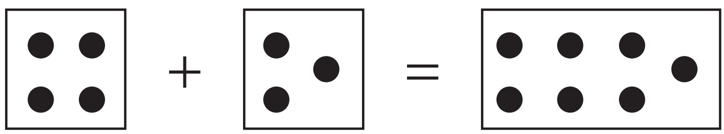
18. The temperature in Anchorage, Alaska, is $^-15$ °F. The temperature rises 12 °F during the day. What is the new temperature? (It may help to draw a vertical number line.)

19. The temperature in Seattle, Washington, is 13 °C. The temperature then dips 15 °C. What is the temperature now? (It may help to draw a vertical number line.)

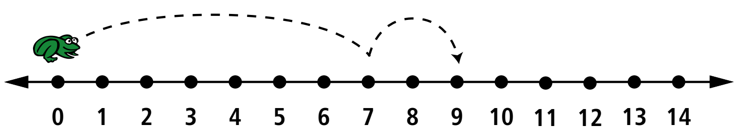
2 MULTIPLYING AND DIVIDING

SECTION 2.1
SKIP COUNTING WITH INTEGERS

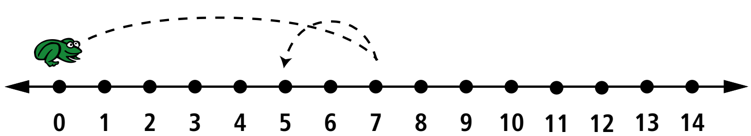
Addition is a mathematical operation for combining integers. Pictorially, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:



We also use our number line model to describe addition and subtraction. For example, we model the addition problem 7 + 2 on the number line, with cars or with frog leaps.



How might 7 – 2 look on the number line?
Does it look like this?

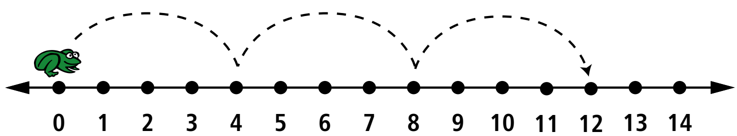


We know that skip counting by 3's generates the list 3, 6, 9, 12, 15, 18, 21, 24, 27, …, which continues indefinitely. Skip counting provides a model for multiplication that we can represent on a number line.

On the number line, the frog's jumps correspond to the numbers we are adding. In order to multiply, we can think of a frog that jumps along the number line. For example, when you multiply $4 \cdot 3$,

- the result of the multiplication is called the **product**
- the first **factor** indicates which direction the frog should face and the length of each jump
- the second factor indicates the number of jumps

The picture below models the multiplication $4 \cdot 3 = 12$. Notice the frog is facing in the positive direction because the first factor, 4, is positive. The frog takes 3 jumps, and each jump is 4 units long. The final location is the product 12.



EXPLORATION 1: FROG JUMP MULTIPLICATION

Copy and fill Table 2.1a in which each jump is 4 units long.

Table 2.1a

Length of Jump (factor)	Number of Jumps (factor)	Frog's Location (product)
4	0	0
4	1	4
4	2	8
4	3	
4	4	
4	5	
4	6	
4	10	
4	20	
4	n	

Does this table look familiar? You might recognize these numbers from a multiplication table of 4's where the pattern is $4 \cdot 1 = 4$; $4 \cdot 2 = 8$; $4 \cdot 3 = 12$; $4 \cdot 4 = 16$. You can think of (4) (3) as (4 units per jump) (3 jumps) = 12 units.

Copy and fill the skip counting Table 2.1b as you did in Table 2.1a, but this time use jumps of length 7.

Table 2.1b

Length of Jump	Number of Jumps	Frog's Location
7	0	0
7	1	7
7	2	
7	3	
7	4	
7	5	
7	6	
7	10	
7	20	
7	n	

Multiplication of 7 and 12, often written as 7×12, can also be written as $7 \cdot 12$, 7*12, or (7)(12).

PROBLEM 1

Compute the following products. Explain how you arrive at your answer.

a. (8)(6)

b. (8)(5)

c. (8)(3)

d. (8)(7)

Let's summarize the frog model:

> The first factor tells us the length of each jump, and the second factor tells us the number of jumps.

Using the frog model, complete the following table using the length 15.

Table 2.1c

Directed Length of Jump	Number of Jumps	Frog's Location
15	0	0
15	1	15
15	2	
15	3	
15	4	
15	6	
15	10	
15	20	
15	100	
15	n	

Use this table to compute the following products:

a. (15)(7)

b. (15)(30)

c. (15)(12)

d. (15)(1000)

EXPLORATION 2

In McAllen, Texas, the temperature rises an average of 3°F per hour over a twelve hour period from 1 a.m. to 1 p.m. The temperature at 7 a.m. is 72°F. Let x be the number of hours after 7 a.m.

a. Make a table that shows a relationship between the time and temperature over the twelve hours. (Hint: make a row for time, a row for the number of hours since 1 a.m. and a row for the temperature).

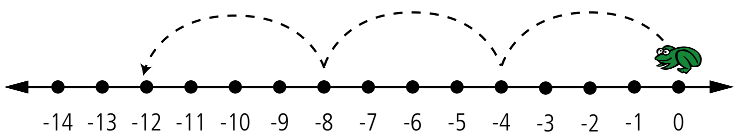
b. What was the temperature at 3 a.m.?

c. What x-value corresponds to 12 p.m.? What is the temperature at 12 p.m.? How many hours will it take for the temperature to rise 12 degrees? What time will that be?

d. When is the temperature 63°F?

e. Is it possible to use multiplication to help determine the temperature in parts b and c? If so, explain how.

You learned how to add positive and negative integers in the first chapter. Is there a way to think about multiplying a negative integer times a positive integer? You can use the frog model to multiply $^{-}4 \cdot 3$, as shown below.



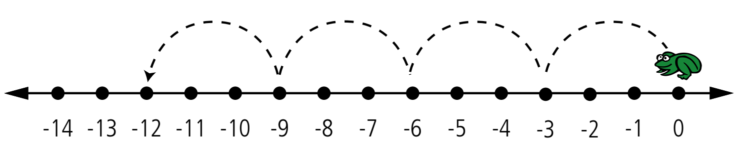
Copy and fill the skip counting Table 2.1d as you did in Table 2.1a, but this time use jumps of directed length $^{-}4$.

Table 2.1d

Directed Length of Jump	Number of Jumps	Frog's Location
$^{-}4$	0	0
$^{-}4$	1	$^{-}4$
$^{-}4$	2	
$^{-}4$	3	
$^{-}4$	4	
$^{-}4$	5	
$^{-}4$	6	
$^{-}4$	10	
$^{-}4$	20	
$^{-}4$	n	

Using the pattern demonstrated in this table, compute the product $^{-}3 \cdot 4$.

The picture below models the product $^{-}3 \cdot 4$. The first factor tells us which direction the frog should face and the length of each jump; the second factor tells us the number of jumps.



The frog is facing left because we are modeling $^{-}3$ units per jump.

Use the number line to compute the following products:

a. $(^{-}3)\,(6)$

b. $(^{-}3)\,(5)$

c. $(^{-}3)\,(3)$

d. $(^{-}3)\,(1)$

How can we make sense of the product $(3)(^{-}4)$? This is the first example where the second factor is negative.

The first number, 3 or $^{+}3$, gives the length of each jump, and the direction the frog is facing. Because the number is positive, the frog faces right.

The second factor gives the number of jumps. What do we mean by the number $^{-}4$ as the number of jumps? If we think of the jumps taking place at equal time intervals, we can imagine the frog jumping along a line.

We pick one location, call it 0, and name the time as the "0 jump." When the frog takes its first jump, jump 1, the frog lands at location 3. When the frog takes its second jump, jump 2, the frog lands at location 6.

Let's go back to the 0 location and ask where the frog was on the jump before it arrived at 0. We call this jump $^{-}1$. Because the frog jumps 3 units to the right every jump, the frog must have been at location $^{-}3$, which is 3 units to the left of 0. Two jumps before reaching 0, the frog was at location $^{-}6$. We can now fill the table below.

Table 2.1e

Directed Length of Jump	Number of Jumps	Frog's Location
3	$^{-}6$	
3	$^{-}5$	
3	$^{-}4$	
3	$^{-}3$	
3	$^{-}2$	
3	$^{-}1$	
3	0	0
3	1	3
3	2	6
3	3	

It is now possible to answer the earlier question. What do we mean by ⁻4 jumps? This means we jump backward in time, or simply jump backward.

Use the number line to compute the following products. Verify that your answers agree with the table.

a. (3)(⁻6)

b. (3)(⁻5)

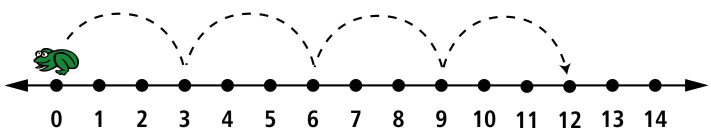
c. (3)(⁻3)

d. (3)(⁻1)

Let's summarize the frog model:

- The first factor tells us which direction the frog should face and the length of each jump.
- The second factor tells us the number of jumps and the direction of the jump. When the second factor is positive, the frog jumps forward; when the second factor is negative, the frog jumps backward.

Using the frog model, compute the product (⁻3) (⁻4). The directed length of each jump is ⁻3. Determine what happens when the frog jumps backward in time.



Copy and fill the following table, starting at the bottom and working up.

Table 2.1f

Directed Length of Jump	Number of Jumps	Frog's Location
⁻3	⁻6	
⁻3	⁻5	
⁻3	⁻4	
⁻3	⁻3	
⁻3	⁻2	
⁻3	⁻1	
⁻3	0	0
⁻3	1	⁻3
⁻3	2	⁻6
⁻3	3	

Use the table to compute the following products:

a. $(^-3)\ (^-6)$

b. $(^-3)\ (^-5)$

c. $(^-3)\ (^-3)$

d. $(^-3)\ (^-1)$

EXERCISES

1. Use the frog model on the number line to compute the following products. As you multiply, visualize the process to verify the accuracy of the products.

   a. $^-4 \cdot 2$

   b. $7 \cdot {}^-5$

   c. $(6)\ (^-7)$

   d. $^-6 \cdot 3$

   e. $11 \cdot {}^-4$

   f. $(^-5)\ (8)$

   g. $^-13 \cdot 3$

   h. $40 \cdot {}^-2$

   i. $(10)\ (^-12)$

   j. What patterns do you observe? In your own words, describe the patterns using complete sentences.

2. Use the number line to demonstrate $(^-2)(^-3)$. Do the same for $(^-4)(^-6)$. What patterns do you see? In your own words, describe the pattern using complete sentences.

3. Use the number line frog model to compute the following products.

   a. $(5)(7)$

      $(5)(^-7)$

      $(^-5)(7)$

      $(^-5)(^-7)$

   b. $(^-6)(8)$

      $(^-6)(^-8)$

      $(6)(^-8)$

      $(6)(8)$

4. We know from experience that when we multiply a positive number by another positive number we will always get a positive number. Using the patterns you discovered in exercises 1-3:

   a. Write a rule for the product of a negative number and a positive number.

   b. Write a rule for the product of a positive number and a negative number.

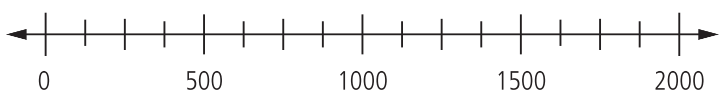
   c. Write a rule for the product of two negative numbers.

5. Evaluate the following products:

   a. $(16)\ (^{-}14)$
   b. $(^{-}24)\ 32$
   c. $^{-}13 \cdot 25$
   d. $(^{-}115)\ (^{-}4)$
   e. $223\ (^{-}13)$
   f. $(^{-}125)\ (^{-}7)$

For exercise 6 - 12, solve each word problem. Write your answers in a complete sentence.

6. Pedro has 17 bags of tater tots that have approximately 80 tots in each bag. Place a point on the number line at the approximate location that indicates about how many tater tots Pedro has. Explain how you arrived at your estimate.



7. Johnny is opening a checking account today. He deposits 8 checks he got for his birthday, each of \$25. How much money will he have in his account after making these deposits? Show two different ways to solve this problem.

8. Andrew is buying tickets to the movies for himself and six friends. Each ticket costs \$9. How much does he have to pay for the tickets?

9. On a November day, a cold front blew into town. The temperature was 70˚F before the temperature dropped by an average of 4˚F each hour. What was the temperature after 7 hours?

   a. Create a table to solve this problem.
   b. Write a numerical expression you can use to solve this problem using addition and multiplication.

10. One morning in San Antonio, the temperature rises for five hours, from 5:00 AM to 10:00 AM. The temperature rises an average of 3˚F per hour. The temperature at 10:00 AM is 88˚F. What was the temperature at 5:00 AM?

    a. Create a table to solve this problem.
    b. Write an expression you can use to solve this problem using addition and multiplication.

11. On a cold day in Roanoke, Virginia, the temperature at 6:00 AM is 10°F. The temperature increases 4°F per hour for the next six hours. What will the temperature be at 12:00 PM?

    a. Create a table to solve this problem.

    b. Write an expression you can use to solve this problem using addition and multiplication.

12. A bee flies by Tommy traveling east at 6 feet per second. Assuming the bee is traveling in a straight line:

    a. How far is the bee from Tommy after 3 seconds?

    b. How many seconds will it take the bee to travel 60 feet?

Spiral Review:

13. The low temperature on Saturday was 5 degrees below zero Celsius. On Sunday, the low temperature was 10 degrees below zero Celsius. Represent each day's low temperature as an integer. On which day was it colder?

14. Jack and Jill collect stamps. Jack has 46 stamps in his collection. He has 15 more stamps than Jill does. Which of the following equations can be used to find ***s***, the number of stamps Jill has in her collection?

    a. $s = 46 + 15$

    b. $s = 46 - 15$

    c. $s = 54(15)$

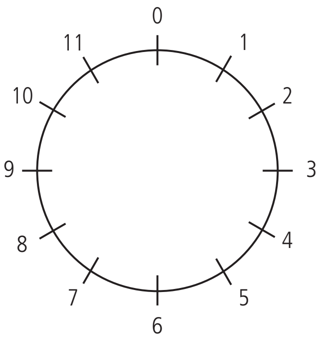
    d. $s = 54 \div 15$

15 **Ingenuity:**

Euclid Middle School put on a play to raise money for the school's drama club. The school sold adult tickets for $5 each and child tickets for $3 each. The total value of the tickets sold was $560. If the school sold exactly as many child tickets as it sold adult tickets, how many tickets did it sell overall?

16. **Investigation:**

Draw a circle and label twelve points on the circle from 0 to 11 as shown below:



Suppose a frog begins at the point 0 and makes several jumps, jumping the same number of units clockwise each time. For example, the frog might make 3 jumps, jumping 2 units each time. If it did this, it would land at the points 2, 4, and, 6.

a. Suppose the frog started at 0 and made 2 jumps, jumping 4 units clockwise each time. Where would it finally land?

b. Suppose the frog started at 0 and made 3 jumps, jumping 5 units clockwise each time. Where would it finally land?

c. Suppose the frog started at 0 and made 5 jumps, jumping 7 units clockwise each time. Where would it finally land?

d. Suppose the frog started at 0 and made 7 jumps, jumping 11 units clockwise each time. Where would it finally land?

e. Suppose we instead marked 60 points on a circle, and labeled them 0 through 59. If the frog started at 0 and made 16 jumps, jumping 59 units clockwise each time, where would it finally land?

2.2

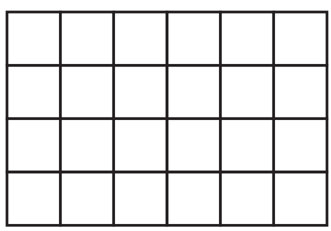
SECTION 2.2
AREA MODEL FOR MULTIPLICATION

In the previous section, we explored multiplication using the frog model of skip counting on the number line, or repeated addition. This is also called the **linear model** for multiplication. In addition to the linear model, we can represent multiplication as area.

EXPLORATION 1

A bird refuge is in the shape of a rectangle 4 miles long and 6 miles wide. Draw a visual representation of this refuge on grid paper, using 1 centimeter = 1 mile, and use it to determine the area of this rectangle. Explain how you use the grid to compute the area. Multiply 4 miles by 6 miles using the traditional algorithm only after you have an answer using the visual representation.

To multiply 4 by 6, consider the picture of the rectangle below. The area of a rectangle is the number of unit or 1×1 squares that it takes to cover the figure with no overlaps and no gaps. What is the area of the rectangle below, assuming that each square in the grid has area 1 square unit?



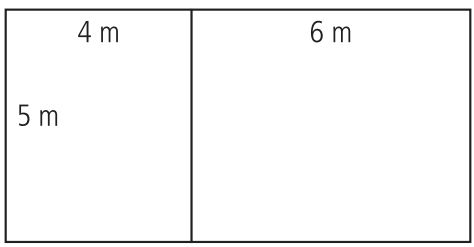
The area is 24 square units. The dimensions of this rectangle, which is 4 units wide and 6 units long, are 4 units in width and 6 units in length. Sometimes the dimensions are abbreviated and used to describe the rectangle: the rectangle is 4 units wide and 6 units long or there are 4 rows and 6 columns. The area can be computed by summing the areas of the columns: $4 + 4 + 4 + 4 + 4 + 4 = 4 \cdot 6 = 24$. We can also think of this area as the sum of the area of the rows: $6 + 6 + 6 + 6 = 6 \cdot 4 = 24$. The rectangle is called a 4 by 6, or a 6 by 4, rectangle because the area is computed as the product $4 \cdot 6 = 6 \cdot 4 = 24$. We call this relationship where the order in which we multiply does not affect the product the **commutative property of multiplication**.

PROPERTY 2.1: COMMUTATIVE PROPERTY OF MULTIPLICATION
For any numbers x and y, $$x \cdot y = y \cdot x.$$

Remember, when we add 4 to itself 6 times, it is the same as when a frog jumps 6 times on a number line, with each jump 4 units long. The visual representation of area above is another model that describes multiplication.

EXAMPLE 1

The Elliots are constructing a small building that is one room wide and two rooms long. Each room is 5 meters wide. The front room is 4 meters long, and the back room is 6 meters long. What is the floor space of each room? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? The floor plan below shows the layout:



SOLUTION

The area of the room on the left is calculated by (5m)(4m) = 20 square meters, or 20 sq. m. The area of the room on the right is (5m)(6m) = 30 square meters. The area of each room is called the partial product. Adding the partial products gives you the total area. The total area is the sum of the areas of the two rooms:

$$20 \text{ square meters} + 30 \text{ square meters} = 50 \text{ square meters.}$$

Another way to compute the total area is to consider the larger rectangle and its width and length:

$$(5 \text{ meters})(4 \text{ meters} + 6 \text{ meters}) = (5 \text{ meters})(10 \text{ meters}) = 50 \text{ square meters.}$$

Notice that $5(4 + 6) = (5 \cdot 4) + (5 \cdot 6)$ and gives the same total area of 50 square meters.

PROBLEM 1

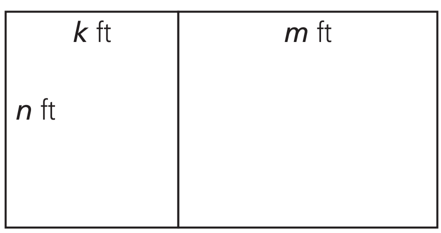
The Redfield family is constructing a small building that is one room wide and two rooms long. Each room is 8 feet wide. The front room is 9 feet long, and the back room is 12 feet long. What is the floor space of each room? What is the length of the building? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? Create the floor plan that shows this situation.

PROBLEM 2

The Gonzalez family is constructing a small building that is one room wide and two rooms long. Each room is 11 feet wide. The front room is 8 feet long, and the back room is 13 feet long. What is the floor space of each room? What is the length of the building? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? Create the floor plan that shows this situation.

EXAMPLE 2

Suppose the dimensions of the Elliots' building have not been decided yet. We need a formula for the areas. Call the width of the building n feet and the lengths of rooms 1 and 2, k and m feet respectively. Find the area of each room and the building's total area.



SOLUTION

The area of room 1 is (n ft)(k ft) = $n \cdot k$ square ft. We often abbreviate square feet with sq. ft.

The area of room 2 is (n ft)(m ft) = $n \cdot m$ sq. ft.

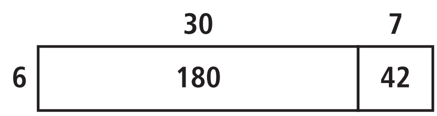
The area of the building is $n(k + m)$ sq. ft.

Remember, the area of the building can also be computed as the sum of the areas of the two rooms, $(n \cdot k + n \cdot m)$ sq. ft.

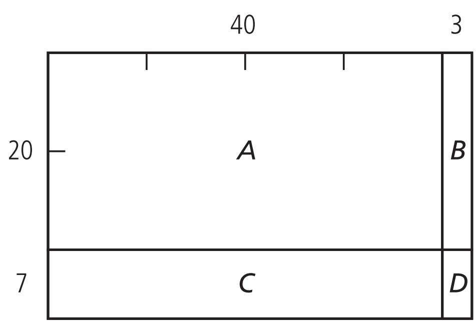
So, $n(k + m) = n \cdot k + n \cdot m$. We call this relationship the **distributive property**. This property tells us how addition and multiplication interact.

PROPERTY 2.2: DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION
For any numbers k, m, and n, $$n(k + m) = n \cdot k + n \cdot m.$$

You have already learned to multiply two-digit and three-digit numbers. Now you can use the area model and the distributive property to explore this process carefully. Begin by modeling the product of a one-digit number and a two-digit number. To multiply $6 \cdot 37$, use place value to write the product $6 \cdot 37$ as $6(30 + 7)$. By the distributive property, $6 \cdot 37 = 6(30 + 7) = 6 \cdot 30 + 6 \cdot 7 = 180 + 42 = 222$.



Visualize the product of 27×43 as area with the picture below:



Area of $\boldsymbol{A} = 20 \cdot 40 = 800$;

Area of $\boldsymbol{B} = 20 \cdot 3 = 60$;

Area of $\boldsymbol{C} = 7 \cdot 40 = 280$;

Area of $\boldsymbol{D} = 7 \cdot 3 = 21$.

The total area is 800 + 60 + 280 + 21 = 1161.

You can extend the same process to multiply 27 by 43 using the distributive property, or in a vertical format.

$$\begin{aligned} 27 \cdot 43 &= (20 + 7)(40 + 3) \\ &= 20(40 + 3) + 7(40 + 3) \\ &= 20 \cdot 40 + 20 \cdot 3 + 7 \cdot 40 + 7 \cdot 3 \\ &= 800 + 60 + 280 + 21 = 1161 \end{aligned}$$

$$\begin{array}{r} 27 \\ \times\ 43 \\ \hline 21 \\ 60 \\ 280 \\ +\ 800 \\ \hline 1161 \end{array}$$

EXAMPLE 3

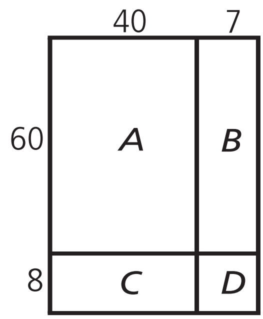
Find the product of 68 and 47 using the area model. Identify each rectangular region with the partial products found using the distributive property.

SOLUTION

Multiply 68 by 47 using the distributive property.

$$\begin{aligned} 68 \cdot 47 &= (60 + 8)(40 + 7) \\ &= 60(40 + 7) + 8(40 + 7) \\ &= 60 \cdot 40 + 60 \cdot 7 + 8 \cdot 40 + 8 \cdot 7 \\ &= 2400 + 420 + 320 + 56 = 3196 \end{aligned}$$

Visualize the product as area with the following picture:



Area of $\boldsymbol{A} = 60 \cdot 40 = 2400$;

Area of $\boldsymbol{B} = 60 \cdot 7 = 420$;

Area of $\boldsymbol{C} = 8 \cdot 40 = 320$;

Area of $\boldsymbol{D} = 8 \cdot 7 = 56$.

The total area is $2400 + 420 + 320 + 56 = 3196$.

PROBLEM 3

Find the product of $56 \cdot 39$ using the area model. Identify each rectangular region as one of the partial products found using the distributive property.

Suppose you are multiplying three numbers such as 2, 3, and 4. How can you multiply $2 \cdot 3 \cdot 4$? One way is to first multiply $(2 \cdot 3)$ then multiply the product 6 with 4. Notice the result of $(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$. Another way is to look at $3 \cdot 4$ first and then multiply the product 12 by 2 so that you have $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$. Notice that the two ways, $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ give the same product 24.

Try multiplying the three numbers $3 \cdot 4 \cdot 6$ as $(3 \cdot 4) \cdot 6$ and as $3 \cdot (4 \cdot 6)$. Were your two products equal? Your product using either computation is 72. This relationship is formally referred to as the associative property of multiplication.

PROPERTY 2.3: ASSOCIATIVE PROPERTY OF MULTIPLICATION
For any numbers k, n, and m, $$(k \cdot m) \cdot n = k \cdot (m \cdot n)$$

We also state the special property that 1 has in multiplication,

PROPERTY 2.4: MULTIPLICATIVE IDENTITY PROPERTY
For any number m, $(m \cdot 1) = 1 \cdot m = m.$ We call 1 the **multiplicative identity**.

EXAMPLE 4

Identify the multiplication property that is used to relate the equivalent expressions.

1. $4 \cdot (7 + 9) = 4 \cdot 7 + 4 \cdot 9$
2. $100 \cdot 25 = 25 \cdot 100$
3. $267 \cdot 1 = 1 \cdot 267 = 267$
4. $(7 \cdot 25) \cdot 100 = 7 \cdot (25 \cdot 100)$

EXERCISES

1. Use the indicated property and write an equivalent number expression.
   a. Use the commutative property of multiplication to write an equivalent number expression to $(^-2)(^-5)$. What number does this equal?
   b. Use the associative property of multiplication to write an equivalent number expression to $(^-3 \cdot 2) \cdot 4$. What number does this equal?
2. Use the area model and the distributive property to compute the following products. Indicate the area of each interior part in your model.
   a. (3)(5)
   b. (4)(3)
   c. (26)(4)
   d. (37)(24)
3. Compute the following:
   a. (42)(33)
   b. (128)(24)
   c. (512)(36)
   d. (709)(16)
   e. (831)(10)
   f. (58)(29)

4. Draw rectangle *A* with length 25 cm and width 12 cm. Draw rectangle *B* with length 12 cm and width 25 cm. Explain why these rectangles have the same area.

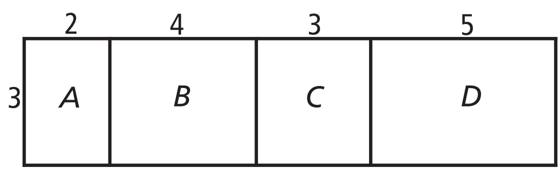
5. Ramses has 38 framed posters he plans to sell for $24.89 each. Estimate the amount of money Ramses will make.

Give two examples to illustrate each of the following multiplication properties:

6. Eddie ordered 12 boxes of playing cards. There are 52 cards in each box. How many cards did Eddie order? What if he ordered 29 boxes? 36 boxes? 48 boxes?

7. Mrs. Guerra purchased 18 packages of 24 pencils. She already had 28 pencils. How many pencils does she now have?

8. Ronnie has a large rectangular area that is fenced to create 4 smaller rectangular sections. The plans below show the dimensions of the area.



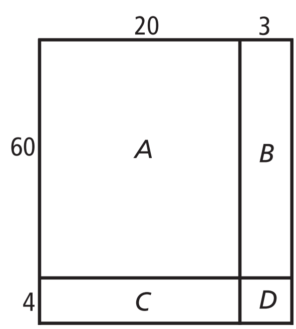
a. Find the partial products of each section.

b. What is the total area?

c. How is the total area related to the areas of the smaller sections?

d. Write equivalent expressions for the area using the distributive property.

9. Compute the area of the large rectangle below.

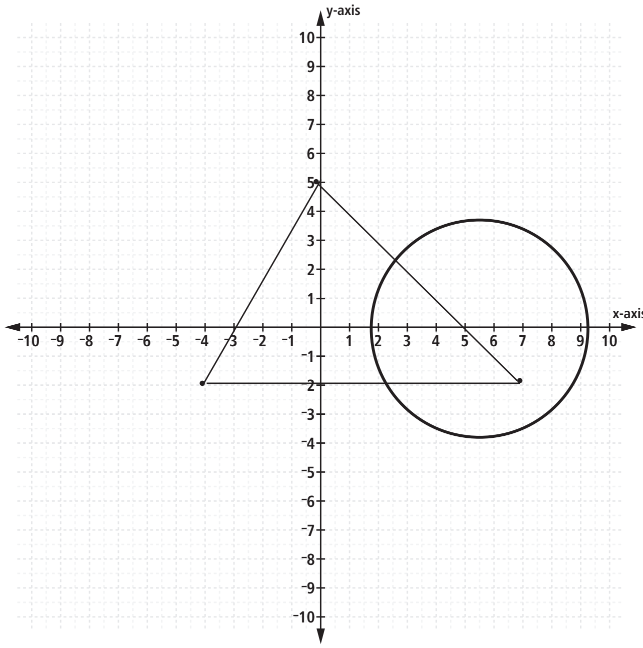


10. Calculate the sum $1 + 2 + 3 + \ldots + 6 + 7 + 8$.

11. Calculate the sum $2 + 4 + 6 + \ldots + 12 + 14 + 16$. How does problem 10 help?

Spiral Review

12. Name 2 points that are inside the triangle but outside the circle.



13. Melissa made cookies. The recipe required less than $\frac{1}{3}$ cup of nuts. Which of the following fractions is less than $\frac{1}{3}$?

a. $\frac{1}{2}$ b. $\frac{3}{4}$ c. $\frac{2}{5}$ d. $\frac{1}{6}$

14. **Ingenuity:**

Recall that when we multiply the expression $(a + b)(x + y)$, we get $ax + ay + bx + by$. This expression has four terms: ax, ay, bx, and by.

a. Multiply the expression $(a + b + c)(x + y)$. How many terms does your answer have? Draw a rectangular area model for the product.

b. Multiply the expression $(a + b + c)(x + y + z)$. How many terms does your answer have?

c. Multiply the expression $(a + b)(m + n)(x + y)$. To do this, first find the product $(a + b)(m + n)$, and then multiply the result by $(x + y)$. How many terms does your answer have?

d. Without multiplying the expression $(a + b + c)(k + m + n)(x + y + z)$, is it possible to tell how many terms the product has? Explain your reasoning.

15. **Investigation:**

Use the area model to compute the following products:

a. $102 \cdot 103$

b. $104 \cdot 105$

c. $101 \cdot 108$

d. $107 \cdot 109$

Discuss any connections you see between the factors given and the products you found in parts a through d.

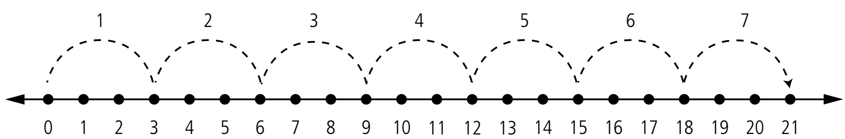
2.3

SECTION 2.3
LINEAR MODEL FOR DIVISION

Just as with multiplication, we will explore the operation of division. We will start by looking at some models to better understand how division works.

ACTIVITY: MODELS FOR DIVISION

You divided a class of 21 by 3 in the activity. One method involved subtracting 3 objects at each step from the original group and counting the number of times it takes to distribute all 21 objects. Another way to think about this problem is to add groups of 3 until you have 21. Skip count by 3's to accumulate objects until you have the desired number, 21. The number of skips that it takes to get to 21 is the result of 21 divided by 3.



To skip count by 3's, count 3, 6, 9, 12, 15, 18, 21, and so on. You know that $21 = 3 \cdot 7$ because we must skip count 7 steps by 3's to get to 21. The inverse of the multiplication statement is $21 \div 3 = 7$, which means when you divide 21 by 3 the result is 7, because 21 is decomposed into 7 skips of 3 units per skip. This is equivalent to 7 groups of 3. We call 3 the **divisor**, 7 the **quotient**, and 21 the **dividend**.

When the divisor divides evenly into the dividend, or the remainder is zero, the word **factor** is used interchangeably with divisor. Looking for the quotient when 21 is divided by 3 is the same as looking for the missing factor $\boldsymbol{x}$ that satisfies $3 \cdot \boldsymbol{x} = 21$. The $\boldsymbol{x}$ that satisfies this equation is the quotient and represents the number of skips of length 3 it takes to reach 21. We call this the **missing factor model**. It is the reverse of the multiplication process.

We should note here that the word "factor" can be a noun that means divisor, as above, where 3 is a factor of 21. It can also be a verb. When we say, "Factor 21," we mean write 21 as a product of two or more positive integers. In this case, write $21 = 3 \cdot 7$ to factor 21 into a product of two numbers, 3 and 7, which are both factors of 21. We will talk more about this in Chapter 3.

PROBLEM 1

Use a number line with the appropriate scale and the skip counting model to compute the following quotients:

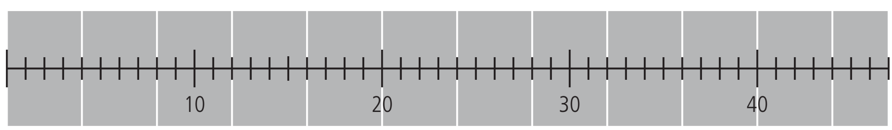
a. $56 \div 7$ b. $91 \div 13$ c. $210 \div 15$

EXAMPLE 1

Robin has 47 feet of ribbon. She wants to cut this ribbon into 4-foot strips for decorations. How many 4-foot strips of ribbon can she make? How much ribbon will be left over, if any?

SOLUTION

In order to make the 4-foot strips, Robin rolls out all of the ribbon and marks off 4-foot lengths. She then skip counts the number of pieces she needs to cut and finds that $4 \cdot 11 = 44$. Therefore, 47 feet divided into 4-foot pieces equals 11 pieces with 3 feet of ribbon left. In other words, $47 \div 4$ is 11 with a remainder of 3. The **remainder** is the amount left over after division when the divisor does not divide the dividend exactly. The remainder is a number greater than or equal to 0 but less than the divisor.



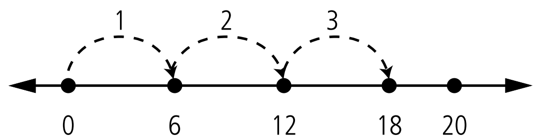
Notice that when using the remainder, the solution is $47 = 4 \cdot 11 + 3$. For now, any number left after dividing you may leave as a remainder.

EXPLORATION 1

Mr. Garza has 20 pieces of candy. He wants to divide the candy equally among 6 children. How should he distribute the candy?

One way to distribute the candy is to think of this process in steps. In step 1, give each child 1 piece of candy. This means Mr. Garza has $20 - 6 = 14$ pieces of candy left. In step 2, Mr. Garza gives each child a second piece of candy. He now has $14 - 6 = 8$

pieces of candy left. In step 3, Mr. Garza gives each child a third piece of candy. He now has $8 - 6 = 2$ pieces of candy left. He can no longer give an equal number of pieces to each of the 6 children, so he stops. It took 3 steps to equally distribute as many pieces of candy as Mr. Garza could. That means each child received 3 candies. Write this as $20 = 3 \cdot 6 + 2$. Picture this as a linear model by skip counting to divide 20 by 6, which corresponds to the counting 3 skips of length 6: $3 \cdot 6 = 18$, 2 units short of 20.



In division, the problem involves the dividend and the divisor, and the task is to compute the quotient. In the linear model, the dividend is the total length. There are two possible cases:

1. Know the length of each jump and call it the divisor. Find the quotient, which in this case is the number of jumps that equal the total length.
2. Know the number of jumps and call it the divisor. Find the quotient, which in this case is the length of each jump.

In multiplication, the problem starts with the length of each jump and the number of jumps. The answer is the accumulated length of all the jumps. We can think of the division process as the reverse of multiplication.

EXERCISES

1. Evaluate the following quotients, and write the associated multiplication fact. Use the linear model or long division, if needed.

a. $42 \div 7$	d. $54 \div 6$	g. $231 \div 7$
b. $24 \div 3$	e. $29 \div 1$	h. $649 \div 11$
c. $64 \div 8$	f. $48 \div 4$	i. $1824 \div 24$

2. Write the associated multiplication fact, making the remainder as small as possible:

a. $45 \div 4$	b. $39 \div 7$	c. $24 \div 5$

d. $56 \div 7$ f. $75 \div 8$ h. $497 \div 9$

e. $63 \div 10$ g. $539 \div 8$ i. $410 \div 6$

3. Estimate each quotient to determine in which range it belongs: between 1-10, between 10-100, or between 100-1000.

a. $48 \div 6$ c. $272 \div 4$ e. $1491 \div 7$

b. $272 \div 16$ d. $964 \div 9$ f. $1190 \div 17$

Solve each problem from 4 -10. Write your answers in complete sentences.

4. Peter has 624 oranges. He wants to place the oranges into equal amounts in bags. Each bag will contain 24 oranges. Estimate the number of bags he can make with the number of oranges he has.

5. Audrey invited 5 friends to a dinner party. She wants to place a small vase with flowers in front of each dinner plate. She has 38 flowers.

   a. How many flowers can she use for each vase so that each has the same number of flowers?

   b. Will she have any flowers left to add to the centerpiece? If so, how many?

6. Erica and her friends are making bead bracelets. It takes 13 beads to make each bracelet. If they have 105 beads, how many bracelets will they be able to make? Will any beads be left over? If so, how many?

7. Madison and her two friends are playing a card game that contains 92 cards. The game requires her to deal out all the cards so that each player gets an equal amount. What is the maximum (most) number of cards each of them could be dealt? How many cards, if any, would be leftover in this case?

8. There are 21 students in Mrs. Padron's 3rd period class. One day she comes to class with a bag of colorful pencils. She gives each student an equal number of pencils and discovers that she has 11 pencils left over. She knows that the bag had fewer than 100 pencils to begin with.

   a. What is the largest possible number of pencils the bag could have contained originally? How many pencils would she have given each student in that case?

b. What other possible solutions could this problem have?

9. Uncle Bill is installing a sprinkler system in his backyard. The system requires 13-inch sections of PVC pipe. He purchases one long 8-foot piece of PVC pipe. How many 13-inch sections can he cut from this long piece? How much pipe will be left over after he cuts the sections? (Hint: 1 ft = 12 in).

10. Evaluate:

    a. $(15 + 15 + 15) \div 3$

    b. $(214 + 214 + 214 +214) \div 4$

    c. $(31 + 31 + 31 + 31 + 31 + 31) \div 6$

    What patterns do you notice?

    Use the pattern you noticed to complete the next two expressions.

    d. $(412 + 412 + 412 + 412) \div 2$

    e. $(58 + 58 + 58 + 58 + 58 + 58) \div 2$

    What pattern do you notice now?

Spiral Review

11. At the Elk lodge there are a total of 12 tables in the dining room. 7 of the tables seat 5 people each, and 5 of the tables seat 8 people each. What is the maximum number of people who can sit at the tables in the dining room?

12. Sam counted 27 buses in the school parking lot. If the buses hold between 30 and 36 students, which is the best estimate of the total number of students on the buses?

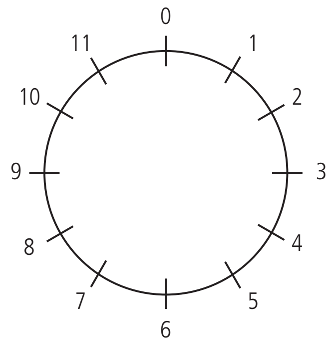
    a. 800 b. 809 c. 900 d. 975

13. **Ingenuity**:

    A certain auto race consisted of 250 laps around an oval track. The only drivers who led laps during the race were Dale, Jeff, and Jimmie. Jeff led three times as many laps as Dale, and Jimmie led twice as many laps as Jeff. In how many laps did Jimmie lead?

14. **Investigation**:

Draw a circle, and label twelve points on the circle from 0 to 11, as in the Investigation in section 2.1.



Suppose a frog begins at the point 0, and hops clockwise around the circle, hopping one unit at a time. For example, if the frog hops 14 times, then it will make one complete lap around the circle, and finally land at the point 2.

a. Suppose the frog hopped 17 times. How many laps around the circle would it make, and where would it finally land?

b. Suppose the frog hopped 28 times. How many laps around the circle would it make, and where would it finally land?

c. Suppose the frog hopped 45 times. How many laps around the circle would it make, and where would it finally land?

d. Suppose the frog hopped 100 times. How many laps around the circle would it make, and where would it finally land? Is there a way to figure this out without actually keeping track of all 100 hops on the circle?

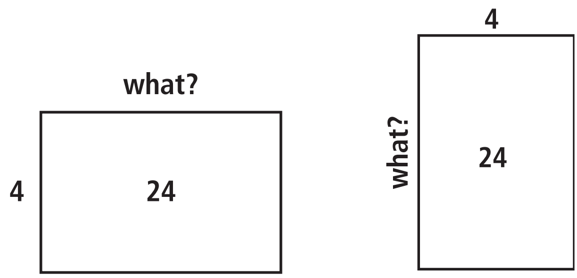
e. Suppose we instead marked 60 points on a circle, and labeled them 0 through 59. If the frog started at 0 and made 314 hops, where would it finally land?

15. **Challenge**:

Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left over. They put the coins back, ignore one pirate, and split the coins again, only to find that there are 3 coins left over. So, they ignore 2 pirates and try again. This time, the coins split evenly. What is the least number of coins there could have been?

SECTION 2.4
THE DIVISION ALGORITHM

Another way of thinking of division is by using the area model. This is similar to the missing factor model. To divide 24 by 4, draw a length of 4 and ask what the width x is to equal a total area of 24. What you are doing is looking for the missing factor: 24 = 4 · (what?), and 24 = (what?) · 4. This is an example of the commutative property of multiplication.



You know that division is the reverse operation for multiplication, just as subtraction is the reverse operation for addition. What do we mean by this?

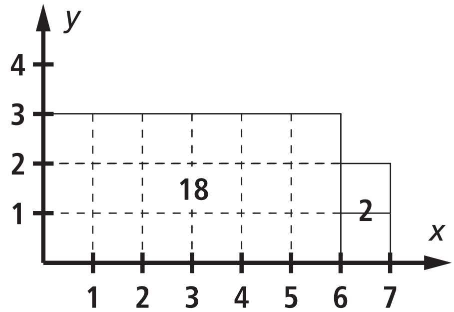
Begin with the number 12. Add 3 to get 15. To undo the addition, you need to subtract 3 from 15 and return to the original number 12. Similarly, in the example above, you found the number 6. Multiply by 4 to obtain 24. That is, $24 = 6 \cdot 4$. To undo this multiplication, divide 24 by 4 and return to the original number because $24 \div 4 = 6$.

EXAMPLE 1

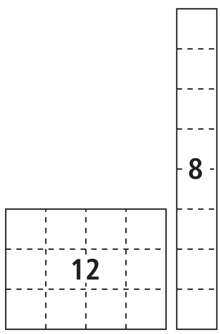
Using the area model, what is $20 \div 3$?

SOLUTION

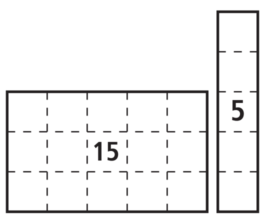
Begin with a length of 3 on the y-axis. If we mark off a length of 6 on the x-axis, the area of the rectangle is 18. We compute this as $18 = 3 \cdot 6$. To get an area of 20, we must add 2 more square units to the end of the rectangle. That means $20 \div 3$ has quotient 6 with remainder 2 because this corresponds to the calculation $20 = 3 \cdot 6 + 2$.



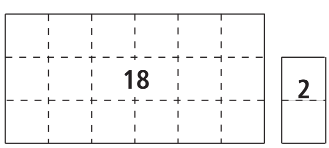
Why is the quotient 6 and the remainder 2? Why not say the quotient is 5 and the remainder is 5? Why not say the quotient is 4 and the remainder 8 since $20 = 3 \cdot 4 + 8$? How do we decide between the different quotients and remainders?



If we think of 20 as $3 \cdot 5 + 5$,



the picture shows that we could break up the last column into pieces of lengths 3 and 2.



Adding this extra 3 to the rectangle is represented by the calculation $3 \cdot 6 + 2$. The picture on the previous page shows we can break up that last column into two pieces, one of length 3 and another piece of length 2. By adding these extra pieces of length 3 to the rectangle, we have the same calculation $3 \cdot 6 + 2$.

By using the area model, we write a in the form of the calculation $a = d \cdot q + r$, where d is the height of the rectangle, q is the length of the rectangle along the x-axis, and r is the remainder, or the height of the last column, added to equal a. This is the division algorithm where d is the divisor, q is the quotient, and r is the remainder.

Are there any restrictions on r and d? Yes! When examining the above example, 20 can be written in several different ways:

$$20 = 3 \cdot 4 + 8$$
$$20 = 3 \cdot 5 + 5$$
$$20 = 3 \cdot 6 + 2$$

The smaller the values of r the closer we are to seeing if 20 is a multiple of 3. Only when we get to $r = 2$, do we see that 20 is not a multiple of 3. There will be some remainder, namely 2. So, one condition to put on r and d is that $r < d$, and r must be greater than or equal to zero ($r \geq 0$).

The divisor d must be positive because r is not negative, and d is greater than r. We write this with our inequalities as follows: Because $r < d$, then $d > r$. And because $r \geq 0$, then $d > r \geq 0$ and $d > 0$. With this added restriction, we write $20 = 3 \cdot 6 + 2$.

We state the formal division algorithm:

THEOREM 2.1: DIVISION ALGORITHM

Given two positive integers a and d, we can always find unique integers q and r such that $a = dq + r$ and $0 \leq r < d$. We call a the ***dividend***, d the ***divisor***, q the ***quotient,*** and r the ***remainder***.

In our previous example with $20 = 3 \cdot 6 + 2$, the dividend $a = 20$, the divisor $d = 3$, the quotient $q = 6$, and the remainder $r = 2$.

PROBLEM 1

Compute the following division problems by writing the corresponding division algorithm and sketching a picture that explains what the algorithm represents.

a. $43 \div 6$ b. $58 \div 9$ c. $87 \div 12$

EXERCISES

1. Draw the area model for each of the following, then use the division algorithm to compute the quotient.

   a. $29 \div 6$
   b. $38 \div 5$
   c. $80 \div 16$
   d. $47 \div 8$
   e. $68 \div 9$
   f. $110 \div 8$

2. Using the area model, predict whether the quotient is between 1 and 10, between 10 and 100, or between 100 and 1000 by estimating the length of the rectangle's base.

   a. $812 \div 40$ b. $217 \div 31$ c. $2401 \div 12$

3. If each mp3 file takes up 8 MB of space, how many files can you fit on a 700 MB CD? How much space, if any, will you have left over?

4. Use graph paper to model $45 \div 6$ using the area model.

5. Pamela has 63 quarters. She goes to the bank and trades them for dollar bills. How many dollar bills will she get? How many quarters, if any, will she have left?

6. Jeremy has 50 square tiles, each of which is 1 foot by 1 foot. He would like to construct a path that is 3 feet wide. How long can he make the path? How many tiles, if any, will he have left?

7. Danny wants to fill 43 bags with candy. He has 1591 pieces of candy he will use. Estimate which range best describes the number of pieces of candy he can put in each bag: 1- 10, 10 - 50, 50 - 100, or more than 100. Explain how you reached this estimate.

8. Evaluate the following expressions:

   a. $(2 \cdot 7) \div 2$

   b. $(4 \cdot 3 \cdot 6) \div (4 \cdot 3)$

   c. $(6 \cdot 2 \cdot 4) \div 4$

   d. $(9 \cdot 3 \cdot 5) \div (9 \cdot 3)$

   e. $(6 \cdot 5 \cdot 4) \div (5 \cdot 4)$

   f. $(4 \cdot 3 \cdot 2) \div (2 \cdot 4 \cdot 3)$

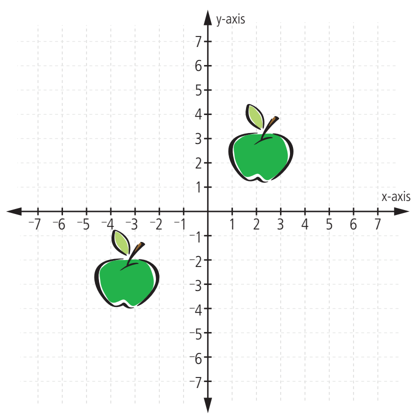
   g. Write rules to describe any patterns you noticed in parts a - f. What causes these patterns?

   h. Using the patterns you observed above, compute the following: $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) \div (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)$.

9. Given two positive integers m and n, explain why the area and linear models of division from $m \div n$ give the same results.

Spiral Review

10. Which transformation is shown below?



11. Mrs. Murphy needs to replace batteries in 30 calculators. Each calculator uses 4 batteries. The batteries are sold in packages of 10. How many packages of batteries does Mrs. Murphy need to buy?

12. **Ingenuity:**

    Answer the following questions:

    a. Suppose we made a 4×4 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 3 tiles each, how many sets would we have?

b. Suppose we made a 7×7 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 6 tiles each, how many sets would we have?

c. Suppose we made a 77×77 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 76 tiles each, how many sets would we have?

d. Suppose we made an $n \times n$ square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of $n - 1$ tiles each, how many sets would we have?

13. **Investigation:**

Use the area model to answer the following question:

a. Ms. Reyes wants to divide her class of 33 students into groups of four or five students each, with as few five-student groups as possible. How many of these groups will be five-student groups?

b. Later that day, Ms. Reyes goes to a teacher appreciation luncheon. There are 83 teachers present at the luncheon. The organizer of the luncheon wants to organize the teachers into tables of six or seven teachers each, with as few seven-teacher tables as possible. How many tables will there be, and how many of these tables will have seven teachers?

14. **Challenge:**

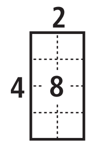
When a number n is divided by 11, the quotient is 11 with a possible remainder. When n is divided by 10, the quotient is 12 with a possible remainder. When n is divided by 9, the quotient is 13 with a possible remainder. What is the quotient when n is divided by 8?

SECTION 2.5
LONG DIVISION

We have seen how closely related multiplication and division are. For example, we know $8 \div 4 = 2$ because $4 \times 2 = 8$. Also recall that in the long division form, the multiplication fact is rewritten as

$$\begin{array}{r} 2 \\ 4\overline{)8} \end{array}$$

The area model looks like this:

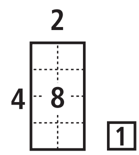


We have the **dividend** 8 "under" the **quotient** 2, and the **divisor** 4 is to the left of the dividend.

By changing the dividend to 9, our problem becomes $4\overline{)9}$. Because $8 \div 4 = 2$, $9 \div 4$ must be more than 2. In the long division form,

$$\begin{array}{r} 2 \\ 4\overline{)\ 9} \\ -8 \\ \hline 1 \end{array}$$

The area model looks like this:

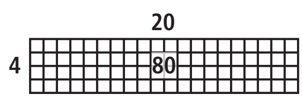


The quotient is 2, and the **remainder** is 1.

Consider the problem $4\overline{)80}$, where the dividend is not simply 8 but 8 tens. The quotient is then 2 tens, or 20, because $4 \times 20 = 80$. In the long division form,

$$\begin{array}{r} 20 \\ 4\overline{)\ 80} \\ -80 \\ \hline 1 \end{array}$$

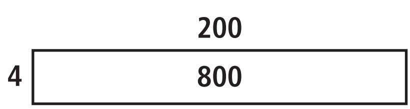
The area model is:



Use the long division form to evaluate the problem $4\overline{)800}$ with 8 hundreds. Do you agree the answer is 2 hundreds, or 200? In the long division form,

$$\begin{array}{r} 200 \\ 4\overline{)800} \end{array}$$

The area model is:

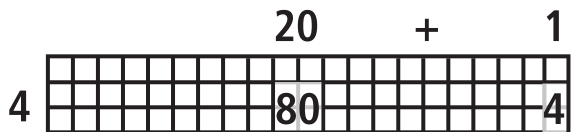


Note that the above picture is not to scale. If you were to draw it to scale and leave the height unchanged, imagine how long the rectangle would be.

A more complex problem is $4\overline{)84}$. One way to think of this problem is to notice the place values and observe that 84 = 80 + 4. You know that:

Putting $\begin{array}{r} 20 \\ 4\overline{)80} \end{array}$ and $\begin{array}{r} 1 \\ 4\overline{)4} \end{array}$ together gives $\begin{array}{r} 21 \\ 4\overline{)84} \end{array}$.

Here is the area model for this problem:



or

$$\begin{array}{r} 1 \\ 20 \\ 4\overline{)\ 84} \\ \underline{-80} \\ 4 \\ \underline{-4} \\ 0 \end{array}$$

This is called the **scaffolding** method because the different partial quotients are first computed and stacked, then combined, much like a scaffold is used in constructing a building.

PROBLEM 1

Use this scaffolding method to compute the following quotients. You may sketch a picture of the corresponding area model if it helps.

a. $52 \div 4$

b. $960 \div 6$

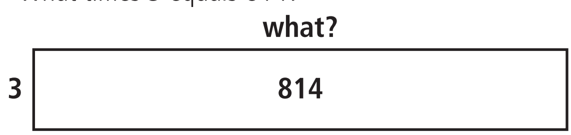
If you remember the common algorithms for adding, subtracting, and multiplying, you know to start working with the smaller place values and then work up to the larger place values. Think about working the following problems, and pay close attention to the place you start and the direction you move in your computation.

$\begin{array}{r} 3108 \\ \underline{+\ \ 15} \end{array}$ $\qquad$ $\begin{array}{r} 3108 \\ \underline{-\ \ 15} \end{array}$ $\qquad$ $\begin{array}{r} 3108 \\ \underline{\times 15} \end{array}$

In dividing, however, we know that it is more common to start with the largest place value to determine the quotient and then gradually include the smaller place values. Using these numbers, we will compute 3108 ÷ 15. First, though, let's look at the division problem 814 ÷ 3, or $3\overline{)814}$, and use it as an example to see

- how long division works,
- how it relates to multiplication, and
- what representation helps us to better understand the long division procedure.

One way to think about this problem is to consider the related multiplication statement. Because the division problem is, "What does 814 ÷ 3 equal?" the related multiplication statement reads, "What times 3 equals 814?"



What is 814 ÷ 3? Look at the Scaffolding method. To use the Scaffolding method, we will subtract multiples of the divisor until we have the smallest remainder. Your work may look slightly different from someone else's, but your quotient and remainder should be the same.

```
      1
     10
     30
     30
    100
    100
3 ) 814
   -300
    514
   -300
    214
   - 90
    124
   - 90
     34
    -30
      4
     -3
      1
```

$$100 + 100 + 30 + 30 + 10 + 1$$

$$814 \div 3 = 271 \text{ remainder } 1$$

The model has reached the sum 813, but because 1 more is needed to reach the dividend 814, 1 is the remainder. The result of the long division can be written as 271 r 1.

Determine what $813 \div 3$ equals. If you found the quotient 271, you are correct.

Now carefully compute $3108 \div 15$, showing the scaffolding method and the area model. Notice how close the scaffolding method is to the long division method.

PROBLEM 2

Use the scaffolding method to compute the following quotients:

a. $685 \div 11$

b. $378 \div 12$

c. $5642 \div 28$

d. $2153 \div 14$

e. $1341 \div 16$

EXERCISES

1. Compute the following quotients using scaffolding long division, if necessary. Verify your answer using multiplication. You might also want to check using your calculator, if you are unsure.

   a. $72 \div 8$
      $720 \div 8$

   b. $868 \div 14$
      $8680 \div 14$

   c. $736 \div 23$
      $7360 \div 23$

2. There are 1230 people attending a concert. A shuttle bus that seats 12 people will transport them to the stadium. Arne says they will need buses in the hundreds. Barak says they will need buses in the thousands. Who is right and why do you say so?

3. Compute the following quotients and remainders. Check your answer with a visual model. Write a relationship of dividend, quotient, divisor, and remainder using the division algorithm.

a. 123 divided by 12

b. 475 divided by 2

c. 209 divided by 16

d. 870 divided by 5

4. Use one of the division methods you have learned to compute the following quotients:

a. $39 \div 7$

b. $68 \div 15$

c. $315 \div 9$

d. $673 \div 41$

e. $1512 \div 36$

f. $8318 \div 27$

5. Sara is going to rent some movies for a party. Each movie rental costs $9, and she has $75 to spend. How many movies can she rent?

6. Israel will drive 530 miles to McAllen from Dallas this summer. If he is driving 125 miles per day, how many days will it take him to get to McAllen?

7. Melissa needs $425 to buy a new purse. If she earns $40 per day, how many days of work will it take Melissa to buy her purse?

8. A football stadium has 4800 seats. During the last game of the season, the stadium made $28,800.00. How many seats were filled, if each seat sold for $12.00?

9. Fossum Middle School has cafeteria tables that seat 10 students each. If there are 273 students going to lunch, how many tables will be needed?

10. Ana, Steven, Ruth, and Mark were raising money for a talent show. Ana raised $45, Steven raised $60, Ruth raised $48, and Mark raised $12. They needed to buy uniforms that cost $35 each. Will they be able to buy 4 uniforms? If yes, how much extra money will be left? If not, how much more money do they need to raise?

11. a. Donna paid $60 for five CDs of equal cost. How much did each CD cost?

b. Shirley paid $36 for twelve hot dogs. How much did she pay for each hot dog?

c. Gary paid 99 cents for three pieces of candy. How much did he pay for each piece?

12. Rolinda has \$279 to spend on video games. Each game costs \$45. What is the greatest number of games Rolinda can buy? How much money will Rolinda have left?

13. Calculate the following:

a. $27 \div 3$	e. $24 \div 4$
b. $270 \div 3$	f. $240 \div 4$
c. $2700 \div 3$	g. $33 \div 11$
d. $27000 \div 3$	h. $3300 \div 11$

i. Write rules to describe any patterns you noticed in parts a – h. What causes these patterns?

Spiral Review

14. Casey, Cathy, and Chasity each bought food for a faculty party. Casey spent \$7 more than Cathy. Cathy spent \$5 less than Chasity. Chasity spent \$10. What is the total amount of money spent on food?

15. Paige earns \$12 each week walking her friend's dog. Which of the following is the best estimate of how much money she will earn in 38 weeks of walking the dog?

a. \$450 b. \$460 c. \$470 d. \$480

16. **Ingenuity:**

Solve the following "missing digit" puzzles. In each puzzle, each letter represents a digit from 0 to 9.

a. Suppose that $AA2 \div 4 = 8A$. What digit does A represent?

b. Suppose that $1B28 \div 3B = B2$? What digit does B represent?

c. Suppose that $6CD \div C = DC$? What digit do C and D represent?

17. **Investigation:**

On Monday, Mr. Jensen brought a bag of small candies to school for his algebra class to enjoy. He divided the candies evenly among his 22 students. Each student got 9 candies, and Mr. Jensen was left with 13 candies, which he ate.

a. Mr. Jensen realized that he did not bring enough candies to class on Monday, so on Tuesday, he brought three bags of candies. Each bag had the same number of candies as the bag he brought on Monday. If Mr. Jensen divided these candies evenly among his 22 students and kept the remainder for himself, how many candies did each student get, and how many did Mr. Jensen get? (Assume that Mr. Jensen always gives his students as many candies as he can, provided that each student has the same number.)

b. Mr. Jensen was happy with the number of candies he brought on Tuesday, so he brought the same number again on Wednesday. This time, however, there were only 21 students in class. How many candies did each student get, and how many did Mr. Jensen get?

c. Can you find a way to do parts a and b without figuring out how many candies are in a bag?

18. **Challenge:**

While performing a trick of long division, a mathemagician made some of his digits disappear. Alas, he cannot reconstruct the missing digits... until he remembers that one of them is a 7. Fill in the blanks to complete the calculation.

```
      _4
   ____
 _)1_3
−  __
   ‾‾
    __
  − 2_
   ___
     3
```

SECTION 2.R
CHAPTER REVIEW

1. Use the number line frog to compute:

   a. $15 \cdot 5$
   b. $(8)16$
   c. $15(7)$
   d. $(11)(9)$

2. Use the area model to compute:

   a. $12 \cdot 3$
   b. $5 \cdot 7$

3. Jimmy bought 12 tickets to the carnival. Each ticket cost \$15. How much did Jimmy pay?

4. There are 580 students at Fossom Middle School. Each math class will have at most 24 students. What is the minimum number of math classes at Fossum Middle School?

5. Lorianne is buying large bags of bubble gum for a gumball machine. The large bags contain 560 gumballs. She purchases 24 large bags. How many gumballs will she have?

6. Compute the following using multiplication properties such as the commutative property to make the computation easier. Show your work.

   a. $25 \cdot 52 \cdot 4$
   b. $25 \cdot 83 \cdot 12$
   c. $202 \cdot 20 \cdot 5$
   d. $221 \cdot 50 \cdot 2$

7. Explain how to compute using the distributive property to make the computation easier.

   a. $92 \cdot 8$
   b. $98 \cdot 36$

8. Draw a 3×6 rectangle and call it rectangle A. Draw a 7×11 rectangle and call it rectangle B. Then fill in the table.

Rectangle	Length	Width	Area
A			
B			

2.R

9. Use the linear model to find the quotient that makes the remainder as small as possible:

   a. $21 \div 3$
   b. $120 \div 11$
   c. $143 \div 4$
   d. $94 \div 28$

10. Use the area model to find the quotient that makes the remainder as small as possible:

    a. $24 \div 6$
    b. $71 \div 9$
    c. $44 \div 12$
    d. $210 \div 24$

11. Predict whether the quotients of the following division problems are between 0.1 and 1, between 1 and 10, between 10 and 100, or between 100 and 1000. Explain your prediction and then do the division to check your answer.

    a. $217 \div 37$
    b. $738 \div 33$
    c. $5521 \div 97$
    d. $364 \div 3$

12. Find the quotient and remainder using scaffolding.

    a. $216 \div 3$
    b. $210 \div 4$
    c. $819 \div 32$
    d. $989 \div 40$
    e. $3921 \div 73$
    f. $7882 \div 80$

13. Find the quotient and remainder using long division.

    a. $311 \div 3$
    b. $611 \div 5$
    c. $2122 \div 13$
    d. $3412 \div 12$
    e. $3412 \div 31$
    f. $1202 \div 48$

14. Nama has \$150 and wants to buy video games. Each video game costs \$27. How many video games can she buy?

15. Perform the following operations and calculate the sum, difference, product, or quotient as appropriate.

    a. $13 + (^-4)$
    b. $^-203 + 4$
    c. $13 \cdot (^-4)$
    d. $13 - (^-4)$
    e. $^-4 \cdot (^-13)$
    f. $20 \div 4$
    g. $203 \div 4$
    h. $^-203 \cdot 4$
    i. $^-203 - 4$

2.R

FACTORS AND MULTIPLES

3

SECTION 3.1
FACTORS, MULTIPLES, PRIMES, AND COMPOSITES

One of the most important concepts in mathematics is the idea of ***divisibility***. Suppose you have 14 marbles, and you want to give the same number of marbles to each of three friends. Is it possible to give each friend the same number and have none left?

You can let each person have 4 marbles, but there are two left. This process is called division. You divide 14 by 3 to get the quotient 4 with remainder 2. This is equivalent to the calculation $14 = 3 \cdot 4 + 2$. In building the multiplication table for 3, skip counting by 3, starting at 0, does not list 14.

On the other hand, if you have exactly 12 marbles, you can give each friend 4 marbles, and everybody has an equal number of marbles. This process corresponds to $12 = 3 \cdot 4$.

What does this have to do with divisibility? We know that 14 objects cannot be divided equally among 3 people. Another way to say this is

$$\begin{array}{r} 4 \\ 3\overline{)14} \\ \underline{-12} \\ 2 \end{array}$$

- 14 is not ***divisible*** by 3.
- 3 is not a ***factor*** of 14.
- 14 is not a ***multiple*** of 3.

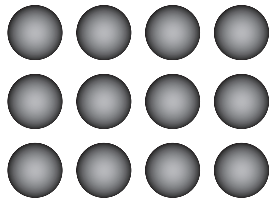
On the other hand, we can divide 12 things equally among 3 people. Mathematically,

$$\begin{array}{r} 4 \\ 3\overline{)12} \\ \underline{-12} \\ 0 \end{array}$$

- 12 is ***divisible*** by 3.
- 3 is a ***factor*** of 12.
- 12 is a ***multiple*** of 3

Although there is no integer that you can multiply by 3 to equal 14, there is the integer 4 that you can multiply by 3 to equal 12.

For example, we know that 12 is divisible by 3 because $12 = 3 \cdot 4$. We can show this using our marble example:



Factor	$\cdot$	Factor	$=$	Product
3	$\cdot$	4	$=$	12

Notice that the twelve marbles can be arranged in a rectangular array. This suggests another way for us to see the divisors of a positive integer.

DEFINITION 3.1: DIVISIBILITY

Suppose that n and d are integers, and that d is not 0. The number n is divisible by d if there is an integer q such that $n = d \cdot q$. Equivalently, d is a **factor** of n, and n is a **multiple** of d.

We'll experiment with this in the following activity.

EXPLORATION 1: THE POSSIBLE RECTANGLE MODEL

Materials: You will need graph paper for this activity and the Possible Rectangles Chart.

1. For each positive integer n from 1 to 30, make as many rectangles with integer side lengths as you can that have area equal to n square units. Count a rectangle only once if the factors are the same. For example, $3 \cdot 4$ and $4 \cdot 3$ will be counted only once.

2. Organize the data in a table provided by your teacher. In the first column, write the positive integer n. In the second column, write the number of rectangles possible with area n. In the third column, list all the possible dimensions of the rectangles. In the fourth column, list all the possible lengths of sides of the rectangles, in increasing order. For example, we have filled in the results for the value of $n = 4$ on the table.

3. What do you notice in the table so far?
4. Continue the table for n from 31 to 50.
5. Looking at the extended table, do the patterns continue?
6. Looking at a given number n, what do you notice about the numbers in the last column for this value of n?
7. What do we call the numbers in the last column in relation to n? For each rectangle, the dimensions form a factor pair, such as 3 and 6 for $n = 18$. If you put all the factors in the last column in order, such as 1, 2, 3, 4, 6, 12 for $n = 12$, how do the factor pairs line up?
8. What do you notice about the number 1? Find any other numbers that have this same property, if possible.
9. Circle the values of n that generate only one rectangle.

   How many factors does each of these have?

   How would you describe the circled numbers, excluding 1?
10. Use a different color pen or marker to box the values of n that have an odd number of positive divisors.

Notice that all of the factors in our chart are positive. Generally, when talking about *factors*, we just mean the positive factors.

The numbers that have only two positive factors play a special role in mathematics and have a special name.

DEFINITION 3.2: PRIME AND COMPOSITE
A **prime** number is an integer p greater than 1 with exactly two positive factors: 1 and p. A **composite** number is an integer greater than 1 that has more than two positive factors. The number 1 is the multiplicative identity; that is, for any number n, $n \cdot 1 = n$. The number 1 is neither a prime nor a composite number.

Your work with the Possible Rectangle Table (PR Table) allows you to see the relationships between a given number, n, the number of rectangles possible with n as its area, and the number of factors n has. In particular, you can see from your PR Table that the prime numbers between 1 and 40, in increasing order, are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. These are the only integers greater than 1 and less than or equal to 40 that had exactly one rectangle possible and hence, exactly two factors. Let's consider a larger number and determine whether it is prime or not.

PROBLEM 1

Is 119 prime or composite?
Here is another approach that is not as geometric, but it is systematic.

EXAMPLE 1

Is the number 171 prime?

SOLUTION

To see if there are any other factors of 171 between 1 and 171, begin to divide 171 by numbers less than 171, in increasing order beginning with 2.

Divide by	Quotient	Remainder	Factor?
2	85	1	no
3	57	0	yes

Dividing 171 by 3 gives quotient 57 and remainder 0, showing that 3 is a factor of 171. This is enough information to conclude that 171 is composite because 171 has not just 1 and itself as factors, but it also has 3 as a factor.

PROBLEM 2

Is the number 127 prime? Explain why or why not.

EXPLORATION 2

In small groups or individually, determine whether the following numbers are prime or composite. Try to devise as many time-saving strategies as you can, so you don't have to check every integer between 1 and the target number.

a. 51	d. 99	g. 171
b. 67	e. 113	h. 131
c. 81	f. 123	i. 323

Let's explore some strategies for finding all the factors or divisors of a given positive integer n. In particular, if n is a positive integer and k is a positive integer, how do we determine whether k is a factor of n; or equivalently, whether n is a multiple of k? The method used in the Possible Rectangle Activity works well for small numbers but does not work as well for larger numbers. So, we want to find a method that can be used when we have to deal with large numbers.

EXAMPLE 2

Is 9 a factor of the number 112? Equivalently, is 112 a multiple of 9?

SOLUTION

Starting with the second question, we could skip count by 9 to determine whether 112 is a multiple of 9. However, this is not an efficient strategy. Instead, ask if there is a positive integer q such that $112 = 9 \cdot q$. You can answer this using long division, which results in a quotient of 12 and a remainder of 4. This means that $112 = 9 \cdot 12 + 4$. The goal is to find an integer q so that $112 = 9q$. If you skip counted by 9, you would not land on 112. That is, there is no integer q for which $112 = 9q$. Therefore, 9 is not a factor of 112.

EXAMPLE 3

Is 18 a factor of 144? Equivalently, is 144 a multiple of 18? Use a T-chart of factor pairs to solve this problem.

SOLUTION

144	
1	144
2	72
3	48
4	36
6	24
8	18
9	16
12	12

From the T-chart, you can see that $8 \cdot 18 = 144$ and therefore 18 is a factor of 144.

PROBLEM 3

a. Is the number 105 divisible by 15?

b. Is every number divisible by 15 also divisible by 3 and 5? Explain.

What distinction in the remainders did you notice between the examples of the divisible case and the not divisible case? Check with a few other examples to confirm that the distinction holds in those cases.

EXPLORATION 3: SIEVE OF ERATOSTHENES

This Exploration is based on an ancient method attributed to a famous Greek mathematician, Eratosthenes of Cyrene. The process involves letting a certain kind of number pass through the sieve leaving only another kind of number left in the sieve. Try the Exploration, and see for yourself.

1. Use the grid of the first 100 natural numbers in the rows of ten handout.
2. Mark out the number 1. We will see why in the next section.
3. Using a colored pencil or marker, circle the number 2, and then mark out every remaining multiple of 2 until you have gone through the whole list. What is a mathematical term for the marked out numbers?
4. From the beginning, with a different colored pencil or marker, circle the first number that is not marked out and not circled. Then, mark out all remaining multiples of

that number. Notice that some numbers are crossed out by two different colored pencils.

5. Repeat this process until you have gone all the way through the list.
6. Make a new ordered list of all the circled numbers. What do these numbers have in common? How is this list of numbers related to patterns from the possible rectangle activity?
7. You might have noticed that in the third round, some of the multiples of 3 were already crossed out in the second round. Find 3 such numbers. Why did this happen?

You might have noticed that your investigation involving the Sieve of Eratosthenes allowed you to find the prime numbers less than 100 quickly. After what number did you notice that all the numbers on the Sieve was either circled or crossed out? Can you explain how the process worked? At what point is it possible to know that all the numbers left in your chart are prime?

EXPLORATION 4: DIVISIBILITY RULES

Use the Sieve of Eratosthenes to explore the following:

1. What pattern do you notice about numbers that are multiples of 2? Make a conjecture for a rule to determine whether or not a number is divisible by 2.
2. What pattern do you notice about numbers that are multiples of 5? Make a conjecture for a rule to determine whether or not a number is divisible by 5.
3. What pattern do you notice about numbers that are multiples of 10? Make a conjecture for a rule to determine whether or not a number is divisible by 10.
4. What pattern do you notice about numbers that are multiples of 3? Make a conjecture for a rule to determine whether or not a number is divisible by 3.
5. What pattern do you notice about numbers that are multiples of 9? Make a conjecture for a rule to determine whether or not a number is divisible by 9.
6. What pattern do you notice about numbers that are multiples of 6? Make a conjecture for a rule to determine whether or not a number is divisible by 6.

We summarize the divisibility rules that can be useful for determining whether any given number is divisible by numbers 2, 3, 5, 6, 9, 10.

If this is true about a number:	Then the number is divisible by:
The ones digit is even	2
The sum of the digits is divisible by 3	3
The last digit is 0 or 5	5
The number is divisible by 2 and 3	6
The sum of the digits is divisible by 9	9
The number ends in 0	10

EXPLORATION 5: SQUARE NUMBERS

1. Find all the factors of 36. How many factors of 36 are there?
2. Pair the factors so that the product is 36. How many pairs do you have that use 2 different factors? How many pairs use 2 of the same factors?
3. Find all the other numbers between 1 and 100 with an odd number of factors. What might these numbers be called?
4. Draw a number line, and locate the first 7 square numbers. Draw the associated square shapes below the number line.

EXERCISES

1. Find all the factors of the following numbers. You may wish to use a t-chart or table.

   a. 18 d. 56 g. 90 j. 17 m. 49

   b. 24 e. 6 h. 120 k. 51 n. 71

   c. 48 f. 32 i. 30 l. 45 p. 43

2. Classify the numbers from Exercise 1 as prime or composite. Then, write a sentence explaining how you know numbers are prime or composite.

3. In each of the following problems, values of n and d are given. Determine whether d is a factor of n. If d is a factor of the product n, find an integer q such that $n = dq$. Explain how you use any patterns or prior knowledge that help you answer the questions. Use long division only on the starred items.

	d (factor)	n	Is d a factor of n? Explain.
a.	5	26	
*b.	4	52	
c.	1	38	
*d.	8	77	
e.	35	0	
f.	4	81	
g.	13	195	
h.	6	78	
*i.	14	159	
j.	9	135	

4. Find all the prime numbers between the given pairs. Verify that your choice is a prime. Example: Find the prime numbers between 70 and 80. List: ~~70~~, 71, ~~72~~, 73, ~~74~~, ~~75~~, ~~76~~, ~~77~~, ~~78~~, 79, ~~80~~. So, only the numbers not crossed are prime numbers.

   a. 100 and 110 b. 110 and 130 c. 130 and 150

5. Find the first 10 multiples of the following numbers:

   a. 3 d. 8 g. 12 j. 24

   b. 4 e. 9 h. 13 k. 25

   c. 7 f. 11 i. 15

6. a. What is the least multiple of 6 that is greater than 43?

   b. What is the least multiple of 7 that is greater than 50?

7. a. What is the least multiple of 12 that is greater than 132?

   b. What is the least multiple of 15 that is greater than 120?

8. Sara has a large number of nickels in her purse.

   a. Can she pay for a pack of gum that costs $0.85? If so, how many nickels will she need? Explain your answer in terms of either factors or multiples. (In other words, write a sentence explaining your answer using the words factor or multiple.)

   b. If Sara only had dimes in her purse, could she pay for the same pack of gum with exact change? If so, how many dimes will she need? Explain your answer in terms of either factors or multiples.

9. Ms. Soto wants to make a wall decoration from the 60 feet of colorful ribbon she bought. She wants to use all the ribbon and wants each piece she cuts to be of equal length. List every possible number of ribbons she can cut using only whole numbers, and the length of each.

10. The local newspaper will be printing the names of students who made the honor roll for the entire year. They will split the names on the list into equal columns. There are 120 students on the list, and the newspaper wants to have fewer than 10 columns. How many different columns can they use, and how many names will be in each column?

11. a. Using your data from the Possible Rectangle Exploration, write all the factors of 30 in order from least to greatest.

    b. There is a natural way to pair up the positive factors of 30. What do you notice from the rectangle model that can help you pair up the factors? What factor pairs did you get?

12. The numbers 1, 2, 4, 5, 7, and 10 are six factors of 140 in numerical order. How can you use this information to find larger factors of 140? Find all those factors.

13. Numbers like 536 and 712 are divisible by 4, while 378 is not. Make a conjecture for a rule to determine whether a number is divisible by 4.

Spiral Review:

14. At 6 a.m. the temperature was $^-2^\circ$ C. At noon, the temperature had risen to 11°C. By how many degrees did the temperature increase?

15. Four friends attended a concert and agreed to share the cost equally. The total of the tickets was \$120, the rental car was \$65 and snacks and drinks were \$50. Which expression can be used to represent the amount, C, each friend should have paid?

    a. $C = (120 + 65 + 50)(4)$

    b. $C = (120 + 65 + 50) \div 4$

    c. $C = 120 + 65 + 50 + 4$

    d. $C = 120 + 65 + 50 - 4$

16. **Ingenuity:**

    In this problem, we will discover how to find long sequences of consecutive composite numbers.

    a. Verify that the number 60 is divisible by 2, 3, 4, 5, and 6

    b. Explain how we know that the numbers 60 + 2, 60 + 3, 60 + 4, 60 + 5, and 60 + 6 are all composite. More generally, if N is a positive integer that is divisible by 2, 3, 4, 5, and 6, explain how we know that $N + 2$, $N + 3$, $N + 4$, $N + 5$, and $N + 6$ are all composite.

    c. Based on your work in part b, you now have a sequence of five consecutive composite numbers: 62, 63, 64, 65, and 66. Now, find a sequence of nine consecutive composite numbers.

    d. Suppose you had a powerful calculator that could compute products as large as you wanted to compute. Explain how you would find a sequence of one thousand consecutive composite numbers.

17. **Investigation:**

    Anne and Sam are playing a number game. One of them writes down a number, and the two then write the number as a product of integers greater than 1. The winner is the player who writes the number as the product of the most integers. For example, if the number is 60, and Sam writes the number as 5×12, while Anne writes the number as $3 \times 2 \times 10$, then Anne is the winner, since she used three integers and Sam used only two.

a. Suppose the number is 80, and Sam writes $80 = 4 \times 4 \times 5$. Is there a product that Anne could write down that would beat Sam's product?

b. Suppose the number is 125, and Anne writes $125 = 5 \times 5 \times 5$. Is there a product that Sam could write down that would beat Anne's product?

c. Write down a product, using as many factors as possible, for the number 84. Is there more than one product that has the largest possible number of factors? If so, do these products have different factors?

d. Write down a product, using as many factors as possible, for the number 600.

18. **Challenge:**

Find the smallest prime number p such that $x^2 - y^2 + 1 = p$ where x and y are both multiples of 8 and x is larger than y.

SECTION 3.2
EXPONENTS AND ORDER OF OPERATIONS

In Section 2.1, we modeled multiplication by repeatedly adding an integer to itself. There are also situations in which it is useful to multiply a number repeatedly by itself.

EXAMPLE 1

Escherichia coli bacteria are more commonly known as E. coli. A scientist places one of the living bacteria in a petri dish. The number of bacteria in the dish doubles each hour. How many bacteria are living in the dish after 1 hour? 2 hours? 3 hours? 5 hours? n hours?

SOLUTION

Because the number of bacteria doubles each hour, after 1 hour there will be 2 bacteria. After 2 hours, there will be $2 \cdot 2 = 4$ bacteria. We can write this information using **exponential notation** as

Number of bacteria after the 1st hour $= 2 = 2^1$

Number of bacteria after the 2nd hour $= 2 \cdot 2 = 2^2 = 4$

Number of bacteria after the 3rd hour $= 2 \cdot 2 \cdot 2 = 2^3 = 8$

Continuing this pattern, the number of bacteria living in the petri dish after 5 hours is equal to $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$. If we let n = number of hours, there are 2^n bacteria after n hours.

Use a calculator to answer the following questions about Example 1:

1. How many bacteria will there be after 10 hours?
2. We need at least 10,000 bacteria for an experiment. When can we harvest this many bacteria?

DEFINITION 3.3: EXPONENTS AND POWERS

Suppose that n is a whole number. Then, for any number x, the nth **power** of x, or x to the nth power, is the product of n factors of the number x. This number is usually written x^n. The number x is usually called the **base** of the expression x^n, and n is called the **exponent**.

EXAMPLE 2

Rewrite the following repeated multiplication in exponential notation. Identify the base and exponent for each expression.

a. $2 \cdot 2 \cdot 2 \cdot 2$

b. $5 \cdot 5 \cdot 5$

c. $7 \cdot 7$

d. $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

e. $n \cdot n \cdot n \cdot n$

SOLUTION:

a. 2^4 The base is 2, exponent 4

b. 5^3 The base is 5, exponent 3

c. 7^2 The base is 7, exponent 2

d. 10^5 The base is 10, exponent 5

e. n^4 The base is n, exponent 4

PROBLEM 1

Rewrite the following expressions using repeated multiplication. Find the values for parts a through d.

a. 4^3

b. 3^4

c. $2^3 \cdot 3^2$

d. $5^2 \cdot 7^2$

e. m^3

f. $p^2 \cdot q^5$

EXPLORATION 1

By using the definition of exponential notation and multiplication, we see that:
$3^4 \cdot 3^6 = (3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{10} = 3^{4+6}$.

Compute the following products, showing all your work.

a. $3^2 \cdot 3^3$

b. $2^2 \cdot 2^3$

c. $3^3 \cdot 3^2$

d. $10^3 \cdot 10^5$

What pattern do you observe when multiplying numbers in exponential form with the same base? Explain.

PROBLEM 2

Compute the product: $2^5 \cdot 2^4$.
The pattern leads to the multiplication property for exponents.

PROPERTY 3.1: MULTIPLICATION OF POWERS

Suppose that x is a number and a and b are whole numbers. Then,

$$x^a \cdot x^b = x^{a+b}$$

Use your observation to determine the product in the problem below.

PROBLEM 3

Use these properties to rewrite the following products and then compute them. Use a calculator to compute both equivalent forms.

a. $4^3 \cdot 4^5$

b. $2^5 \cdot 2^5$

c. $10^3 \cdot 10^4$

d. $3^2 \cdot 4^3$

Special Cases: What do 4^1 and 4^0 equal?

We note that $4 \cdot 4 = 4^2 = 4^{1+1} = 4^1 \cdot 4^1$, so 4^1 must be the same as 4. We can use the same process for any number x: $x \cdot x = x^2 = x^{1+1} = x^1 \cdot x^1$, so $x^1 = x$.

What does 4^0 equal? Because $4 \cdot 4^0 = 4^1 \cdot 4^0 = 4^1 + 0 = 4^1 = 4 = 4 \cdot 1$, we see that multiplying by 4^0 is the same as multiplying by the number 1. We therefore assume that for any positive integer n, $n^0 = 1$.

Consider the number 4638, which we read as four thousand six hundred thirty eight.

Using place value and our notation of exponents, we can rewrite 4638 using expanded notation in the following way:

$$4 \cdot 1000 + 6 \cdot 100 + 3 \cdot 10 + 8 \cdot 1 = 4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$$

Start with the expression $4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$, or in calculator notation, $4 \times 10^3 + 6 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$. In what order can we perform the calculations in this expression so the sum equals 4638?

EXPLORATION 2

Compute the following, showing all your work.

$20 - 10 \div 2 + 3^3 - 9$

ORDER OF OPERATIONS

To compute 2^4 on a calculator, we enter 2^4. To multiply 3 by 5, we enter 3×5. How would we enter the following into a calculator:

a. $1 + 2^3$

b. $3 + 2 \cdot 5$

c. $2 \cdot 4^3$

d. $5 - 2 + 4$

What will the results be? Can you explain what the calculator is doing? You might wonder why the calculator does not always perform these calculations from left to right, as we read them. We can see that the order the calculator uses, which is called the **order of operations**, is natural by examining our place value system.

We summarize below the order in which mathematical operations are performed:

Order of Operations

- Compute the numbers inside any **p**arentheses.
- Compute any **e**xponential expressions.
- **M**ultiply or **d**ivide as they occur from left to right.
- **A**dd or **s**ubtract as they occur from left to right.

Why do these two problems have different solutions?

a. $7 \cdot 8 - 6 \div 2$

b. $7 \cdot (8 - 6) \div 2$

EXAMPLE 3

Generate equivalent numerical expressions to simplify the following expressions:

a. $3 - 5 + 4$

b. $2^3 \div 4 \cdot 2$

c. $(7 - 2 + 3) + 2 \cdot 3^2$

SOLUTION

a. This problem only has addition and subtraction, so we do the operations from left to right, first subtracting, then adding.

$$3 - 5 + 4 = (3 - 5) + 4 = {}^{-}2 + 4 = 2$$

b. First we simplify the exponent.

$$2^3 \div 4 \cdot 2 = 8 \div 4 \cdot 2$$

Then we do the multiplication and division, from left to right. Multiplication and division are done left to right in order, just like addition and subtraction are done left to right in order.

$$8 \div 4 \cdot 2 = (8 \div 4) \cdot 2 = 2 \cdot 2 = 4$$

c. We begin by simplifying inside the parentheses. We first subtract, then add, from left to right

$$(7 - 2 + 3) + 2 \cdot 3^2 = (5 + 3) + 2 \cdot 3^2 = 8 + 2 \cdot 3^2$$

Next we simplify the exponent, $3^2 = 9$, so the above becomes:

$$8 + 2 \cdot 3^2 = 8 + 2 \cdot 9$$

We then multiply, since we always multiply or divide before adding or subtracting:

$$8 + 2 \cdot 9 = 8 + 18.$$

Finally, we add or subtract, left to right.

$$8 + 18 = 26$$

PROBLEM 4

Compute the following, showing all of your work:

$4 + 2^3 \cdot 3 - (17 - 5) \cdot 2 + (17 - 5) \div 2$

EXERCISES

1. Rewrite each of the following multiplication expressions into expressions using exponents.

   a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

   b. $3 \cdot 3 \cdot 3 \cdot 3$

   c. $4 \cdot 4 \cdot 4 \cdot 5 \cdot 5$

   d. $(2 \cdot 2 \cdot 2 \cdot 2) + (3 \cdot 3 \cdot 3)$

   e. $n \cdot n \cdot n \cdot n \cdot n$

   f. $a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b$

2. Expand and compute each of the following. Tell which expression is greater or if they are equal.

   a. 4^3 or 3^4

   b. 3^6 or 6^3

   c. 5^3 or 10^2

   d. 4^2 or 2^4

3. Evaluate the following expressions:

   a. $3^2 + 3$

   b. $2^3 + 3^2$

   c. $3^2 \cdot 3$

   d. $2^3 \cdot 3^2$

4. Evaluate the following numerical expressions using Order of Operations:

   a. $22 - 3^2 + 6$

   b. $22 - (3^2 + 6)$

5. Rhonda has one sheet of paper. She cuts it into thirds and stacks the three sheets. If she completes this process a total of 5 times, how many sheets thick will the resulting stack be? Fun fact: she only has to complete the process 27 times before the stack reaches the moon.

6. Calculate the following:

   a. 10^1

   b. 10^2

   c. 10^3

   d. 10^8

   e. Explain how you would calculate 10^{200}.

7. Evaluate the following numerical expressions using Order of Operations:

   a. $7 \times 10^2 - 53 + (8 \times 3^2)$

   b. $7016 + (3 \times 10^2) \times 10 - 10^2 \div 10$

8. Six students are having a Skittles eating contest. The first student eats 1 Skittle. Each student after that must eat 4 times as many Skittles as the previous student. How many Skittles does the sixth student eat?

9. Evaluate the following numerical expressions using Order of Operations:

   a. $4 \times 2 + 48 \div 6 - 2^2 \times 4$

   b. $5^3 - 100 + 4\,(7 - 4) \div 6$

10. Compute the following:

   a. $(2 + 3)^2$ c. $(3 + 4)^3$ e. $(3 \cdot 5)^3$

   b. $(3 + 5)^3$ d. $(2 \cdot 3)^2$ f. $(3 \cdot 4)^3$

Spiral Review:

11. Rocky, the squirrel, is busy storing pecans for the winter. Sadie, the golden retriever, is busy finding Rocky's hidden pecans. The table below shows the number of pecans stored by Rocky and the number of pecans eaten by Sadie over the last 4 days. Which expression best describes the number of pecans still stored for the winter after the 4th day?

Day	Number Buried	Number Eaten
Monday	22	5
Tuesday	24	8
Wednesday	21	11
Thursday	15	4

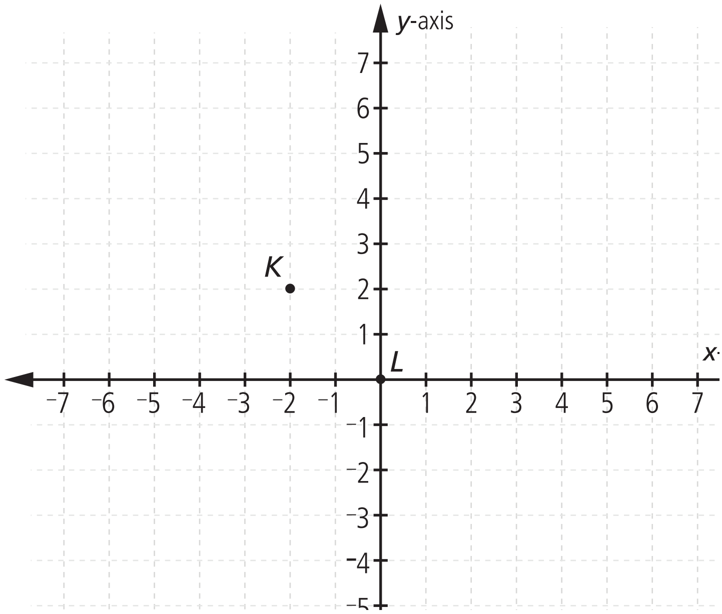
   a. $^-22 + 5 - 24 + 8 - 21 + 11 - 15 + 4$

   b. $22 + 5 + 24 + 8 + 21 + 11 + 15 + 4$

   c. $22 + 24 + 21 + 15 - 5 - 8 - 11 - 4$

   d. $22 + 24 + 21 + 15 - 5 + 8 + 11 + 4$

12. If point *K* is translated 5 units to the right and 3 down, what will point *K*'s new coordinates be?



13. What do the numbers 30, 12, 90, 60, and 24 have in common? List all common traits.

14. **Ingenuity:**

    Insert parentheses in the following expression so that the value of the expression is as large as possible.

    $5 + 3 \times 7 + 6 - 4 \times 2 + 6$

    For example, one way to insert parentheses would be:

    $(5 + 3) \times 7 + 6 - 4 \times (2 + 6) = 30$.

15. **Investigation:**

    Suppose that Donny has three shirts and three pairs of pants that he can wear to school. Donny has poor fashion sense and is willing to wear any shirt with any pair of pants, even if the colors do not go well together.

    a. Come up with three colors for Donny's shirts and three colors for his pairs of pants (use your imagination). Make a list or chart showing all of the combinations of a shirt and a pair of pants that Donny can wear to school. How many possible combinations are there?

b. Suppose that, in addition to having three shirts and three pairs of pants, Donny also has three baseball caps that he can wear to school. How many ways are there for Donny to choose a shirt, a pair of pants, and baseball cap to wear to school?

c. Suppose that Donny also has three jackets that he can wear. How many ways are there for Donny to choose a shirt, a pair of pants, a baseball cap, and a jacket to wear?

d. What pattern do you notice in parts a through c? Write a sentence or two explaining your findings.

SECTION 3.3
PRIME FACTORIZATION

One reason we are so interested in prime numbers is that they are the building blocks of the integers. In the previous section, we learned that a prime number is a positive integer greater than 1 that can be written as a product of two positive integers in only one way ignoring the order of the factors. For example,

$$13 = 13 \cdot 1 = 1 \cdot 13.$$

We cannot write 13 as a product of two positive integers without using the number 13 itself. In this way, the number 13 cannot be divided into smaller equal whole parts. We can, however, use the number 13, together with other prime numbers, to form many other numbers:

$$13 \cdot 2 = 26$$

$$13 \cdot 3 = 39$$

$$13 \cdot 5 = 65$$

$$13 \cdot 7 = 91$$

Primes are combined in various ways to form different positive integers. In some cases, you might use a certain prime factor more than once when building a number:

$$4 = 2 \cdot 2 = 2^2$$

$$8 = 2 \cdot 2 \cdot 2 = 2^3$$

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

$$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$

In a similar way, every positive integer greater than 1 that is not prime can be written as the product of prime factors. In other words, each positive integer can be identified by its prime factors and the number of times each of these factors occurs. For example, n is a positive integer that is composed of 3 factors of 2, 1 factor of 3, and 2 factors of 5. What is the exact value of n? Does it matter if n is written as $2 \cdot 3 \cdot 5 \cdot 2 \cdot 5 \cdot 2$ or $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$? Is there an accepted way to organize these factors as a product of n?

We can answer these questions with the following:

THEOREM 3.1: FUNDAMENTAL THEOREM OF ARITHMETIC

If n is a positive integer, $n > 1$, then n is either prime or can be written as a product of primes

$$n = p_1 \cdot p_2 \cdot \ \dots \ \cdot p_k ,$$

for some prime numbers $p_1, p_2, \dots, p_k$ such that $p_1 \le p_2 \le \dots \le p_k$, where k is a natural number. In fact, there is only one way to write n in this form.

Positive integers greater than 1 are either prime or products of primes. Note that some of these primes might be repeated, as in the examples above. But the Fundamental Theorem of Arithmetic, or FTA, also tells us that there is only one way to decompose a given integer into prime factors with the factors written in order. The Fundamental Theorem of Arithmetic is also referred to as the Unique Prime Factorization Theorem. That is, if two different people correctly write a positive integer as a product of prime factors, their products always contain exactly the same prime factors, whether the order is the same or not. When using the FTA, however, the prime factors should be in increasing order.

The prime factorization of an integer gives us useful information about its factors. Factoring a number into primes usually takes some trial and error, but there is a technique that makes the process easier. Let's look at an example.

EXAMPLE 1

Find the prime factorization of 60.

SOLUTION

When looking for prime factors of a positive integer, it is useful to have a list of prime numbers. Look at the Sieve of Eratosthenes from the activity in Section 3.1 to confirm that the first few primes are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \dots$$

Work your way through these primes to see if any of them are factors of 60. Divide 60 by 2 to get a quotient of 30 and a remainder of 0. So, 2 is a factor of 60, with

$$60 = 2 \cdot 30.$$

It is tempting to go on to the next prime in our list, but remember that a prime might appear more than once in a prime factorization. So before you continue, notice that 30 is even and realize that 2 is a factor of 30. Dividing, you will find that $30 = 2 \cdot 15$. So,

$$60 = 2 \cdot 2 \cdot 15.$$

Using the multiplication facts, divide 15 by 3 to find that $15 = 3 \cdot 5$. So,

$$60 = 2 \cdot 2 \cdot 3 \cdot 5.$$

Because each of the factors 2, 2, 3, and 5 is prime, you are through. You have written 60 as a product of prime factors. A useful way to write your result is to use exponents:

$$60 = 2^2 \cdot 3 \cdot 5$$

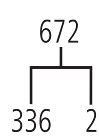
The process of finding the prime factors of a number is called prime factorization. We also use the same term to describe the result of the process. For example, the prime factorization of 60 is $2^2 \cdot 3 \cdot 5$.

EXAMPLE 2

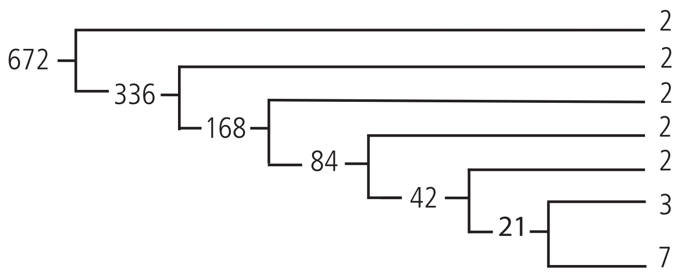
Find the prime factorization of 672.

SOLUTION

If you want, use the same step-by-step process you used in the previous example. But there is a faster way to write the information. You can track the prime factors using a **tree diagram** or **factor tree**. You know that 2 is a prime factor of 672 because 672 is even, so start by dividing 672 by 2:



Continue factoring by 2 until the remaining number is odd:



Remember that $7 \cdot 3 = 21$, so you know that the last 2 factors in the process are 7 and 3. The prime factorization of 672 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^5 \cdot 3 \cdot 7$.

Notice that the processes used in Examples 1 and 2 are related. The second process is the same as the first, except that it organizes the factoring process more visually using a factor tree, which can be helpful when working with larger numbers. Notice that the tree can be arranged vertically or horizontally.

PROBLEM

Find the prime factorization of each number below and write the answer using exponents.

a. 240   b. 306

We can also use prime factorization to generate equivalent numerical expressions.

EXAMPLE 3

Use prime factorization to simplify the following expression and leave your answer in factored form:

a. $15^2 \cdot 3^2$ b. $10^4 \div 2^3$

SOLUTION

a. The most straightforward way to simplify this is to use order of operations:

$15^2 \cdot 3^2 = 225 \cdot 9 = 2025.$

However, there is another way using prime factorization. We factor $15 = 3 \cdot 5$.

So, $15^2 \cdot 3^2 = (3 \cdot 5)^2 \cdot 3^2$. Now we use properties of exponents,

$(3 \cdot 5)^2 = (3 \cdot 5) \cdot (3 \cdot 5) = 3^2 \cdot 5^2$

So, $(3 \cdot 5)^2 \cdot 3^2 = 3^2 \cdot 5^2 \cdot 3^2 = 3^2 \cdot 3^2 \cdot 5^2 = 3^4 \cdot 5^2 = 81 \cdot 25 = 2025$.

Either way gives you the same answer!

b. Again, we could use order of operations, $10^4 \div 2^3 = 10000 \div 8 = 1250$. A different way is to use prime factorization,

$$10^4 \div 2^3 = \frac{(2 \cdot 5)^4}{2^3} = \frac{(2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5)}{2 \cdot 2 \cdot 2}$$

$$= \frac{(2 \cdot 2 \cdot 2) \cdot 2 \cdot (5 \cdot 5 \cdot 5 \cdot 5)}{2 \cdot 2 \cdot 2}$$

$$= \frac{(2 \cdot 2 \cdot 2)}{(2 \cdot 2 \cdot 2)} \cdot \frac{2 \cdot (5 \cdot 5 \cdot 5 \cdot 5)}{1}$$

$$= 2 \cdot (5 \cdot 5 \cdot 5 \cdot 5)$$

$$= 1250$$

While order of operations at first may seem much simpler, when you have large exponents, using prime factorization can sometimes make the problem easier. In the exercises, try using both ways, and see which you like better.

EXERCISES

1. Factor each of the following integers into primes. Write the integer as a product of its prime factors, using exponents when there are repeated prime factors.

a. 10	d. 31	g. 64	j. 51	m. 169
b. 8	e. 28	h. 72	k. 77	n. 360
c. 25	f. 60	i. 48	l. 84	o. 1225

2. The prime factorization of a number can be used to find a **perfect square**, which is the square of an integer. Look at the numbers given in Exercise 1. Are any of these perfect squares? If so, which ones? How can you use their prime factorization to determine whether each is a perfect square?

3. A **perfect cube** is an integer n that can be written in the form $n = k^3$, where k is an integer. Some examples of perfect cubes are

$$0^3 = 0, \ 1^3 = 1, \ 2^3 = 8, \ 3^3 = 27, \ 4^3 = 64, \ \ldots.$$

   How can you use the prime factors of a number to determine whether it is a perfect cube?

4. What is the smallest positive integer that has three different prime factors?

5. Write the prime factorizations of 125 and 144. Looking at your factorizations, explain what the answers have in common and what their main difference is in terms of what you learned in Exercises 2 and 3.

6. Numbers like 25 and 120 are divisible by 5. Make a conjecture (educated guess) about what numbers divisible by 5 look like.

7. Make a conjecture about what numbers divisible by 10 look like.

8. Determine as efficiently as possible whether each of the following numbers is prime or composite. Prove your answer with a factor pair if you believe the number is composite.

a. 105	c. 117	e. 213
b. 113	d. 153	f. 239

9. Find all the prime factors of each of the following numbers: 2, 4, 8, 16, and 32. Write your prime factorization as a list of primes using exponents. Using the pattern you observed, find the prime factorizations for 64 and 128.

10. Use prime factorization to simplify the following expressions. Leave your answer in prime factored form.

a. $2^{10} \div 2^2$	c. $(3^4)^3$	e. $10^3 \cdot 12^2$
b. $6^4 \cdot 3^2$	d. $35^4 \div 5^2$	

Spiral Review:

11. List 42, 39, $^{-}5$, $^{-}39$, 15, and $^{-}16$ in order from least to greatest.

12. What is the value of the expression below?

    $6 + 6\,(12 \div 3)^2$

13. **Ingenuity:**

    For the purposes of this problem, we say that a positive integer is **balanced** if it has exactly as many odd divisors as it has even divisors. How many of the integers from 80 to 90 (inclusive) are balanced?

14. **Investigation:**

    In this Investigation, we'll explore a useful technique for counting the divisors of a positive integer. Let us begin with the number 72 as an example. We know that the prime factorization of 72 is $72 = 2^3 \times 3^2$.

    a. Make a table of the divisors of the number 72, as shown below. For each divisor, find the prime factorization, and write it in the form $2^m \times 3^n$, where m and n are whole numbers. For this exercise, if a prime p is not part of a divisor's prime factorization, write it as p^0. For example, since 2 is not a factor of 9, we will write $9 = 2^0 \times 3^2$ in the table. Be sure to include the divisors 1 and 72 in your table. Your table may need to be bigger than the one shown.

Divisor	Prime Factorization
9	$2^0 \times 3^2$
6	$2^1 \times 3^1$
24	$2^3 \times 3^1$

    b. Now that you have recorded all of the divisors of the number 72 in the table, see if you can find a nice way to order the rows the table according to the prime factorizations of the divisors. (Your ordering will not necessarily have the divisors in increasing order.)

c. What do you notice about the prime factorizations of the divisors of 72? How many divisors are there? How could you have predicted this by looking at the prime factorization of 72?

d. Based on your work in parts a through c, can you predict how many divisors the number 100 will have? How about the number 112?

e. How many divisors does the number 180 have?

15. **Challenge:**

Find the smallest positive integer whose prime factorization uses each odd digit (1, 3, 5, 7, 9) exactly once.

SECTION 3.4
COMMON FACTORS AND THE GCF

We know that when we multiply 3 and 8 to obtain the product 24, the numbers 3 and 8 are factors of 24. Notice that $2 \cdot 12$ also equals 24, and 2 and 12 are factors of 24. You have discovered several other numbers that are factors of 24. In this section, we examine how we can use what we know about factors to determine factors common to two or more numbers. When is it necessary to find common factors? Let's examine the following situation to see.

EXPLORATION 1

To prepare for a frog-jumping contest, Fernando decided to train a group of his fellow frogs. Each frog was trained to jump a certain length along a number line starting at 0. He trained a 1-frog to jump a distance of 1 unit in each hop. He also trained a 2-frog to jump 2 units, a 3-frog to jump 3 units, and so on. The frogs always start at the zero point on the number line. Now Fernando wants to know which frogs will land on certain locations on the number line.

1. Which of his frogs will land on both the locations 24 and 36?
2. Which is the longest jumping frog that will land on both 24 and 36? Explain why this answer makes sense.
3. What is the longest jumping frog that will land on both 20 and 32?
4. What is the longest jumping frog that will land on both 24 and 25?

If a frog lands on 24, then the length of its jump is a factor of 24. So, if a frog lands on both 24 and 36, then the length of its jump is a factor of 24 and 36.

DEFINITION 3.4: COMMON FACTOR AND GCF

Suppose m and n are positive integers. An integer d is a **common factor** of m and n if d is a factor of both m and n. The **greatest common factor**, or **GCF**, of m and n is the greatest positive integer that is a factor of both m and n. We write the GCF of m and n as GCF(m, n).

If you revisit Exploration 1, you will see that in question 2, the GCF of 24 and 36 is 12. From question 3, the GCF of 20 and 32 is 4, and the GCF of 24 and 25 is 1.

EXPLORATION 2

Suppose you have two types of string with different lengths. The cotton string is 120 inches long, and the nylon string is 72 inches long. Determine every possible integer length both of the strings can be cut so that each piece is the same length and there are no string pieces leftover. Make a list of all the possible common lengths. What is the longest common length possible? In this Exploration, each piece must be cut into a positive integer length with no fractions and no string left.

There are several different ways to calculate the GCF of two numbers. Here is one way that reinforces the term Greatest Common Factor.

EXAMPLE 1

Find the GCF of 15 and 25 using a T-chart.

SOLUTION

Find the GCF of two numbers by first listing all the factors of each of the numbers. Find all the common factors of the two numbers, then choose the greatest.

15	
1	15
3	5

25	
1	25
5	5

The common factors of 15 and 25 are 1 and 5, and the GCF of 15 and 25 is 5 because it is the larger of the two common factors.

EXAMPLE 2

List the common factors of 27 and 32, then find the GCF.

SOLUTION

Use the same method you used in the previous example. Using a T-chart, first list the factors of each number.

27	
1	27
3	9

32	
1	32
2	16
4	8

27: 1, 3, 9, 27

32: 1, 2, 4, 8, 16, 32

In this case, there is only one common factor: 1. Therefore, the GCF of 27 and 32, written also as GCF(27, 32), is 1. There is a term to describe a relationship between numbers whose GCF is 1.

DEFINITION 3.5: RELATIVELY PRIME
Two integers m and n are **relatively prime** if the GCF of m and n is 1.

From the definition above, the numbers 27 and 32 are relatively prime. Notice that neither 27 nor 32 are prime numbers. If we consider two prime numbers like 3 and 7, what is their GCF? Check a few more examples. Make a generalization about the GCF of any two prime numbers.

PROBLEM 1

Find the common factors and GCF of the following pairs of numbers. State whether the numbers are relatively prime or not.

a. 15 and 22 b. 39 and 65 c. 7 and 11

Find the GCF of a pair of larger numbers like 108 and 168 using the process of first finding factors, then the common factors, and eventually the greatest common factor.

First, list all the factors of 108. Then, list all the factors of 168. Make sure you have 12 factors for 108 and 16 factors for 168. There are many factors to find. If you do not have them all listed, go back and find them.

Determine all the factors the two numbers have in common. You should find that the common factors are 1, 2, 3, 4, 6, 12. From this list, you can see that the greatest common factor of both 108 and 168 is 12. As you discovered, this method for finding the GCF works well. However, the more factors the numbers have, the more time it takes to make the list of factors for each number. Fortunately, prime factorization makes finding the GCF of two numbers easier.

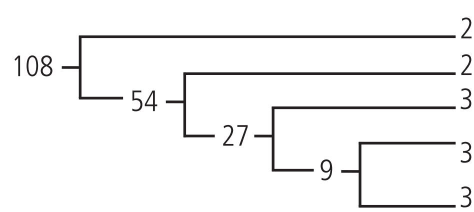
Here is an example of the efficiency of prime factorization.

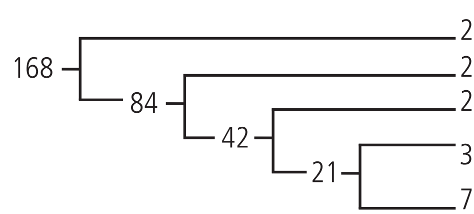
EXAMPLE 3

Find the GCF of 108 and 168 using prime factorization.

SOLUTION

Start by finding the prime factors of 108 and 168.
Use factor tree diagrams to do this:





Recall that the prime numbers are the building blocks of the integers. If you want to find the GCF of two integers, find the building blocks, or prime factors, the two numbers have in common. Remember, when doing this, it is helpful to write out the prime factors of each number in an organized way using exponents.

$$108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3$$

$$168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7$$

You have written the prime factors of 108 and 168 with and without exponents, lining up the equal prime factors. It is clear which factors the two numbers have in common:

$$\begin{array}{llllll} 108 = & \boxed{2 \cdot 2} & & \cdot \boxed{3} \cdot 3 \cdot 3 & & = 2^2 \cdot 3^3 \\ 168 = & \boxed{2 \cdot 2} & \cdot\, 2 & \cdot \boxed{3} & \cdot\, 7 & = 2^3 \cdot 3 \cdot 7 \end{array}$$

The prime factors have two 2's and one 3 in common. So $2 \cdot 2 \cdot 3 = 12$ is the greatest common factor of 108 and 168. It is also possible to check the common factors using long division. If there were a larger common factor, the quotients would have even more common prime factors, or higher powers of common prime factors, or both. However, the quotient after dividing 108 by 12 is 9, and the quotient after dividing 168 by 12 is 14. It is easy to check that 9 and 14 have no common factor other than 1, and so 9 and 14 are relatively prime.

EXAMPLE 4

Use the prime factorization method to find the GCF of 18 and 35.

SOLUTION:

The prime factorizations are $18 = 2 \cdot 3^2$; $35 = 5 \cdot 7$. Notice that there are no common prime factors. However, we should remember that a factor of any number is 1. The GCF of two numbers with no common prime factors will have 1 as its GCF.

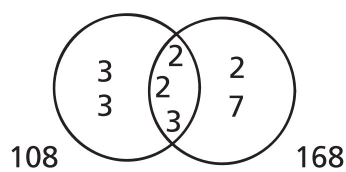
PROBLEM 2

Use the prime factorization method to find the GCF of each pair of numbers:

a. 105 and 225 b. 56 and 60 c. 42 and 65

A visual representation called a **Venn diagram** might help you to see how the GCF of numbers like 108 and 168 is constructed from the prime factorization. To make the Venn diagram, draw a circle for each number you are considering. Inside each circle write all its prime factors. If there are common prime factors, then the circles will intersect and the common primes will be in the area common to both, or in the intersection of the circles. If the circles have no common prime factors, then you conclude that the only common factor is 1. Remember, 1 is always a common factor for any pair of integers, and it is the greatest common factor if there are no other common factors.

The Venn diagram will look like this, and the factors for the GCF can be found in the circles' overlap. GCF (108, 168) = 2 · 2 · 3 = 12.



PROBLEM 3

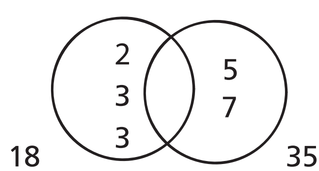
Compute the GCF of 196 and 210. Use a Venn diagram and the method from Example 3, and decide which you prefer.

EXAMPLE 5

Let us revisit the problem of finding the GCF of 18 and 35 again.

SOLUTION

The Venn Diagram approach will show that there are no common prime factors in the two circles.



In such cases the GCF will be 1.

EXERCISES

1. For each pair of integers below, find all of the common factors of the two integers. Then find the GCF of the two integers.

   a. 15 and 10
   b. 12 and 36
   c. 21 and 24
   d. 30 and 41
   e. 55 and 70
   f. 45 and 72
   g. 64 and 80
   h. 120 and 144

2. In each part of this exercise, the prime factorizations of two numbers are given. First, use the prime factorizations to find the GCF of the two numbers. Then, compute (find the value of) the two numbers from their prime factors.

   a. $2 \cdot 2 \cdot 2$
      $2 \cdot 2 \cdot 3$
   b. $2 \cdot 3 \cdot 3$
      $2 \cdot 3 \cdot 5$
   c. $2 \cdot 2 \cdot 3^3$
      $3 \cdot 3 \cdot 7$
   d. $2 \cdot 3 \cdot 7$
      $2 \cdot 5 \cdot 5$
   e. $2 \cdot 3 \cdot 7$
      $5 \cdot 11$
   f. $2^3 \cdot 3^2$
      $2^2 \cdot 3 \cdot 11$
   g. $3 \cdot 5^2 \cdot 7$
      $2 \cdot 3 \cdot 7$
   h. $a^3 \cdot b^2$
      $a^2 \cdot b \cdot c$

3. For each pair of integers below, find the GCF of the two integers using prime factorization.

   a. 16 and 40
   b. 7 and 17
   c. 60 and 72
   d. 63 and 75
   e. 125 and 200
   f. 144 and 168
   g. State which of the above pairs of numbers is relatively prime. Explain your reasoning.

4. What is the GCF of two prime numbers p and q? Explain your reasoning.

5. Josh has 12 basketball cards, 20 football cards, and 24 baseball cards. He wants to separate them into mini albums that contain an equal number of each type of sports card.

   a. What is the greatest number of mini albums he will be able to make if all cards are to be used?

b. How many of each type of sports card will he put into each mini album?

6. Julianne is making necklaces to sell at the fair. She wants to use all the beads she has left from another project. This includes 30 blue beads, 48 red beads, and 36 yellow beads. How many of each color of bead will each necklace contain if she uses all the beads and makes each necklace identical?

7. Flora's flower shop makes balloon arrangements. She has 32 yellow balloons, 24 orange balloons, and 16 red balloons. Each arrangement must have the same number of each color. What is the greatest number of arrangements that Flora can make if every balloon is used?

8. Hiroko has 42 green jelly beans, 28 blue jelly beans, and 14 orange jelly beans. She wants to share her jelly beans evenly among her friends. What is the greatest number of friends Hiroko can divide the jelly beans among?

9. Tina hosted a party for a certain number of guests. She prepared a vegetable tray with 14 celery sticks and 21 carrot sticks. If all the vegetables were eaten, and each guest had the same number of celery sticks and the same number of carrot sticks as Tina had, how many guests were at the party?

10. a. Make a list of all the amounts of money less than a dollar that you can make with an endless supply of quarters and dimes.

    b. How many cents is a quarter worth? How many cents is a dime worth? What is the GCF of these two numbers?

    c. What is the relationship between your answers in parts a and b?

11. Find the GCF of 10^6 and 6^{10}.

Spiral Review:

12. Holly bought breakfast for herself and 2 friends. The cost of each breakfast was between \$6 and \$7, including tax. Which of the following could not be a total cost for the breakfasts that Holly bought?

    a. \$19.20 b. \$17.80 c. \$20.50 d. \$18.25

13. Bryan works at the grocery store and earns \$11 per hour. If Bryan works 38 hours

each week, which expression could be used to determine his total earnings for 1 year?

a. 11(38)

b. 11(52)

c. 11(38)(52)

d. 11(38)(12)

14. **Ingenuity:**

Suppose that the product of two integers is 63,000. What is the greatest possible value of their GCF?

15. **Investigation:**

Suppose we want to find the GCF of the numbers 8407 and 8457. Remember that in the frog-jumping model of the GCF, the GCF of 8407 and 8457 is the number of the highest-numbered frog that will land on both 8407 and 8457 on the number line, assuming that all frogs start at the number 0.

a. Explain why no frog numbered higher than 50 can land on both 8407 and 8457.

b. Can the 33-frog land on both 8407 and 8457? How about the 29-frog? The 25-frog? The 17-frog? The 10-frog? What do you notice?

c. Make a list of the frogs with numbers from 1 to 50 that could land on both 8407 and 8457.

d. Assuming that all frogs start at 0, which of the frogs you listed in part c actually do land on both 8407 and 8457?

e. What is the GCF of 8407 and 8457?

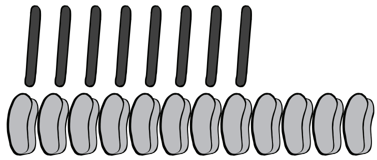
f. What is the GCF of 54218 and 54418?

16. **Challenge:**

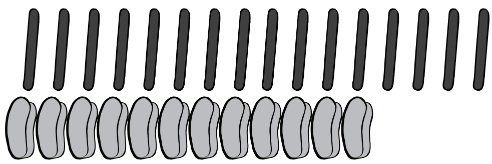
Find the GCF of 123,456,789,011 and 314,159.

SECTION 3.5
COMMON MULTIPLES AND THE LCM

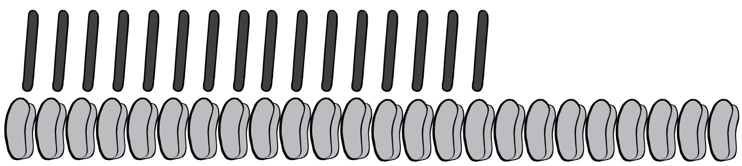
Have you noticed that hot dogs often come in packages of eight, and hot dog buns come in packages of twelve? When people plan to cook hot dogs, they tend to buy one package of hot dogs and one package of buns. But if they do this, they are left with four extra buns.



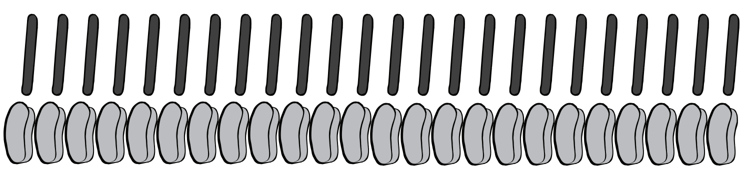
Some people who pay for the extra hot dog buns don't want to waste them. What can they do? They could buy another package of eight hot dogs:



But now there are four extra hot dogs without buns. If they buy more buns:



There are eight buns without hot dogs, even more extra buns than the first time. Will this process ever end? Try buying one more package of hot dogs:



Aha! We have finally reached a point where we have exactly the same number of hot dogs and buns. Of course, in order to get there, the customers had to buy two packages of buns and three packages of hot dogs. Maybe they can freeze the rest.

What happened mathematically with the hot dogs and buns? One way to organize the number of hot dogs and the number of buns is:

Hot dogs:	8	16	**24**	32	40	48	56	64	72…
Buns:		12	**24**	36		48	60		72…

Look for some positive integer N where anyone could buy exactly N hot dogs and N buns. Finding the number gives the consumer the number of hot dogs and the number of buns to buy so that they come out evenly.

It is easy to see that 24 is the smallest positive integer that is on both lists. So, the smallest value of N for both items is 24. Notice that the numbers on the first list are the multiples of 8. This makes sense because it is only possible to buy 8 hot dogs at a time. The numbers on the second list are the multiples of 12 because buns come only in packages of 12. So, 24 is the smallest positive integer that is a multiple of both 8 and 12. Mathematicians have a term for this:

DEFINITION 3.6: COMMON MULTIPLE AND LCM

Integers a and b are positive. An integer m is a **common multiple** of a and b if m is a multiple of both a and b. The **least common multiple**, or **LCM**, of a and b is the smallest positive integer that is a common multiple of a and b. We write the LCM of a and b as LCM(a, b).

3.5

Notice that 48 and 72 are also common multiples of 8 and 12, but not the least.

As we found with the GCF, there are several ways to find the LCM of two numbers. Try some examples:

EXAMPLE 1

Find the LCM of 5 and 7.

SOLUTION

One way to find the LCM of two numbers is to list the positive multiples of each number in increasing order until you find an integer that is on both lists. For example, start by writing the first fourteen positive multiples of 5 and the first 10 positive multiples of 7:

Multiples of 5:

5, 10, 15, 20, 25, 30, **35**, 40, 45, 50, 55, 60, 65, 70, …

Multiples of 7:

7, 14, 21, 28, **35**, 42, 49, 56, 63, 70, …

Notice that 35 is on both lists. 70 is also on both lists. In fact, if we continued listing the multiples, we would find other common multiples of 5 and 7. However, the smallest positive integer that is a multiple of both 5 and 7 is 35. That means 35 is the LCM of 5 and 7.

EXAMPLE 2

Radio station KISS broadcasts a commercial every 22 minutes. WILD broadcasts a commercial every 12 minutes. If the two stations broadcast their commercials at 3:20, when is the next time their commercials will air at the same time?

SOLUTION

Try the same method on 22 and 12 as you did in Example 1. Start by writing the first ten multiples of each number:

Multiples of 22:

22, 44, 66, 88, 110, 132, 154, 176, 198, 220, …

Multiples of 12:

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, …

There appear to be no common multiples from the lists. However, the two lists do not stop after ten entries; they go on forever. So, to find a common multiple of 22 and 12, try extending the two lists:

Multiples of 22:

22, 44, 66, 88, 110, **132**, 154, 176, 198, 220, 242, 264, 286, 308, …

Multiples of 12:

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, **132**, 144, 156, 168, …

The number 132 is on both lists. Therefore, 132 is the LCM of 22 and 12. The LCM was on the original list of multiples of 22, but it was necessary to extend the list of multiples of 12 to reach 132.

Remember that there are 60 minutes in 1 hour. 132 minutes is equal to 120 minutes + 12 minutes or 2 hours and 12 minutes. We add this time to 3:20 and get 5:32 as the next time their commercials will air at the same time.

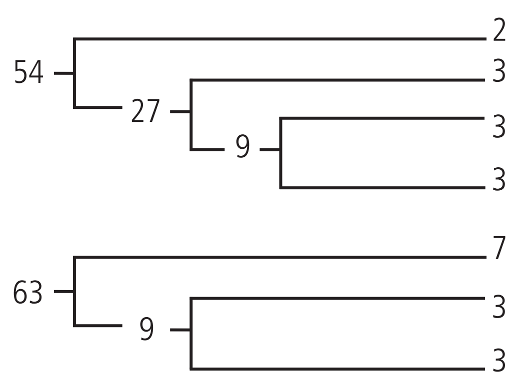
As you have seen, with very large numbers, it might be necessary to write many multiples to find their LCM. This can be very time-consuming, so it would be better to have a more efficient method for computing the LCM of two numbers, just as you did when looking for the GCF. Fortunately, prime factorization comes to the rescue again.

EXAMPLE 3

Find the LCM of 54 and 63.

SOLUTION

First factor 54 and 63, using prime factorization:



The prime factorizations are $54 = 2 \cdot 3^3$, and $63 = 3^2 \cdot 7$. Again, it is useful to express the prime factors with the exponents, aligning like factors:

$$54 = 2 \cdot 3^3 \cdot 7^0$$

$$63 = 2^0 \cdot 3^2 \cdot 7$$

To find a number that is a multiple of both 54 and 63, the multiple must have the prime building blocks that both 54 and 63 have. Before continuing, consider how you might be able to construct a multiple of 54 and 63 using the prime factors for each of the numbers. How can you construct the smallest such multiple?

Now look at the method for finding the LCM of numbers using their prime factorization. The factors of each are as follows: 54: $3 \cdot 3 \cdot 3 \cdot 2$ and 63: $3 \cdot 3 \cdot 7$. By examining the two sets of prime factors, you can see that a common multiple must include 2, 3, and 7. However, $2 \cdot 3 \cdot 7 = 42$ is not a multiple of 54 or 63. Because 54 has three factors of 3 and 63 has two factors of 3, to include both numbers, use three factors of 3. Why won't two factors of 3 be enough? Now multiply the factors 2, 3^3, and 7.

$$2 \cdot 3^3 \cdot 7 = 378$$

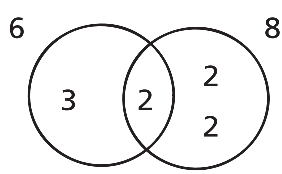
This is the smallest integer that contains all of the building blocks, or prime factors, in both sets of prime factors. Therefore, 378 is the LCM of 54 and 63.

PROBLEM 1

Find the LCM of each pair of numbers below using the prime factorization method:

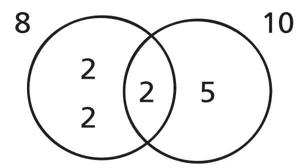
a. 36 and 42 b. 17 and 18 c. 13 and 52

Another approach to solving for the LCM is to look at the Venn diagram, as you did in Section 3.4, when you worked with GCFs. Examine the prime factors of 6 and 8. The Venn diagram includes the prime factors for each number in their respective circles. Note the common factors in the overlapping part of the circles.



A multiple of 6 requires factors of 2 and 3; a multiple of 8 requires 2, 2, and 2. So, a multiple of 6 and 8 requires one 3 and three 2s as factors. The shortest list of factors that will produce a multiple of both is $2 \cdot 2 \cdot 2 \cdot 3$. The greatest common factor is 2 and we avoid double counting it in finding the LCM. Thus, to get the LCM of 6 and 8, take the product of the highest power of all the factors that occur in either number, that is $2^3 \cdot 3$, to get the LCM of 24. ***Remember to use the highest power of any prime for the LCM just as you used the lowest power of any common prime for the GCF***.

Try the Venn diagram approach to find the LCM of 8 and 10.



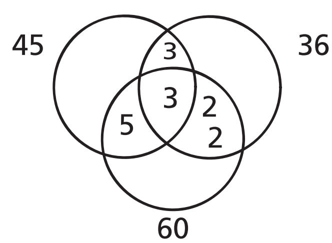
You will find that the Venn diagram is more practical when the numbers are larger and there are three numbers or more. For example, to find the least common multiple of 45, 36, and 60, write all the prime factors and notice which factors are common.

$$45 = 3^2 \cdot 5$$

$$36 = 2^2 \cdot 3^2$$

$$60 = 2^2 \cdot 3 \cdot 5$$

We can represent this information as a Venn diagram:



There is a factor of 3 common to both 45 and 36 so their intersection of the circle for the factors of 45 and the circle of factors of 36 contains a 3. 60 also has a factor of 3 so 3 is in the intersection of all three circles. Both 45 and 60 have a common factor of 5 but 36 does not so 5 is in the intersection of just the 45 circle and the 60 circle. The factor of 4 or 2^2 is in both 60 and 36, so 2^2 is in the intersection of the 60 circle and the 36 circle.

After you have separated the factors into the different regions in the Venn diagram, by multiplying all the numbers in the circles and their intersections you will get the LCM of 45, 36, and 60. Is your answer 180? If so, you are correct.

To summarize these investigations, the LCM of two integers a and b is equal to the product of all the primes that occur in the prime factorizations, raised to the highest exponent that appears in either factorization.

EXERCISES

1. Find the LCM of the following pairs of numbers by listing multiples of the two numbers until you find the first multiple common to both lists.

   a. 4 and 7
   b. 5 and 15
   c. 6 and 8
   d. 9 and 11
   e. 12 and 60
   f. 7 and 13

2. Find the LCM of the given numbers either by listing multiples of the two numbers until you find the first multiple common to both lists or by using the prime factorization of each number.

   a. 12 and 20
   b. 8 and 9
   c. 16 and 40
   d. 24 and 56

3. Were there any pairs of integers in the previous exercises where the integers were relatively prime? Explain whether you have to check every prime factor of both integers to find that the two integers are relatively prime.

   Based on the evidence in the previous exercises, if p and q are different prime numbers, what is the LCM of p and q? Use what you know about the common factor of two relatively prime numbers to explain why your previous answer makes sense.

4. Find the LCM for each of the following pairs of numbers:

   a. 11 and 88
   b. 7 and 84
   c. 39 and 13
   d. Look for a pattern in computing the LCM for a-c. Make a conjecture that explains this pattern.

5. Prime factors of two integers appear in each part of this problem. Compute the two integers from their prime factors. Use the prime factorization method from this section to find the LCM of the two numbers.

a. $2 \cdot 2 \cdot 2$
   $2 \cdot 3$

b. $2 \cdot 5$
   $3 \cdot 7$

c. $2 \cdot 2 \cdot 5$
   $2 \cdot 2 \cdot 3$

d. $2 \cdot 2 \cdot 2 \cdot 2$
   $3 \cdot 3 \cdot 3$

6. The prime factors of two integers appear below. Compute the two integers from their prime factors. Find the GCF and LCM for each pair of numbers.

a. $2^2 \cdot 3$
   $2 \cdot 3^2$

b. $3^2 \cdot 5$
   $3^3 \cdot 5^2$

c. $2^2 \cdot 11$
   $2 \cdot 3^2 \cdot 5$

d. $2^3 \cdot 3^2 \cdot 5$
   $2 \cdot 3^2 \cdot 5^3$

7. For each pair of integers below, use prime factorization to find the GCF and LCM.

a. 48 and 72 b. 90 and 60 c. 120 and 144

8. Jackie attends dance class every 7 days, and Lorianne attends every 3 days. If they both attended class today, in how many days will they both attend class together again?

9. Kristi, Nama, and Karen like to go work out at the recreation center. Kristi goes every other day, Nama goes every third day, and Karen goes every fifth day. They made an arrangement to carpool on the days they all go on the same day. If they all start today, find the next 3 times they will go together. Make a table to help you find the answers.

10. For your birthday party, you are planning to invite 45 friends. Invitations come in packages of 15, and party favors come in groups of 9. What is the least number of packages of invitations and party favors you need to buy in order to have enough for each friend and nothing left over?

11. Mr. Brown wants to reward students at his school for good behavior by giving them a fancy pencil and an eraser. Erasers come in containers of 45, while pencils come in containers of 60.

a. What is the least number of containers of each that Mr. Brown needs to purchase in order to have enough to have the same number of pencils as erasers?

b. How many of Mr. Brown's students would receive this reward? Explain.

12. Thomas is baking cookies for a party. Alice, Brad, Carlos, and Diana told Thomas that they definitely plan to attend. Eric told Thomas that he would like to attend, but he is not sure he can make it. Thomas wants to bake enough cookies so that he and all his guests can have the same number of cookies, whether or not Eric shows up. What is the smallest number of cookies that Thomas can bake for his party?

13. Randy is baking cookies for a family get-together at his house. He wants to bake enough cookies so that each person at the party, including himself, gets the same number of cookies. Randy knows that there are seven family members who will definitely come to the party. In addition, Randy has an aunt, an uncle, and two cousins who might or might not come. Because they are in the same family, either all four of these people will show up, or none of them will. What is the smallest number of cookies that Randy can cook if he wants to guarantee that everyone gets the same number of cookies, with no leftovers?

14. Use a Venn diagram to find the GCF and LCM for 15, 36, and 48.

15. For Junior Math Camp, Patty had to purchase snacks for 100 students. She plans to buy granola bars and juice bottles. The granola bars come in boxes of 12, and juice bottles come in cartons of 8. She needs to purchase the smallest number of boxes of granola bars and cartons of juice bottles so that each student gets exactly one granola bar and one juice bottle with the same number of granola bars and juice bottles left over.

    a. What is the least number of boxes of granola bars and cartons of juice bottles Patty needs to buy to have an equal amount of each?

    b. How many students will this feed? Is it enough?

    c. How many total boxes and cartons need to be purchased in order to feed all students and have the same number of granola bars and juice boxes left over?

Spiral Review:

16. Mrs. Waters has 28 desks in her classroom. She wants to arrange the desks into groups so that each group has the same number of desks. List all the possible numbers of desks that could be in each group.

17. What is the value of each expression below?

    a. $48 - 5 \cdot (3+5)$ b. $3(9 - 2)^2 + 4(5)$

18. **Ingenuity:**

    Suppose that m and n are positive integers such that the GCF of m and n is 20, and the LCM of m and n is 1800. What are the possible values of m and n? Find as many possible values as you can.

19. **Investigation:**

    The table below lists two variables m and n. Copy and fill out this table. In the column labeled GCF(m, n), write the GCF of m and n. In the column labeled LCM(m, n), write the LCM of m and n. As you fill out the table, what do you notice?

m	n	GCF(m, n)	LCM(m, n)
3	7		
4	6		
5	15		
8	15		
10	36		
20	48		
30	50		
30	65		
77	81		
96	100		

20. **Challenge:**

    The LCM of 1, 2, 3, ..., 98, 99 ends in how many zeros?

SECTION 3.R
CHAPTER REVIEW

1. List the factors of each of the following numbers using T-charts:

   a. 36   b. 64   c. 57   d. 29

2. List five multiples of 8 that are greater than 50.

3. Write a true statement for each of the following words (multiple, factor, and divisible) using the numbers 4, 5, and 20.

   a. ______________________________

   b. ______________________________

   c. ______________________________

4. Use the table below to indicate if each of the numbers in the left column is divisible by 2, 3, 5, 6, 9, and 10.

	2	3	5	6	9	10
873						
78						
280						

5. Determine whether each number is prime or composite. If composite, write a factor pair that proves your answer.

   a. 87   b. 124   c. 91   d. 121   e. 79

6. Find the prime factorization of each number using a prime factor tree. Write your answer using exponents if possible. If not possible, explain why.

   a. 72   b. 350   c. 169   d. 67

7. Find the greatest common factor of each pair of integers using one of the methods studied in Chapter 3.

   a. 20 and 25   c. 12 and 36

   b. 45 and 60   d. 16 and 17

8. Find the least common multiple of each pair of integers using one of the methods studied in Chapter 3.

   a. 8 and 12   c. 14 and 21

   b. 10 and 18   d. 15 and 20

9. Find the greatest common factor and the least common multiple for each of the following factorizations.

   a. $2 \cdot 2 \cdot 2 \cdot 2$ and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ GCF: LCM:

   b. $2 \cdot 2 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 5$ GCF: LCM:

   c. $2 \cdot 2 \cdot 3 \cdot 5$ and $2 \cdot 2 \cdot 3$ GCF: LCM:

10. Chandra is planning to have a party. Cookies are sold in packages of six. Juice drinks are sold in packages of eight. She is trying to figure out how many friends she is able to invite so that each friend gets exactly one cookie and one drink with nothing left over. How many friends can Chandra invite to her party?

11. One day, Nathan and Jeremy decide to help their neighbor Carlos keep his yard clean over the summer. Since Jeremy has more free time, he is planning to continue cleaning the yard every 4 days, while Nathan is planning to clean the yard every 7 days. When will be the second time both Jeremy and Nathan clean Carlos' yard on the same day?

12. Christopher is having friends over for a fun day of water games. He has 12 water guns and 18 water balloons. If all the water guns and water balloons are handed out and each person has the same number of water guns and water balloons, how many people were playing? How many water guns and water balloons did each person have?

13. Veronica is making balloon bouquets for a party. She has a package of 24 yellow balloons, 36 red balloons, and 60 blue balloons. If she makes identical bouquets using all the balloons, what is the largest number of balloon bouquets she can make? How many balloons of each color will each bouquet have?

FRACTIONS

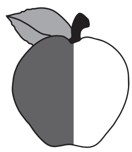
4

SECTION 4.1
MODELS FOR FRACTIONS

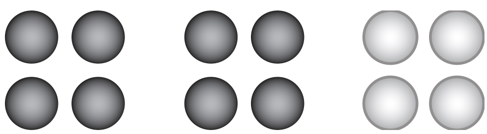
In this section you will examine some models that represent fractions.

First, think of examples of fractions that you have seen.

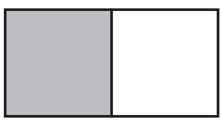
There are two basic ways to represent fractions: as a part of a whole or part of a group. Picture one apple divided into two equal parts. The shaded part represents half the apple.



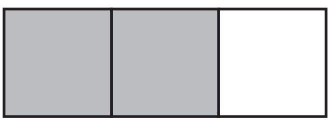
Next, divide a number of marbles into three equal groups. Two-thirds of the marbles is two of these three equal groups:



Now think of a fraction as part of a pan of brownie. This is called the **area** or **brownie model** for a fraction. Pictures representing the numbers one-half and two-thirds can look like these:



and



Of course, the pan need not always be a rectangle.

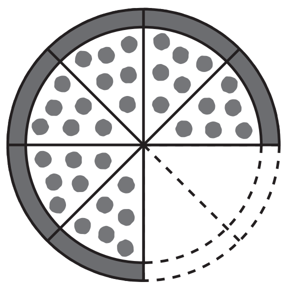
We write the two fractions mathematically as:

One half = $\frac{1}{2}$

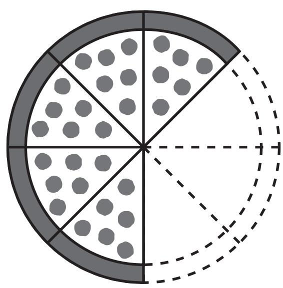
Two thirds = $\frac{2}{3}$

We use two numbers in writing a fraction, the **numerator** and the **denominator**. The numerator is above the denominator; it indicates how many parts are shaded. "Numerator" comes from the same root as "number." It counts the number of parts. The denominator tells how many parts the whole or group is divided into. "Denominator" comes from the same root as "name." It names the parts that the whole is divided into, like halves or thirds.

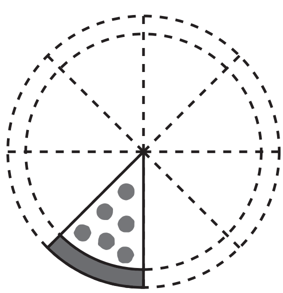
If you've ever had pizza, you probably noticed that it is already divided into several pieces that are roughly equal. Draw a circle to represent a pizza and draw lines to subdivide it into 8 equal pieces. Depending on how hungry you are, you choose to eat a certain number of pieces of pizza. You might have just two pieces, or two eighths of the pizza:



If you're a little hungrier, you might have three pieces, or three-eighths of one pizza:



And if you're extremely hungry, you might eat seven pieces, or seven-eighths of a pizza.



Use the area model to draw the fractions below. Draw the whole as a rectangle because it is easier to divide into equal pieces. For each fraction, identify the numerator and denominator. Then write the fraction mathematically.

a. Three-fourths
b. Two-fifths
c. One-fifth
d. Three-tenths

ACTIVITY: FOLDING PAPER

Materials: You will need several sheets of paper for this activity. Each sheet represents one whole.

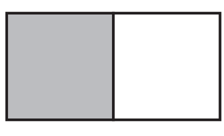
Step 1: Fold the paper to represent the number $\frac{1}{2}$. Write $\frac{1}{2}$ on each of the two parts of the folded paper. Is there more than one way to represent $\frac{1}{2}$?

Step 2: Use a new sheet to create $\frac{1}{4}$. How many parts equal to $\frac{1}{4}$ are there in the whole sheet of paper? What fraction represents three of these parts? What represents two parts of the paper?

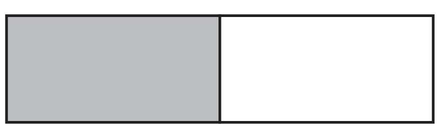
Step 3: Use a new sheet of paper to make a folded piece that has eight equal parts. Identify and make a list of as many fractions involving the denominator 8 as you can. Which of these fractions represent the same fractional part as the fractions in Step 1 and 2 with different denominators?

Step 4: Fold this same sheet of paper once more to make sixteenths. How many times did you fold the paper?

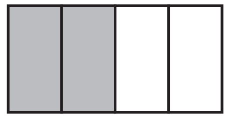
It is possible for two fractions with different numerators and denominators to represent the same amount. Consider the following example:

$\frac{1}{2}=$ 

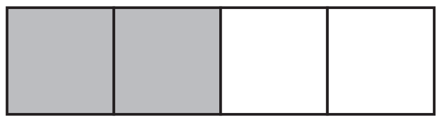
Notice that we can draw a larger picture, but it still represents the same part of the whole.

$\frac{1}{2}=$ 

If we divide a whole into 4 equal parts, 2 of the 4 parts can be written as the fraction

$\frac{2}{4} =$ 

or we can draw it as the larger picture

$\frac{2}{4} =$ 

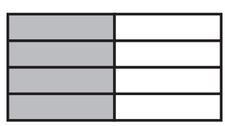
Are these both a representation of $\frac{2}{4}$? They do not look the same. So what is the difference?

Shading 2 equal parts out of 4 is equivalent to shading 1 part out of 2. This means the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are **equivalent**. Another way to show that the fraction $\frac{1}{2}$ is equivalent to the fraction $\frac{2}{4}$ is to take the picture representing $\frac{1}{2}$ and draw a horizontal slice as shown below:

$\frac{2}{4} =$ 

The horizontal slice doubles the numerator and also doubles the denominator, the number of parts the whole is divided into.

Suppose we make three horizontal cuts in the original rectangular model for $\frac{1}{2}$ to form equal sized pieces. What fraction is shaded?



Like the example above, the picture represents both $\frac{1}{2}$ and $\frac{4}{8}$. These fractions are equivalent. We write $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$ because the fractions represent the same part of a whole.

Remember, the "whole" is not just a geometric shape that represents one whole, like a circle or a rectangle, subdivided into equal parts. For example, the whole might be a class that has 8 girls and 8 boys. What fraction of the class is female? Male?

ACTIVITY: EQUIVALENT FRACTION CHART

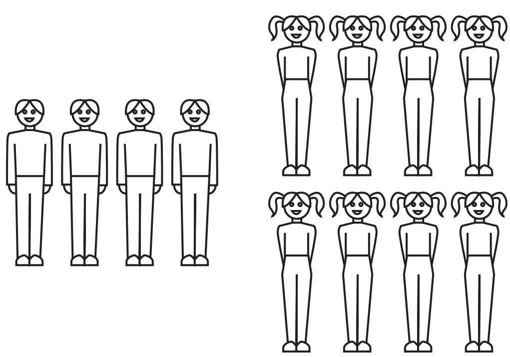
Start a chart of equivalent fractions. In each column, write fractions that are equivalent to each other. Be sure to use the fractions from the paper folding activity as well as any other fractions we introduce from now on in our discussions. You can also use a number line and label the corresponding points with the equivalent fractions. This may be a useful reference as you work with fractions in the sections to follow.

EXAMPLE 1

A class of 12 consists of 4 boys and 8 girls. We know that the fraction of the class that is male can be written $\frac{4}{12}$ and the fraction of the class that is female can be written $\frac{8}{12}$. Write another way to express the fraction of the class that is boys as well as the fraction of the class that is girls, using equivalent fractions.

SOLUTION

If you said $\frac{4}{12} = \frac{1}{3}$ as another way to express the fraction of the class that represents the boys, then you were correct. $\frac{2}{6}$ is also an equivalent fraction of the class that represents boys. In the original fraction, the denominator 12 represented the number of people in the class and the numerator 4 represented the number of boys in the class. Another way to write the fraction of the class that is girls is $\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$.



Remember, two fractions that represent the same part of a whole are called **equivalent fractions**.

This is an example of the **discrete model** for fractions. Discrete means being made up of individual parts, just like groups of students. In this model, we assume we have a collection or group of n objects and subgroup of m objects. The fraction $\frac{m}{n}$ represents the part of the whole group that is in the subgroup. Though $\frac{4}{12} = \frac{1}{3}$ and the two fractions are equivalent, we do not want to leave the impression that there are only 3 students in the class now and that the number of boys decreased from 4 to 1.

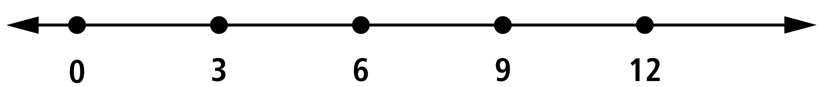
A number line is another way to find equivalent fractions.

EXAMPLE 2

Sandra buys a pack containing a dozen pencils. She sharpens $\frac{3}{4}$ of the pack of pencils. How many pencils did Sandra sharpen?

SOLUTION

You can use a number line and make 4 jumps like this:



Notice that if each jump represents a grouping of the dozen pencils into four groups, then one jump lands at 3 pencils, two jumps at 6, 3 jumps lands at 9 pencils, and 4 jumps lands at12 pencils. 9 pencils is equivalent to $\frac{3}{4}$ of the dozen pencils. We can say $\frac{9}{12} = \frac{3}{4}$.

PROBLEM 1

Pat wants to buy a video game that costs \$60. So far, Pat has raised $\frac{2}{3}$ of the cost. How much money has she raised? How much more money does she need to raise?

EXPLORATION

Find three equivalent fractions for each of the fractions below. You may use paper folding or any other model to visually show your fractions are equivalent fractions.

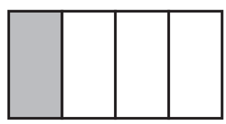
a. $\frac{1}{2}$ b. $\frac{1}{8}$ c. $\frac{1}{4}$ d. $\frac{2}{5}$ e. $\frac{2}{3}$

Do you see a pattern when two fractions are equivalent? How can you make an equivalent fraction from a given fraction without a model?

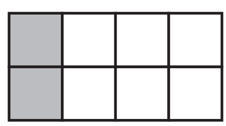
In general, we can find equivalent fractions by multiplying the numerator and denominator by the same number. For example,

$$\frac{1}{4} = \frac{2 \cdot 1}{2 \cdot 4} = \frac{1 \cdot 2}{4 \cdot 2} = \frac{2}{8}.$$

Pictorially,

$\frac{1}{4} =$ 

Multiplying the numerator and denominator by 2 has the effect of doubling the number of slices:

$\frac{1}{4} = \frac{2}{8} =$ 

Multiplying the numerator and denominator by the same number changes the number of shaded parts and the total number of parts by the same factor, yielding an equivalent fraction.

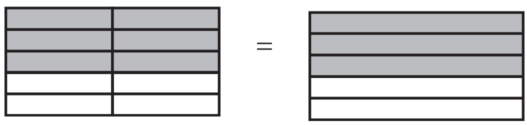
PROPERTY 4.1: EQUIVALENT FRACTION PROPERTY

For any number a and nonzero numbers k and b

$$\frac{a}{b} = \frac{k \cdot a}{k \cdot b} = \frac{a \cdot k}{b \cdot k} = \frac{ak}{bk}$$

We have generated equivalent fractions using the area model by dividing a given representation into smaller equal pieces, and converting 1 part out of 4 parts into 2 parts out of 8 parts. $\frac{1}{4}=\frac{2}{8}=\frac{3}{12}=\frac{4}{16}$ and so on. Notice that in this case, the new denominator is a multiple of the original denominator. Must this always be true?

Many times we will want to find an equivalent fraction with a smaller denominator, if possible. We call this process **simplifying** a fraction. We will do this by using Property 4.1 in reverse. For example, to simplify the fraction $\frac{6}{10}$, we first recognize that $\frac{6}{10} = \frac{2(3)}{2(5)}$. The numerator 6 and the denominator 10 have a **common factor**, 2. Dividing both the numerator and the denominator by the common factor of 2 produces an equivalent fraction. Using the Equivalent Fraction Property, we see that $\frac{6}{10}$ is equivalent to $\frac{3}{5}$.



$$\frac{6}{10} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{3}{5}$$

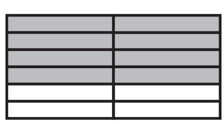
So, we have simplified $\frac{6}{10}$ to the form $\frac{3}{5}$. Notice that we are using the Equivalent Fraction Property, $\frac{a \cdot k}{b \cdot k} = \frac{a}{b}$. A fraction is said to be in **simplest form** if the numerator and denominator have no common factors except 1, in other words, a and b are relatively prime.

EXAMPLE 3

Write $\frac{8}{12}$ in simplest form.

SOLUTION

Look at a rectangular model with $\frac{8}{12}$ shaded in.



Notice that we can also view this fraction as $\frac{4}{6}$.

$$\frac{8}{12} = \frac{4 \cdot 2}{6 \cdot 2} = \frac{4}{6}$$

However, notice that this fraction can also be written equivalently as $\frac{2}{3}$.

$$\frac{8}{12} = \frac{4}{6} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{2}{3}$$

Once you find the factors of each numerator and denominator as shown below,

8	
1	8
2	4

12	
1	12
2	6
3	4

You can identify the greatest common factor, GCF, of the numerator and the denominator and use it to simplify the given fraction by applying the equivalent fraction property so that $\frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3}$.

PROBLEM 2

Write the following fractions in simplest form.

1. $\frac{24}{60}$ 2. $\frac{14}{45}$ 3. $\frac{27}{63}$ 4. $\frac{9}{10}$

The GCF is useful in finding the simplest equivalent fraction.

We will illustrate the usefulness of the prime factorization in computing the simplest form for a fraction. Examine the fraction $\frac{108}{168}$. Knowing the GCF of 108 and 168, we can write the fraction using the equivalent fraction property as

$$\frac{108}{168} = \frac{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 7} = \frac{12 \cdot 9}{12 \cdot 14} = \frac{9}{14}$$

Simplification of fractions is only one useful application of prime factorization.

EXERCISES

1. Find three equivalent fractions for each of the following fractions. You may use paper folding or any other model to determine the equivalent fractions.

   a. $\frac{6}{10}$ c. $\frac{4}{8}$ e. $\frac{15}{18}$

   b. $\frac{1}{9}$ d. $\frac{8}{10}$ f. $\frac{6}{12}$

2. Find four fractions that are equivalent to $\frac{1}{10}$.

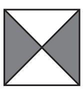
3. Find three equivalent fractions for $\frac{14}{24}$.

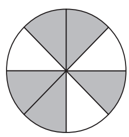
4. For each of the following fractions, find a common factor in the numerator and denominator. Then, use the Equivalent Fraction Property to simplify the fraction.

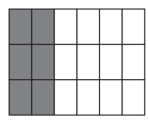
   a. $\frac{18}{24}$ c. $\frac{24}{30}$ e. $\frac{35}{45}$

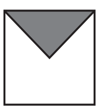
   b. $\frac{21}{28}$ d. $\frac{28}{42}$ f. $\frac{51}{72}$

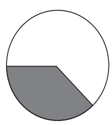
5. What fraction is represented by the shaded portion of each figure below?

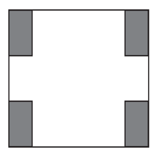
   a. 

   c. 

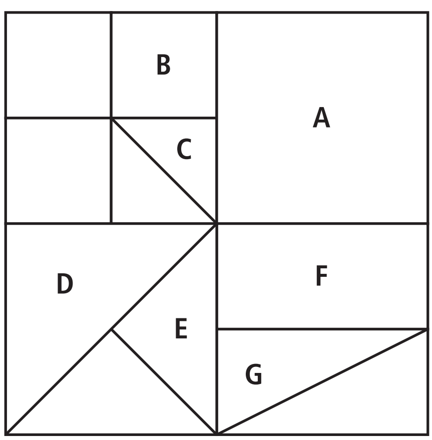
   e. 

   b. 

   d. 

   f. 

6. Determine the fraction that represents each of the labeled regions assuming the large square represents the whole or 1.



For the following exercises, when applicable, use an area or linear model to solve.

7. Jonathan practiced playing the trumpet for 20 minutes. For what fraction of an hour did he practice? Write this fraction in simplest form.

8. Juan practiced trombone for 45 minutes. For what fraction of an hour did Juan practice? What is the simplest equivalent fraction to your answer? Explain.

9. In Mrs. Garcia's class there are 20 girls and 12 boys.

   a. What fraction of the class are girls?

   b. What fraction of the class are boys?

   c. Write both answers as equivalent fractions by simplifying your answers.

10. Mindy, Andrew's cat, had 8 kittens. He says that $\frac{1}{4}$ of them are white. Use the rectangular area model to show how many kittens are white.

11. Sandra has a box of M&M candies. She counted 6 brown M&Ms and says that is $\frac{1}{3}$ of all the M&Ms she has left. How many M&Ms does Sandra have?

12. A new spirit shirt is being sold at Brilliant Middle School. Of the 360 sixth grade students, $\frac{2}{3}$ of them have purchased the new shirts. How many sixth graders have purchased the new shirts? Use the rectangular area model to help you find the answer.

13. If the time is 1:40 p.m. right now, what time will it be in the following situations: (Hint: You might want to use the circular model to help you.)

    a. half an hour later?

    b. $\frac{1}{4}$ of an hour later?

    c. $\frac{1}{3}$ of an hour later?

14. Find an equivalent fraction for $\frac{4}{12}$ that has a larger denominator. Then find 2 equivalent fractions for $\frac{4}{12}$ that have smaller denominators.

15. A class with 24 students has 4 students who do not like peanut butter. What fraction of the class does not like peanut butter? Give at least 3 equivalent fractions for this answer.

16. Write the following fractions in simplest form, if possible.

a. $\frac{42}{72}$ c. $\frac{25}{100}$ e. $\frac{17}{51}$ g. $\frac{9}{16}$ i. $\frac{15}{90}$

b. $\frac{125}{1000}$ d. $\frac{7}{5}$ f. $\frac{12}{13}$ h. $\frac{12}{25}$ j. $\frac{8}{52}$

17. Use paper folding to discover a fraction that represents a part that is larger than $\frac{1}{4}$ and less than $\frac{1}{2}$. Find 2 more fractions between $\frac{1}{4}$ and $\frac{1}{2}$.

18. Find and write a fraction that represents more than $\frac{2}{5}$ and less than $\frac{3}{5}$ of a whole. Explain how you found your answer and why your fraction is correct. Find two more fractions between $\frac{2}{5}$ and $\frac{3}{5}$.

19. Find and write a fraction that represents more than $\frac{1}{2}$ and less than $\frac{2}{3}$. Show that your answer is right.

Spiral Review:

20. Mrs. Murphy handed out 28 test booklets equally among 7 groups of students. Which equation can be used to find t, the number of test booklets each group received?

a. $t = 28 \div 7$ c. $t = 28 \cdot 7$

b. $t = 28 - 7$ d. $t = 28 + 7$

21. List the following integers in order from least to greatest.

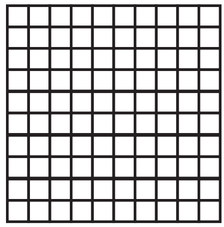
52 50 -6 $^-27$ $^-35$

22. **Ingenuity:**

Of the students in Ms. Garrett's algebra class, exactly $\frac{3}{8}$ are boys and $\frac{5}{8}$ are girls. On Friday, six girls were absent due to a basketball tournament. None of the boys were absent. That day, there were exactly as many girls as there were boys in class. How many students were in Ms. Garrett's algebra class on Friday?

23. **Investigation:**

Willy makes an enormous chocolate bar in the shape of a square composed of 100 bite-size pieces.



He then eats a certain fraction of the bar. For each of the following fractions, determine how many of the 100 bite-size pieces Willy would eat if he ate the given fraction of the bar.

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{3}{4}$

d. $\frac{1}{5}$

e. $\frac{3}{10}$

4.2

SECTION 4.2
COMPARING AND ORDERING FRACTIONS

You often see fractions as being parts of a whole. Can you think of any fractions that represent a very small part of a whole? At other times fractions are almost as big as the whole or larger. What are some examples of such fractions? You may find it useful to be able to estimate whether a fraction is very small, in which case we say that the fraction is close to 0. Or a fraction may be nearly a whole, in which case we say the fraction is close to 1. In another case, a fraction may be very close to $\frac{1}{2}$. Being able to use these estimations will be helpful in determining reasonableness of answers.

We begin with an activity that will help you to compare fractions.

LINEAR MODEL FOR FRACTIONS ACTIVITY

On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now construct a number line from 0 to 1.

Materials: You will need a long strip of paper like a sentence strip or an 18-inch piece of adding machine paper.

1. Mark the left end point of the strip as 0 and the right end point as 1. You may fold this strip or use another strip to fold and transfer points to this master number line. Fold the strip end to end into two equal parts and mark the crease as the midpoint between 0 and 1. What fraction is this midpoint equivalent to? Label the points on the number line as fractions above the line.
2. Fold the strip again and use the creases to mark and label points on the master number line. Because the strip is now folded into 4 equal parts, we label the first point as $\frac{1}{4}$. Label the other points as $\frac{2}{4}$ and $\frac{3}{4}$. Write the equivalent fraction $\frac{2}{4}$ under $\frac{1}{2}$.
3. Repeat this method by folding the strip again into 8 equal parts, transferring the locations to the master number line and labeling the points with fractions. Use this method to locate, mark and label all the eighths on the number line.
4. Compare the number line with a typical foot ruler or yardstick.

5. Label these fractions on the number line.

   a. $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$

   b. $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, \frac{10}{10}$

   c. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

   d. $\frac{1}{6}, \ldots, \frac{5}{6}, \frac{6}{6}$

   e. $\frac{1}{9}, \ldots, \frac{8}{9}, \frac{9}{9}$

6. Use your new number line to determine which benchmark (0, $\frac{1}{2}$, or 1) each fraction is closest to.

   a. $\frac{4}{12}$

   b. $\frac{2}{3}$

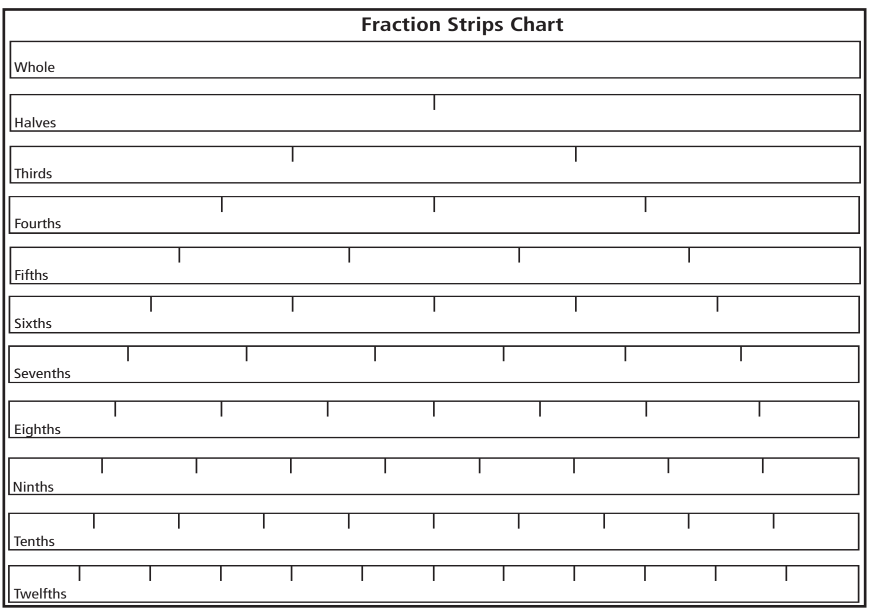
   c. $\frac{5}{16}$

   d. $\frac{8}{12}$

   e. $\frac{4}{5}$

   f. $\frac{5}{50}$

The rows of the Fraction Strips Chart below represent the result of folding the whole into equal parts to represent fractions as we did in the Linear Model Activity. The computer that created it allows the folds to be very accurate. For example, using the Fraction Strips Chart, determine which fraction is greater: $\frac{2}{5}$ or $\frac{3}{8}$.



Given two fractions, how can you determine which of them is greater? We can now locate fractions on the number line. The fraction that is to the right of the other is the greater. What problems might arise from this method? For one, its accuracy depends on the quality of the comparative number lines. It becomes harder as the fractions get closer to the same value.

Use the master number line that you constructed or the Fraction Chart to decide which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$. Explain your answer. Is there another way to explain which is greater?

COMPARING AND ORDERING FRACTIONS

Linear Model:

Use the number line from 0 to 1 or the Fraction Chart to answer the following.

a. Which fraction is greater, $\frac{3}{5}$ or $\frac{4}{9}$? Can you tell how much greater using only the number line or the Fraction Chart?

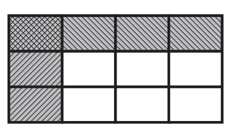
b. Which fraction is greater, $\frac{3}{4}$ or $\frac{4}{5}$?

c. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{8}$?

Area Model:

Use the area model to compare the following pairs of fractions. Use vertical cuts for one fraction and horizontal folds for the second fraction.

a. Which fraction is greater, $\frac{1}{4}$ or $\frac{1}{3}$? How much greater?



b. Which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$? How much greater?

c. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{7}$? How much greater?

Common Denominator Method:

The common denominator method compares the two fractions by rewriting the given fractions equivalently with the same denominators.

a. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{8}$? How much greater? Write equivalent fractions using a common denominator for each: $\frac{2}{5} = \frac{2 \cdot 8}{5 \cdot 8} = \frac{16}{40}$ and $\frac{3}{8} = \frac{3 \cdot 5}{8 \cdot 5} = \frac{15}{40}$. When comparing the numerators of the equivalent fractions you can see that $\frac{16}{40}$ is greater than $\frac{15}{40}$. We say $\frac{2}{5} > \frac{3}{8}$ because $\frac{16}{40} > \frac{15}{40}$. $\frac{2}{5}$ is $\frac{1}{40}$ greater than $\frac{3}{8}$.

b. Which fraction is greater, $\frac{3}{5}$ or $\frac{13}{20}$? By how much?

c. Which fraction is smaller, $\frac{5}{9}$ or $\frac{4}{7}$? By how much?

PROBLEM 1

Determine whether the pairs of fractions are equal. If they are not equal, determine which is greater.

a. $\frac{3}{5}$ and $\frac{12}{50}$ b. $\frac{3}{4}$ and $\frac{7}{8}$ c. $\frac{5}{7}$ and $\frac{7}{10}$

There may be times when you must order more than two fractions. You may use the same common denominator method.

EXAMPLE 1

Lorenz has four different wrench sizes given by their diameter measures: $\frac{5}{8}$ inches, $\frac{11}{16}$ inches, $\frac{3}{4}$ inch, and $\frac{1}{2}$ inch. Determine the order of the wrenches from largest to smallest.

SOLUTION

We note that all the wrenches are larger than $\frac{1}{2}$ except for the $\frac{1}{2}$ inch wrench. Therefore, $\frac{1}{2}$ is the smallest fraction. We must compare the remaining three fractions, $\frac{5}{8}$, $\frac{11}{16}$, and $\frac{3}{4}$. You may wish to use the fraction strip and see that $\frac{5}{8}$ is less than $\frac{3}{4}$. To determine where $\frac{11}{16}$ is situated, you may use benchmarks such as $\frac{1}{2}$, and notice that $\frac{5}{8}$ is $\frac{1}{8}$ larger than $\frac{1}{2}$ while $\frac{11}{16}$ is $\frac{3}{16}$ greater than $\frac{1}{2}$. Additionally, $\frac{3}{4}$ is $\frac{1}{4}$ larger than $\frac{1}{2}$. So the order of the fractions from largest to smallest is $\frac{3}{4}$, $\frac{11}{16}$, $\frac{5}{8}$, then $\frac{1}{2}$.

Another approach is to use common denominators for all four fractions. Using 16, the least common multiple of 8, 16, 4, and 2, we can rewrite each of the fractions equivalently as follows: $\frac{5}{8} = \frac{10}{16}$, $\frac{11}{16}$, $\frac{3}{4} = \frac{12}{16}$, and $\frac{1}{2} = \frac{8}{16}$. The order of the fractions can be made by comparing the numerators. From largest to smallest, we have: $\frac{12}{16} = \frac{3}{4} > \frac{11}{16} > \frac{10}{16} = \frac{5}{8} > \frac{4}{8} = \frac{1}{2}$.

PROBLEM 2

Kassandra has 3 pieces of different length ribbons. She has $\frac{7}{8}$ meter of blue ribbon, $\frac{5}{6}$ meter of green ribbon, and $\frac{2}{3}$ meter of yellow ribbon. She needs to use the longest ribbon for a picture frame and the shortest ribbon for a bracelet. The remaining ribbon is going to be used for a bow for a gift. What color is the ribbon that needs to be used for the picture frame? The bracelet? The bow?

EXPLORATION 1

Draw a number line and locate 0 and $^-1$.

a. Use the number line to locate points with coordinates $^-\frac{1}{2}$, $^-\frac{1}{4}$, $^-\frac{1}{3}$.

b. Determine which of these three numbers, $^-\frac{1}{2}$, $^-\frac{1}{4}$, $^-\frac{1}{3}$ is greatest.

c. Determine which of these three numbers, $^-\frac{1}{2}$, $^-\frac{1}{4}$, $^-\frac{1}{3}$ is smallest.

EXERCISES

1. a. Is $\frac{7}{8}$ closer to 0, $^-\frac{1}{2}$, or 1? Use a number line to prove your answer.

   b. Find a fraction that is even closer to 1 than $\frac{7}{8}$.

   c. Is $^-\frac{3}{8}$ closer to 0, $^-\frac{1}{2}$, or $^-1$? Use a number line to prove your answer.

2. Which is closer to zero, $\frac{2}{7}$ or $\frac{3}{8}$? Explain how you arrived at your conclusion. You may use a number line to prove your answer.

3. The Miller team completed their project in $\frac{2}{3}$ of the time allowed. The Kealing team completed their project in $\frac{3}{8}$ of the time allowed. Each team had equal amount of time to complete the project. Determine which team finished their project first. Explain how you know which team finished first.

4. Ms. Lara is making iced tea and wants to add one cup of sugar to one gallon of unsweetened tea. She can only find measuring cups of $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{1}{2}$. Which measuring cup is closest to one cup?

5. Mr. Reyna wants to find the correct size wrench for a bolt in order to change the oil in his car. The $\frac{9}{16}$ inch wrench is too large. He wants the next smaller size. Which wrench should Mr. Reyna use, the $\frac{5}{8}$ inch or $\frac{1}{2}$ inch wrench?

6. Plot the numbers below on a number line. Then list the fractions in order from least to greatest:

   a. $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{2}{5}$ c. $\frac{3}{8}, \frac{3}{4}, \frac{1}{16}, \frac{4}{32}$

   b. $\frac{1}{4}, \frac{3}{5}, \frac{5}{8}, \frac{7}{10}$ d. $-\frac{4}{6}, \frac{3}{8}, -\frac{1}{2}, \frac{1}{4}$

7. Use the Fraction Chart to discover and represent three fractions that are greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. Explain why each fraction is between $\frac{1}{4}$ and $\frac{1}{2}$ without using the chart.

8. Write the equivalent fraction in simplest form for each of the following fractions.

   a. $\frac{4}{6}$ c. $\frac{12}{16}$ e. $\frac{16}{20}$ g. $\frac{6}{21}$ i. $\frac{24}{30}$

   b. $\frac{9}{12}$ d. $\frac{12}{18}$ f. $\frac{6}{20}$ h. $\frac{18}{30}$ j. $\frac{16}{30}$

9. Compare the two fractions $\frac{4}{5}$ and $\frac{6}{7}$ by plotting them on a number line. Which fraction is greater, $\frac{4}{5}$ or $\frac{6}{7}$? By how much? Explain your reasoning.

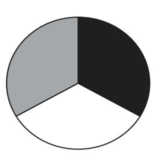
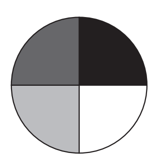
10. Why is it important to know how to simplify fractions?

11. Use the different models or methods learned in this section to compare the following fractions. Use <, >, or = between the two fractions.

    a. $\frac{7}{8}$ $\frac{5}{6}$ c. $\frac{2}{3}$ $\frac{5}{8}$ e. $\frac{1}{12}$ $-\frac{1}{3}$

    b. $\frac{8}{9}$ $\frac{9}{10}$ d. $\frac{4}{7}$ $\frac{5}{9}$ f. $-\frac{3}{4}$ $-\frac{6}{16}$

12. Tianna has $\frac{1}{3}$ of a large pizza left. Desiree has $\frac{1}{4}$ of a large pizza left. Who has more pizza left? (Assume the white region represents uneaten pizza.)

    Tianna:  Desiree: 

13. Use a number line to determine which benchmark (0, $\frac{1}{2}$, 1) each fraction is closest to.

a. $\frac{11}{15}$ b. $\frac{3}{8}$ c. $\frac{0}{5}$ d. $\frac{9}{16}$

14. Evan ate $\frac{2}{7}$ of an apple pie. Pattie ate $\frac{1}{3}$ of the same pie. Evan says he ate less. Is Evan right or wrong? Explain your answer.

15. Lorianne bought two ribbons to make bows. One ribbon was $\frac{14}{16}$ of a yard and the other was $\frac{6}{8}$ of a yard. Which ribbon will create a larger bow?

16. Mr. Trevino's science class measured the distance four toy cars traveled on a yard stick. They found the distances:

Car A went $\frac{5}{6}$ of a yard.

Car B went $\frac{11}{12}$ of a yard.

Car C went $\frac{4}{9}$ of a yard.

Car D went $\frac{18}{36}$ of a yard.

List the distances the cars traveled in order from greatest to least.

17. Compare the two fractions. You may use a number line. Determine if the pair of fractions are equal to each other. If the two fractions are not equal, determine which is greater.

a. $\frac{1}{4}$ and $\frac{25}{100}$ b. $\frac{3}{5}$ and $\frac{13}{20}$ c. $\frac{9}{16}$ and $\frac{8}{15}$

18. The math team was asked to pose for a picture. Rolinda is $5\frac{2}{3}$ feet tall, Melissa is $5\frac{4}{7}$ feet tall, Sarah is $4\frac{7}{8}$ feet tall and Celia is $5\frac{5}{6}$ feet tall. The photographer has them line up from tallest to shortest. In what order should they line up?

Spiral Review:

19. In a competition, Ray, Josh, Tyler, and Aaron were asked to draw toothpicks. Ray's toothpick was $\frac{12}{18}$ inches, Josh's was $\frac{2}{7}$ inches , Tyler's was $\frac{1}{5}$ inch, and Aaron's was $\frac{6}{7}$ inches long. Whose toothpick was closest to a whole inch? Explain your answer.

20. What is the prime factorization of 312?

21. What is the value of the expression $(4 + 4)^2 \div 8 + 3 \cdot 2$?

22. **Ingenuity:**

    Suppose we use the digits 1, 2, 3, 4, 5, and 6 to make the numerator and denominator of a fraction. Every digit must be used exactly once, the numerator and denominator must have three digits each, and the numerator must be less than the denominator. What is the greatest fraction we can make?

23. **Investigation:**

    In parts a through d, three fractions are given. Write the fractions in order from least to greatest. In each part, note the relationship between the last fraction and the first two.

    a. $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}$

    b. $\frac{1}{4}, \frac{1}{5}, \frac{2}{9}$

    c. $\frac{1}{6}, \frac{2}{7}, \frac{3}{13}$

    d. $\frac{3}{4}, \frac{5}{7}, \frac{8}{11}$

    e. What do you notice in parts a through d?

    f. Using what you have discovered, give ten fractions that lie between $\frac{4}{7}$ and $\frac{3}{5}$.

4.3

SECTION 4.3
UNIT, MIXED, PROPER, AND IMPROPER FRACTIONS

Recall on the fraction chart we have $\frac{1}{2}+\frac{1}{2}=1$, $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$. Notice that we have two times $\frac{1}{2}$ equals 1 and three times $\frac{1}{3}$ equals 1.

What would $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ equal?

A **unit fraction** always has 1 in the numerator and the denominator is a positive integer. For example, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on are all examples of unit fractions.

Now extend the addition of unit fractions to make another connection to multiplication. You have seen several models that represent the fraction $\frac{3}{5}$. In the area model, $\frac{3}{5}$ represents three $\frac{1}{5}$'s of a whole. This means $\frac{3}{5}$ is the sum of $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. In the frog model, this is the same as taking 3 jumps of length $\frac{1}{5}$ That is, $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{1}{5}\cdot 3=\frac{3}{5}$. This understanding can be extended to all fractions. For example, the fraction $\frac{5}{9}$ is the same as the sum of 5 copies of $\frac{1}{9}$:

$$\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{9}\cdot 5=\frac{5}{9}.$$

Write the sum of 8 copies of $\frac{1}{5}$. Using the frog model, what is the result of 8 jumps of length $\frac{1}{5}$ each?

Using math with recipes can be fun not only for learning but for eating, too! Usually recipes involve quantities of ingredients and cooking directions. Here is a chocolate chip cookie recipe:

- $2\frac{1}{4}$ cups flour
- $\frac{3}{4}$ cup sugar
- $\frac{3}{4}$ cup brown sugar
- 12 oz chocolate chips
- 1 tsp baking soda
- 1 tsp salt
- 1 tsp vanilla

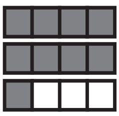
This makes approximately 6 dozen cookies.

Each of the sugar quantities is $\frac{3}{4}$ of a cup. This quantity is called a **proper fraction** because it is a number less than 1.

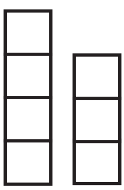
Look at the first ingredient in our recipe, "two and one-fourth cups." Remember, "and" means adding two cups to one-fourth of a cup. Quantities like $2\frac{1}{4}$ are called **mixed fractions** or **mixed numbers** because they consist of an integer like 2, in addition to a fractional part that is less than a whole like $\frac{1}{4}$. It is customary to write the fractional part in simplified form. The mixed fraction $2\frac{1}{4}$ is actually the sum $2 + \frac{1}{4}$. The rest of the recipe contains both fractional parts of cups or teaspoons and numbers of ounces.

Look at the mixed fraction $2\frac{1}{4}$. If you have only a quarter-cup measure, describe how you can measure the correct amount with the quarter cup.

Did you find $2\frac{1}{4}$ equivalent to $\frac{9}{4}$? In fact, what you have found are two ways to write the same quantity: as a mixed fraction, $2\frac{1}{4}$, and as an **improper fraction**, $\frac{9}{4}$. How would you describe improper fractions? Why do you think they are called improper?

$\frac{9}{4}$  $2\frac{1}{4}$

If we have seven quarters, we can think of each quarter as $\frac{1}{4}$ of one dollar. Then we have $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. Because each of the four quarters equals one dollar, you have 1 dollar and 3 more quarters or $1\frac{3}{4}$ is equal to $\frac{7}{4}$.



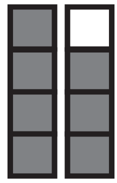
PROBLEM 1

Use a model and repeated addition sentences to find the improper fraction for each of the following mixed fractions:

a. $1\frac{7}{8}$

b. $3\frac{3}{4}$

Another way to think about $\frac{7}{4}$ is to view this fraction as a division problem, 7 divided by 4. If we divide 7 by 4, we have a quotient of 1 with a remainder of 3. Using the area model, we group 4 of the 7 into a rectangle of dimension 4 by 1 because 4 is the divisor. The remaining 3 fill up $\frac{3}{4}$.

$\frac{7}{4}$  $1\frac{3}{4}$

$$\frac{7}{4} = 7 \div 4 = 1\frac{3}{4}$$

PROBLEM 2

Use division to write the improper fraction $\frac{7}{3}$ as a mixed fraction.

PROBLEM 3

State the difference between proper and improper fractions. What is the advantage of using an improper fraction or using a mixed number?

PROBLEM 4

Each student in Ms. Milligan's class is to receive $\frac{1}{3}$ of a pizza for lunch. There are 17 students in her class. How many whole pizzas should Ms. Milligan order? Will there be any portion of a pizza left for Ms. Milligan? If so, how much will be left for her?

EXPLORATION 1

Use the Internet to find a recipe for chocolate chip cookies from another country. Analyze the differences you see.

Interestingly enough, in the United States most recipes are written as fractions. Why do you think American recipes are usually written in fractional form?

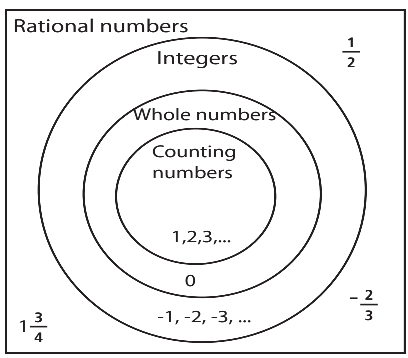
Many times recipes are either too large or too small for our purposes. If you double the recipe, how much sugar will you need? Write the sugar quantity in both mixed fraction and improper fraction form. Also write the chocolate chip quantity in the doubled recipe in terms of pounds.

a. A recipe for pancakes calls for $1\frac{3}{4}$ cups of flour. Locate this point on the number line. Describe the equivalent improper form for the mixed fraction. What does the numerator represent?

b. Jack has three identical pans of brownies and decides to divide each pan into 12 equal pieces. How many brownies pieces does he have in all? Because Jack was very hungry, he ate 2 of the pieces. If you assume each brownie pan represents 1 or a whole, express the amount of brownies that remains in terms of whole pans and pieces.

c. If he takes half of the uneaten brownies to a party, what quantity will he take? Using the area model, draw the brownie quantities he will take and leave. Be sure to include the fact that each pan is divided into 12 pieces.

We now have fractions that include natural numbers, whole numbers, and integers. The fractions we are considering have a numerator and denominator that are integers, except the denominator cannot be 0. The set of fractions of this form $\frac{a}{b}$, with b not zero and a and b integers, are called rational numbers. A Venn Diagram shows how these sets are related.



PROBLEM 5

For the Venn Diagram above, find the following:

a. What is a whole number that is not a counting number? Where would this be in the Venn Diagram?

b. What is an integer that is not a whole number? Where would this be in the Venn Diagram?

c. What is a rational number that is not an integer? Where would this be in the Venn Diagram?

d. Are all integers rational numbers? Explain.

e. Are all counting numbers integers? Explain.

EXERCISES

1. Rewrite each sum as an improper fraction and as a mixed fraction:

   a. $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} =$

   b. $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} =$

   c. $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} =$

   d. $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} =$

2. Convert each of these improper fractions to a mixed fraction. Sketch a number line from 0 to 4 with eighths marked and locate each mixed fraction.

$$\frac{9}{8}, \frac{18}{8}, \frac{12}{8}, \frac{16}{8}, \frac{27}{8}, \frac{11}{8}, \frac{20}{8}$$

3. Write the improper fraction $\frac{17}{4}$ as a mixed fraction. Explain the process you used.

4. Convert each improper fraction to a mixed fraction. Simplify any mixed fractions that are not already in simplest form.

a. $\frac{7}{3}$	d. $\frac{24}{6}$	g. $\frac{15}{6}$	j. $-\frac{24}{18}$
b. $\frac{9}{4}$	e. $-\frac{12}{8}$	h. $\frac{33}{10}$	k. $\frac{39}{12}$
c. $-\frac{6}{5}$	f. $\frac{19}{3}$	i. $\frac{27}{5}$	l. $-\frac{45}{15}$

5. Let W = {whole numbers}, N = {counting numbers}, Z = {integers}, and Q = {rational numbers}.

   The curly brackets { } enclose the elements in a set. Set notation is a shorter way of listing the elements. For example, {1,2,3} is the set containing the three integers 1, 2, and 3. Another way of describing this set is $\{x|x$ is a positive integer less than $4\}$. This notation means "the set consisting of x such that x is a positive integer less than 4." The vertical line | is read as "such that."

a. Draw a Venn Diagram of the four sets listed and show how the sets are related. Label each set.

b. Find at least one number that is in each set that is not contained in its subset.

6. Convert the mixed fraction $2\frac{3}{4}$ to an improper fraction. Explain the process you used.

7. Convert each of these mixed fractions to an improper fraction.

a. $3\frac{4}{5}$

b. $2\frac{3}{8}$

c. $^{-}4\frac{1}{8}$

d. $5\frac{2}{5}$

e. $10\frac{5}{6}$

f. $^{-}9\frac{3}{5}$

g. $6\frac{11}{12}$

h. $13\frac{3}{9}$

8. How many one-eighth cups of flour are in $1\frac{3}{4}$ cups of flour?

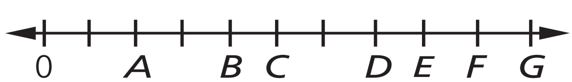
9. Which is greater, $2\frac{2}{3}$ or $\frac{15}{6}$? Explain your reasoning.

10. Which is greater, $\frac{27}{8}$ or $3\frac{1}{4}$? Explain.

11. Aunt Bea is making pudding for dessert. She wants to serve everyone $\frac{1}{4}$ of a cup. If she makes $3\frac{3}{4}$ cups, how many people can she serve?

12. Mr. Kellerman had a box of candy bars. He cut them into fourths to give each student an equal amount of candy. If he distributed $\frac{52}{4}$, how many whole candy bars did he have in the box?

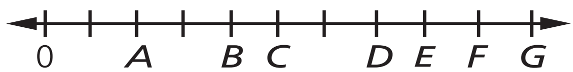
13. For each of the following questions, make a copy of the picture below and use it as a linear model for a fraction bar. Remember to make a separate fraction bar for each part, a - d. Zero has already been placed for you.



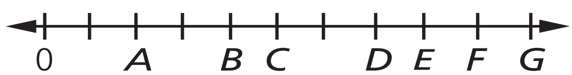
a. If the point A represents the number 1 on the number line, what numbers do the other points represent? Write the values above the points on the number line. You may answer in mixed fractions or improper fractions.

b. If the point ***B*** represents the number 1 on the number line, what numbers do the other points represent? Write the values above the points on the number line. You may answer in mixed fractions or improper fractions.

c. If the point ***C*** represents the number 1 on the number line, what numbers do the other points represent? Write the values above the points on the number line. You may answer in mixed fractions or improper fractions.



d. If the point ***E*** represents the number 1 on the number line, what numbers do the other points represent? Write the values above the points on the number line. You may answer in mixed fractions or improper fractions.



14. Place zero in the center of this number line. Number by halves from $^{-}2$ to 2.

Place the following numbers on the number line. (Hint: Consider simplifying your answers before placing them on the number line.)

a. $\frac{12}{8}$	c. $1\frac{5}{6}$	e. $\frac{24}{12}$
b. $^{-}\frac{5}{4}$	d. $\frac{9}{12}$	f. $^{-}\frac{6}{8}$

Spiral Review:

15. At a dance recital, a prize was given to the person sitting in the chair numbered with the least common multiple of 10, 15, and 20. Find the number of the prize-winning chair.

16. The fraction $^{-}\frac{3}{8}$ is found between which pair of fractions on a number line?

a. $^{-}\frac{21}{22}$ and $^{-}\frac{8}{16}$

b. $^{-}\frac{24}{32}$ and $^{-}\frac{10}{16}$

c. $^{-}\frac{19}{32}$ and $^{-}\frac{9}{16}$

d. $^{-}\frac{24}{32}$ and $^{-}\frac{1}{4}$

17. **Ingenuity:**

While doing a homework assignment on mixed numbers, Dania thought that she discovered a new way to rewrite a mixed fraction as an improper fraction. She moved the whole number part of the mixed fraction into the numerator, as shown in the examples below:

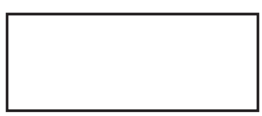
$$3\frac{1}{7} \rightarrow \frac{31}{7}$$
$$4\frac{3}{5} \rightarrow \frac{43}{5}$$

a. Explain why Dania's method is incorrect.

b. Are there any mixed fractions for which Dania's method would give the correct improper fraction? That is, are there any mixed fractions that we can convert to improper fractions simply by moving the whole number part into the numerator as shown above?

18. **Investigation:**

Copy the rectangle below for each part of the problem and suppose that its area is given in each of the problems below. Determine and shade an area of 1 square unit for each of the rectangles.



a. $1\frac{2}{3}$

b. $2\frac{2}{3}$

c. $\frac{5}{4}$

d. $\frac{2}{3}$

SECTION 4.4
ADDITION AND SUBTRACTION OF FRACTIONS

Adding 1 foot to 2 feet equals 3 feet. Combining 1 apple with 2 apples gives 3 apples. In each case, both numbers and units are important. Given these two examples, it seems reasonable to say that the sum of 1 fifth and 2 fifths is 3 fifths. More precisely, in Chapter 2, the linear skip counting model demonstrated that $\frac{3}{5}$ is $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. Using skip counting, it is easy to see that

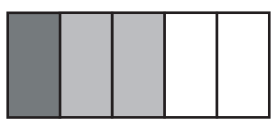
$$\frac{2}{5}+\frac{1}{5}=\left(\frac{1}{5}+\frac{1}{5}\right)+\frac{1}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}.$$

In general, for positive integers m and n, the fraction $\frac{m}{n}$ is the sum of m unit fractions of the form $\frac{1}{n}$. For the rest of this chapter, assume all possible denominators are positive integers.

PROBLEM 1

Compute the sum of $\frac{3}{8}$ and $\frac{2}{8}$. Explain your answer.

Now look at the area model. How is the sum $\frac{1}{5}+\frac{2}{5}$ computed using the area model? Use a candy bar model. Betsy had $\frac{1}{5}$ of a candy bar, and her friend had $\frac{2}{5}$ of a candy bar like Betsy's.



Together, they have $\frac{3}{5}$ of a candy bar. Express this as $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$. Write rules to generalize the previous discussion of adding fractions. The sum of two fractions with like denominators, $\frac{a}{d}$ and $\frac{b}{d}$, is given by: $\frac{a}{d}+\frac{b}{d}=\frac{a+b}{d}$

PROBLEM 2

Find the sum and put in simplest form or if possible write as a mixed fraction. Use models to illustrate these.

a. $\frac{4}{12}+\frac{5}{12}$ b. $\frac{3}{5}+\frac{3}{5}$ c. $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$

The same principle applies when subtracting fractions. So, $\frac{4}{5}-\frac{3}{5}=\frac{1}{5}$

PROBLEM 3

Find the difference and put in simplest form. Use models to illustrate each. Remember the order of operations.

a. $\frac{5}{6} - \frac{2}{6}$ b. $\frac{7}{10} - \frac{2}{10}$ c. $\frac{5}{8} - \frac{3}{8} - \frac{2}{8}$

PROBLEM 4

Compute $\frac{7}{9} - \frac{4}{9}$ and explain how to obtain the answer.

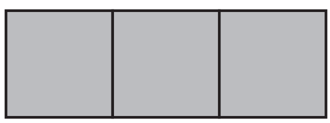
Describe how to subtract fractions with like denominators. Find the difference for $\frac{m}{n} - \frac{k}{n}$. How does your method compare to the addition rule above?

EXAMPLE 1

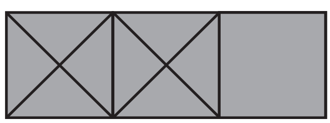
If you eat $\frac{2}{3}$ of a whole candy bar, how much of the candy bar is left? How can you use subtraction of fractions to answer this question?

SOLUTION

Using mathematical fractions, the problem looks like this: $1 - \frac{2}{3}$. In order to perform this calculation, begin by drawing a picture of a candy bar and divide it into three pieces. First convert 1 into the fraction $\frac{3}{3}$.

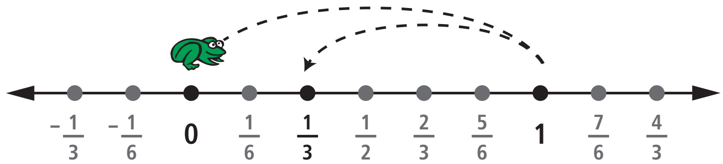
$1 = \frac{3}{3}$ 

What happens when you subtract $\frac{2}{3}$ of the candy bar? × out the portions that are subtracted, and the difference is

$1 - \frac{2}{3} = \frac{1}{3}$ 

Write this as $1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$.

Another way to think of this subtraction uses the frog model. Like the model for subtracting integers, the frog hops 1 unit to the right and then hops backwards a distance of $\frac{2}{3}$ to land on the number $\frac{1}{3}$. This model represents $1 - \frac{2}{3}$.



PROBLEM 5

Compute the difference $2 - \frac{1}{4}$ and illustrate the process with either the area model or the linear model.

Now that you've explored adding and subtracting fractions with like denominators, explore finding the sum of an integer and a fraction, like the addition problem $2 + \frac{3}{5}$. Applying a similar process for this sum as you did for the last sum, there are several ways to combine these two quantities:

$$2 + \frac{3}{5} = 1 + 1 + \frac{3}{5} = \frac{5}{5} + \frac{5}{5} + \frac{3}{5} = \frac{13}{5}, \text{ or}$$

$$2 + \frac{3}{5} = 2\frac{3}{5}.$$

The first method trades one addition problem for another addition problem and results in an answer that is an improper fraction. The second way results in the mixed fraction 2 $\frac{3}{5}$. In Section 4.3, you learned how to convert mixed fraction to improper fractions and vice versa. Now you see that a mixed fraction can also be thought of as a sum. Which form is better, $2\frac{3}{5}$ or $\frac{13}{5}$? Explain your reasoning.

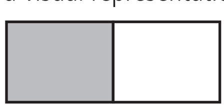
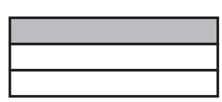
EXAMPLE 2

Explore how to use the ideas just learned to compute the sum of two fractions when the denominators are not the same.

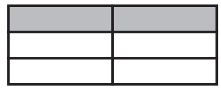
Use the area model to compute the sum $\frac{1}{2} + \frac{1}{3}$.

SOLUTION

Begin by looking at a visual representation.

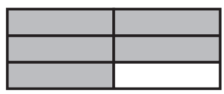
$\frac{1}{2} =$  $\frac{1}{3} =$ 

Is it possible to combine the shaded amounts? Earlier, you discovered that in comparing the fractions $\frac{1}{2}$ and $\frac{1}{3}$, it was helpful to find equivalent fractions for both $\frac{1}{2}$ and $\frac{1}{3}$ to determine which is greater. Modify the picture above to display equivalent divisions of the whole.

$\frac{1}{2} =$  $= \frac{3}{6}$ $\frac{1}{3} =$  $= \frac{2}{6}$

To do this, divide the first model horizontally to represent $\frac{1}{2}$ as 3 parts out of 6 parts. Then, divide the second model vertically to represent $\frac{1}{3}$ as 2 parts out of 6 parts. It is easy to see from the model that $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6}$ and that $\frac{3}{6}$ is greater than $\frac{2}{6}$.

More importantly, it is also easy to see how to add the two fractions in their equivalent forms.



Using the rule for adding fractions with like denominators, the sum is

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

In order to add the fractions, find a common-sized piece so that the two fractions can be written with the same or **common denominator**.

Using the equivalent fraction property, transform the two fractions to fractions with the same denominator:

$$\frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \text{ and } \frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6}$$

The most important thing to remember when adding fractions is to ensure that you have a common denominator.

EXPLORATION

Compute the sum $\frac{1}{3} + \frac{1}{4}$ by first using the area model and then the equivalent fraction property to convert the fractions into equivalent fractions with like denominators.

Find the pattern to add the fractions $\frac{1}{a}$ and $\frac{1}{b}$ when $\boldsymbol{a}$ and $\boldsymbol{b}$ are not the same number and show the process.

PROBLEM 6

Find three common denominators for the fractions $\frac{1}{6}$ and $\frac{1}{4}$. Write each fraction in equivalent forms using the three denominators. What do you notice about these common denominators? Which denominator would be the best choice for computing the sum $\frac{1}{6} + \frac{1}{4}$? Why?

Look at all the denominators you created in Problem 6. Do they have common factors? What is the relationship between every one of the common denominators and the original denominators? Which common denominator did you not have to simplify after adding? Now combine the discoveries about common denominators in Problem 6 with those about common multiples from Section 3.5.

Let's review finding least common multiples.

Find the least common multiple of:

a. 3 and 6
b. 12 and 48
c. 9 and 15
d. 5 and 7
e. 2 and 8
f. 9, 6, and 12

When we add and subtract fractions, having a common denominator is very useful. In order to add $\frac{1}{3} + \frac{1}{6}$, use the equivalent fraction, $\frac{2}{6}$, for $\frac{1}{3}$. The restatement of the problem $\frac{1}{3} + \frac{1}{6}$ to $\frac{2}{6} + \frac{1}{6}$ makes finding the sum of $\frac{3}{6}$ easier to determine.

PROBLEM 7

To find the following sums: (1) find a common multiple for both denominators, (2) use it to find equivalent fractions for each fraction, (3) compute their sum, and (4) simplify your answer, if possible.

a. $\frac{1}{9} + \frac{1}{12}$
b. $\frac{3}{8} + \frac{5}{12}$
c. $\frac{7}{12} + \frac{5}{18}$

DEFINITION 4.1: LEAST COMMON DENOMINATOR

The **least common denominator** of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is the least common multiple of n and m.

In adding or subtracting fractions, the LCM of the denominators produces the least common denominator or LCD. Using the LCD has the advantage of allowing you to work with smaller numbers.

EXPLORATION: MODELING FRACTION ADDITION

1. Draw a linear model to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$. Explain what denominator you used and why.
2. Use an area model, for example a rectangle, to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$.
3. Use another area model, for example a circle, to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$.

EXERCISES

1. Find the least common denominator for the fractions. Write each equivalent fraction using the least common denominator.

   a. $\frac{3}{5}$ and $\frac{6}{7}$
   b. $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$
   c. $\frac{4}{12}$, $\frac{1}{3}$, and $\frac{2}{8}$
   d. $\frac{5}{8}$, $\frac{3}{4}$, and $\frac{9}{16}$
   e. $\frac{3}{4}$ and $\frac{5}{9}$
   f. $\frac{2}{3}$ and $\frac{5}{9}$
   g. $\frac{6}{5}$, $\frac{3}{10}$, and $\frac{2}{15}$

2. Add or subtract the following fractions. Write your answers in simplest form.

   a. $\frac{2}{8} + \frac{4}{8}$
   b. $\frac{3}{10} + \frac{2}{10}$
   c. $\frac{2}{5} + \frac{6}{5}$
   d. $1 - \frac{3}{7}$
   e. $\frac{4}{9} + \frac{5}{9}$
   f. $\frac{9}{3} - \frac{5}{3}$
   g. $4 - \frac{2}{8}$
   h. $\frac{16}{3} + \frac{14}{3}$
   i. $\frac{3}{a} + \frac{4}{a}$
   j. $\frac{8}{m} - \frac{3}{m}$
   k. $\frac{2}{x} + \frac{2}{x}$
   l. $\frac{5}{y} - \frac{5}{y}$

3. Compute the sums or differences. Write your answers in simplest form.

   a. $\frac{2}{3} + \frac{3}{5}$
   b. $\frac{5}{8} - \frac{1}{4}$
   c. $\frac{1}{4} + \frac{1}{3}$
   d. $\frac{6}{16} + \frac{2}{8}$
   e. $\frac{7}{8} - \frac{1}{4}$
   f. $\frac{1}{4} - \frac{1}{5}$
   g. $\frac{3}{8} - \frac{5}{12}$
   h. $\frac{6}{21} - \frac{3}{14}$
   i. $\frac{1}{10} - \frac{1}{20}$

j. $\frac{3}{8} - \frac{2}{7}$

k. $\frac{7}{10} - \frac{1}{6}$

l. $\frac{4}{18} + \frac{2}{9}$

m. $\frac{3}{7} + \frac{5}{8}$

n. $\frac{1}{m} + \frac{1}{n}$

o. $2x + 3x$

4. Julie made a large rectangular cake. She decided to give $\frac{1}{4}$ of it to her neighbor, $\frac{3}{8}$ to her mom, and $\frac{1}{8}$ to her sister as shown in the model below.

   a. Use the model to determine how much of the cake she plans to give away. Write an equation to show how this problem is represented mathematically.

   b. How much of the cake is left for her family?

5. To make a certain shade of purple paint, Jennifer must mix $\frac{2}{3}$ of a gallon of blue paint with $\frac{1}{8}$ of a gallon of red paint. How much paint will she make?

6. Sam is sponsoring a contest to see what fraction of the jellybeans in his jar are red. Paul estimates that $\frac{3}{5}$ of them are red while Mark estimates the amount is $\frac{7}{10}$. Sam knows the answer is $\frac{2}{3}$. Whose estimate was closest to the actual fraction? How much closer is the winner to the amount?

7. Create a number line to resemble the section of the ruler between 0 and 1.

   a. Place and label the following points on the number line you created:

   $A = \frac{1}{8}$ $B = \frac{5}{16}$ $C = \frac{1}{2}$ $D = \frac{15}{20}$

   b. Using your model, find the distance from point A to point B.

   c. Using your model, find the distance from point A to point C.

   d. Using your model, find the distance from point B to point D.

8. Compute and simplify. Express as a mixed fraction, if needed:

   a. $\frac{3}{8} + \frac{1}{4} + \frac{1}{2}$

   b. $\frac{2}{4} + \frac{1}{5} + \frac{6}{10}$

   c. $\frac{1}{3} + \frac{3}{9} + \frac{2}{4}$

   d. $\frac{3}{8} + \frac{2}{3} + \frac{3}{12}$

   e. $\frac{3}{5} + \frac{3}{8} + \frac{3}{4}$

   f. $\frac{4}{12} + \frac{3}{8} + \frac{3}{9}$

9. Compute the following sums. Express each as a simplified mixed fraction.

   a. $\frac{2}{3} + \frac{3}{2}$ b. $\frac{4}{5} + \frac{5}{4}$ c. $\frac{3}{7} + \frac{7}{3}$

Spiral Review:

10. Which statement about the mixed number $1\frac{1}{4}$ is true?

    a. $1\frac{3}{10} > 1\frac{1}{4}$

    b. $2 < 1\frac{1}{4}$

    c. $1\frac{1}{4} > 1\frac{3}{10}$

    d. $1\frac{1}{4} < 1\frac{1}{10}$

11. If Mr. Jones drives at a constant speed of 60 miles per hour, which method can be used to find the number of hours it will take him to drive 300 miles?

    a. Add 60 and 300

    b. Subtract 60 from 300

    c. Multiply 300 by 60

    d. Divide 300 by 60

12. **Ingenuity:**

    Beau's family baked him a cake for his birthday. After Beau blew out the candles, his sister, mother, and father all ate equal-sized pieces of the cake. Since it was Beau's birthday, he got to have a piece twice as big as what each of the others had. All together, Beau and his family ate one half of the cake. What fraction of the cake did Beau eat?

13. **Investigation:**

    Candice is baking a huge batch of cookies for a bake sale. She has $1\frac{5}{8}$ pounds of flour in her house. She decides this is not enough, so she borrows an additional $2\frac{1}{2}$ pounds of flour from Carl, her neighbor. Candice wants to know how many pounds of flour she has now.

    a. Draw a picture that represents this situation.

    b. Using the picture or another method of your choice, figure out how many pounds of flour Candice now has. Express your answer as a mixed fraction.

SECTION 4.5
COMMON DENOMINATORS AND MIXED FRACTIONS

In Section 4.4, the discussion of adding two fractions involved finding a common denominator. In many problems, you probably used the least common denominator (LCD), the LCM of the given denominators. We will discuss the advantages of using the LCD and then develop a systematic approach to computing the LCD.

In Exploration 1 in Section 4.4 you discovered a rule for adding two fractions with unknown and unlike denominators:

$$\frac{1}{a}+\frac{1}{b}=\frac{1\bullet b}{a\bullet b}+\frac{1\bullet a}{b\bullet a}=\frac{b}{a\bullet b}+\frac{a}{a\bullet b}=\frac{b+a}{a\bullet b}$$

Use this pattern to compute the sum in the following example.

EXAMPLE 1

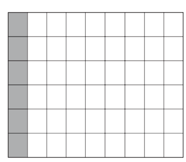
Compute: $\frac{1}{6}+\frac{1}{9}$

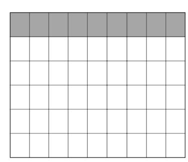
SOLUTION

Common Denominator Method:

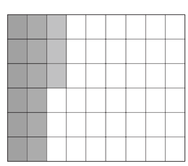
As in the rule above, you can create a common denominator for each sum by multiplying the two given denominators:

$$\frac{1}{6}+\frac{1}{9}=\frac{1\bullet 9}{6\bullet 9}+\frac{1\bullet 6}{9\bullet 6}$$





$$=\frac{9+6}{6\bullet 9}=\frac{15}{54}$$



Are these fractions simplified? Do the numerator and denominator have any common factors? The numerator and denominator can be factored into primes. A factor of 3 is common to both the numerator and the denominator.

$$\frac{15}{54}=\frac{3\bullet 5}{2\bullet 3\bullet 3\bullet 3}=\frac{5}{18}$$

LCD Method:

Notice that the last approach did not involve finding the LCD. Another approach is to first find the LCD of the fractions in each sum or, equivalently, the LCM of the denominators. A method for finding the LCD is to list the multiples of each denominator until the LCM is found. An alternate strategy is using the prime factorization. Look at the prime factorizations of the denominators of the fractions: $6 = 2 \cdot 3$, $9 = 3^2$. Remember the rule for finding the LCM of two numbers from their prime factorizations: take the product of each prime raised to its larger exponent. So, the LCD is $\text{LCM}(6, 9) = 2 \cdot 3^2 = 18$.

Now, in computing the sum $\frac{1}{6} + \frac{1}{9}$, multiply the numerator and denominator of each fraction by a factor that will make the denominator the LCD:

$$\begin{aligned} \frac{1}{6} + \frac{1}{9} &= \frac{1}{2 \cdot 3} + \frac{1}{3^2} \\ &= \frac{1 \cdot 3}{(2 \cdot 3) \cdot 3} + \frac{1 \cdot 2}{3^2 \cdot 2} \\ &= \frac{3 + 2}{2 \cdot 3^2} = \frac{5}{18} \end{aligned}$$

Notice that the final answer is already simplified.

PROBLEM 1

Use the process just developed to compute the LCD for the fractions and then compute the sum:

a. $\frac{1}{40} + \frac{1}{50}$ b. $\frac{3}{8} + \frac{5}{12}$ c. $\frac{7}{10} + \frac{4}{9}$

Formulate a written procedure that describes the process of:

- Finding the LCD for any two fractions,
- Rewriting fractions equivalently using the LCD, and
- Computing sums and differences of two fractions.

EXPLORATION 1

Silvia is baking six sheet cakes for a party. The recipe she is using calls for $3\frac{1}{6}$ pounds of refined sugar and $5\frac{1}{4}$ pounds of unrefined sugar. First use the linear model to give

4.5

an estimate of how much sugar Silvia needs. Then compute how many pounds of sugar Silvia needs. Explain your process for both the estimation and the calculation. Can you use the same process to add other mixed numbers?

Activity: Recipe Project

Bring a recipe from home. Make sure there are at least 8 ingredients. At least 4 of the ingredients should contain fractions and at least 2 should be mixed numbers. Be sure to indicate serving size.

1. Rewrite the recipe so that it would serve twice the number of people as the original recipe.
2. Rewrite the recipe so that it would serve half the number of people as the original recipe.

EXAMPLE 2

Compute the sum $6\frac{3}{5} + 3\frac{5}{7}$.

SOLUTION

There are at least three ways to compute this sum.

1. Improper Fractions:

One approach is to treat this as an ordinary fraction addition problem by converting from mixed to improper fractions and back again. First, convert the mixed fractions to improper fractions:

$$6\frac{3}{5} = \frac{6 \cdot 5}{1 \cdot 5} + \frac{3}{5} = \frac{33}{5} \qquad \text{and} \qquad 3\frac{5}{7} = \frac{3 \cdot 7}{1 \cdot 7} + \frac{5}{7} = \frac{26}{7}$$

Then, find the LCD and compute the sum. Note that in this case the denominators are relatively prime, so the LCD is their product.

$$\begin{aligned} \frac{33}{5} + \frac{26}{7} &= \frac{33 \cdot 7}{5 \cdot 7} + \frac{26 \cdot 5}{7 \cdot 5} \\ &= \frac{231}{35} + \frac{130}{35} \\ &= \frac{361}{35} \end{aligned}$$

Finally, convert the improper fraction to a mixed fraction and simplify. Because the largest multiple of 35 less than 361 is 350, convert 361 to $35 \cdot 10 + 11 = 350 + 11$, or $361 \div 35$ is 10 with a remainder of 11.

$$\begin{aligned}
\frac{361}{35} &= \frac{350 + 11}{35} \\
&= \frac{350}{35} + \frac{11}{35} \\
&= 10 + \frac{11}{35} \\
&= 10\frac{11}{35}
\end{aligned}$$

2. Combining Like Parts:

The improper fractions approach can be cumbersome because it involves working with relatively large numbers. Another approach is to consider each mixed fraction as the sum of an integer and a proper fraction and regroup the whole parts together and the proper fractions together:

$$6\frac{3}{5} + 3\frac{5}{7} = \left(6 + \frac{3}{5}\right) + \left(3 + \frac{5}{7}\right) = (6 + 3) + \left(\frac{3}{5} + \frac{5}{7}\right)$$

This leads to the sum of proper fractions:

$$\begin{aligned}
\frac{3}{5} + \frac{5}{7} &= \frac{3 \cdot 7}{5 \cdot 7} + \frac{5 \cdot 5}{7 \cdot 5} \\
&= \frac{21}{35} + \frac{25}{35} \\
&= \frac{46}{35} \\
&= 1\frac{11}{35}
\end{aligned}$$

Combining these results, the original sum is

$$\begin{aligned}
6\frac{3}{5} + 3\frac{5}{7} &= \left(6 + \frac{3}{5}\right) + \left(3 + \frac{5}{7}\right) \\
&= (6 + 3) + \left(1 + \frac{11}{35}\right) \\
&= 10 + \frac{11}{35} \\
&= 10\frac{11}{35}
\end{aligned}$$

As you can see, in computing the sum of mixed fractions, it is often easier to separate the mixed fractions as whole parts and fractional parts. Add each group and then combine these two partial sums.

3. Vertical Addition:

There is another way to organize and write this same process vertically:

$$\begin{array}{r} 6\frac{3}{5} \\ \underline{+\,3\frac{5}{7}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 6 & \frac{3}{5} \\ \underline{+\,3} & \underline{+\,\frac{5}{7}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 6 & \frac{21}{35} \\ \underline{+\,3} & \underline{+\,\frac{25}{35}} \\ 9 & \frac{46}{35} \end{array} \quad \rightarrow \quad \begin{aligned} &= 9 + \left(1 + \frac{11}{35}\right) \\ &= 10 + \frac{11}{35} \\ &= 10\frac{11}{35} \end{aligned}$$

How would finding the difference between two mixed fractions be different?

EXAMPLE 3

Compute the following differences:

a. $8\frac{4}{5} - 5\frac{3}{10}$ b. $6\frac{3}{5} - 3\frac{4}{7}$ c. $4 - 2\frac{3}{5}$

SOLUTION

a. Use the vertical method from the previous example:

$$\begin{array}{r} 8\frac{4}{5} \\ \underline{-\,5\frac{3}{10}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 8 & \frac{4}{5} \\ \underline{-\,5} & \underline{-\,\frac{3}{10}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 8 & \frac{8}{10} \\ \underline{-\,5} & \underline{-\,\frac{3}{10}} \\ 3 & \frac{5}{10} \end{array} \quad \rightarrow \quad \begin{aligned} &= 3 + \frac{5}{10} \\ &= 3 + \frac{1 \cdot 5}{2 \cdot 5} \\ &= 3\frac{1}{2} \end{aligned}$$

Notice that the fraction $\frac{5}{10}$ in the solution is simplified to $\frac{1}{2}$.

b. Again, use the vertical method. However, a complication arises when attempting to subtract $\frac{5}{7}$ from $\frac{3}{5}$ since $\frac{5}{7}$ is greater than $\frac{3}{5}$.

$$\begin{array}{r} 6\frac{3}{5} \\ \underline{-\,3\frac{5}{7}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 6 & \frac{3}{5} \\ \underline{-\,3} & \underline{-\,\frac{5}{7}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 6 & \frac{3 \cdot 7}{5 \cdot 7} \\ \underline{-\,3} & \underline{-\,\frac{5 \cdot 5}{7 \cdot 5}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 6 & \frac{21}{35} \\ \underline{-\,3} & \underline{-\,\frac{25}{35}} \end{array}$$

To avoid the negative fraction, rename 6 as 5 + 1 and regroup 1, or $\frac{35}{35}$, with the fraction $\frac{21}{35}$.

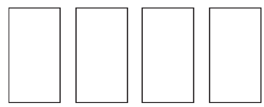
$$\begin{array}{rr} 5 & 1 + \frac{21}{35} \\ \underline{-\,3} & \underline{-\,\frac{25}{35}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 5 & \frac{35}{35} + \frac{21}{35} \\ \underline{-\,3} & \underline{-\,\frac{25}{35}} \end{array} \quad \rightarrow \quad \begin{array}{rr} 5 & \frac{56}{35} \\ \underline{-\,3} & \underline{-\,\frac{25}{35}} \\ 2 & \frac{31}{35} \end{array} \quad \rightarrow \quad \begin{aligned} &= 2 + \frac{31}{35} \\ &= 2\frac{31}{35} \end{aligned}$$

c. Since you need to subtract $\frac{3}{5}$ you must regroup or borrow a whole from 4.

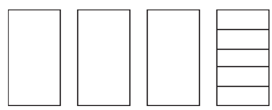
$$\begin{array}{r} 4 \\ -\,2\frac{3}{5} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{rr} 3 & \frac{5}{5} \\ -\,2 & -\,\frac{3}{5} \\ \hline 1 & \frac{2}{5} \end{array} \quad \longrightarrow \quad \begin{array}{l} \\ = 1 + \frac{2}{5} \\ = 1\frac{2}{5} \end{array}$$

We can visualize the stages as follows where we start with Step 1 as shown below. Then regroup 4 as 3 + 1 = 3 + $\frac{5}{5}$ as shown in Step 2. Finally, subtract $2\frac{3}{5}$ as blacked out in Step 3, leaving $1\frac{2}{3}$.

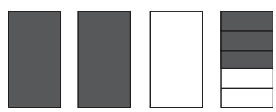
Step 1



Step 2



Step 3



EXERCISES

1. Compute the following sums of mixed fractions using either the horizontal or vertical method. Show all the steps in the process. Simplify your answers if needed.

a. $4\frac{1}{6} + 2\frac{3}{6}$	c. $2\frac{2}{3} + 4\frac{1}{3}$	e. $5\frac{3}{4} + 6\frac{5}{6}$
b. $6\frac{2}{5} + 2\frac{2}{7}$	d. $3\frac{3}{8} + 4\frac{9}{12}$	f. $6\frac{6}{8} + 2\frac{2}{3}$

2. Compute the following differences of mixed fractions using the vertical method. Show all the steps in the process.

a. $6\frac{1}{5} - 3\frac{1}{6}$	c. $6 - 3\frac{2}{6}$	e. $9\frac{3}{6} - 3\frac{5}{8}$
b. $5\frac{3}{10} - 3\frac{5}{8}$	d. $7\frac{1}{8} - 3\frac{5}{8}$	f. $10\frac{4}{12} - 9\frac{7}{8}$

For problems 3 - 9, write an expression that can be used to solve the problem. Then solve the problem and simplify if necessary.

3. Travis has $5\frac{3}{8}$ gallons of orange juice. Alex has $4\frac{1}{4}$ gallons of pineapple juice. How much fruit juice do the boys have together?

4. While training for a marathon, Joseph rode his bike $3\frac{5}{8}$ miles on Monday and $4\frac{3}{4}$ miles on Tuesday. How much farther did he ride on Tuesday than on Monday?

5. Noah is painting a large wall at the gym. He pours $2\frac{2}{6}$ gallons from a 5 gallon container. How much paint is left in the container?

6. Phillip cut $\frac{3}{8}$ of a foot off a piece of PVC pipe that was $5\frac{1}{4}$ feet long. How much pipe does he have left?

7. Frankie ate $2\frac{1}{3}$ servings of carrots, $1\frac{3}{4}$ servings of corn, $\frac{7}{8}$ servings of green beans, and $2\frac{2}{5}$ servings of bananas this weekend. How many servings of vegetables did Frankie eat this weekend?

8. During a recent stormy day, it rained $10\frac{2}{10}$ inches in Austin and $7\frac{1}{2}$ inches in San Marcos. How much more rain fell in Austin than San Marcos?

9. There are 2 fifth grade classes at Midway Elementary. Class A has $\frac{1}{3}$ males and Class B has $\frac{2}{3}$ males. Both classes have PE together. Class A has a total of 24 students and Class B has 30 students. Use what you've learned about fractions to find what fraction of the combined PE class is male.

10. Practice renaming whole numbers as mixed numbers:

    a. $9 = 8\frac{}{3}$ b. $4 = 3\frac{}{4}$ c. $7 = 6\frac{}{8}$

11. Practice subtracting mixed numbers from whole numbers:

    a. $9 - 2\frac{5}{8}$ b. $4 - 1\frac{2}{3}$ c. $7 - 6\frac{3}{4}$

12. Practice subtracting mixed numbers with like denominators:

    a. $9\frac{3}{8} - 2\frac{5}{8}$ b. $4\frac{1}{3} - 1\frac{2}{3}$ c. $7\frac{1}{4} - 6\frac{3}{4}$

13. Practice subtracting mixed numbers with unlike denominators:

    a. $8\frac{1}{9} - 2\frac{5}{6}$ b. $9\frac{3}{10} - 4\frac{1}{2}$ c. $4\frac{5}{6} - 3\frac{8}{9}$

Spiral Review:

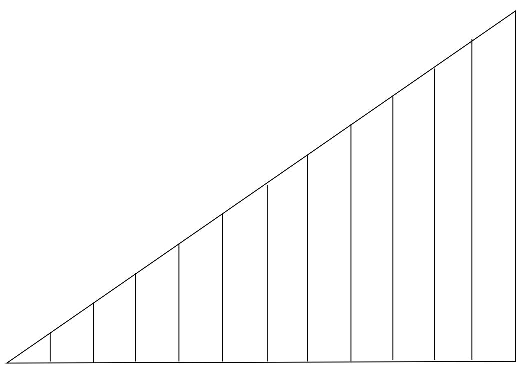
14. What is the least common multiple that Pat can use to add three fractions with

denominators of 3, 8, and 9?

15. Marco's homeroom teacher ordered 3 pizzas (pepperoni, cheese, and sausage) as a reward for their hard work. Each pizza was divided into 8 pieces. The students ate 5 pieces of pepperoni, 6 pieces of cheese, and 7 pieces of sausage. What portion of the pizza was not eaten?

16. **Ingenuity:**

Doug Bob, a farmer, plans to paint gold stripes on one side of his tool shed. The side of his tool shed is triangular, as shown in the following picture, and is 3 meters tall at its highest point. Doug Bob plans to paint eleven stripes. The first stripe will be $\frac{1}{4}$ meter tall, and each successive stripe will be $\frac{1}{4}$ meter taller than the previous one, with the last stripe being $2\frac{3}{4}$ meters tall. What is the total length of the stripes Doug Bob will paint?



17. **Investigation:**

Add the following pairs of fractions, and in each case, write your answer as a mixed fraction. See if you can find a pattern in your answers.

a. $\frac{2}{3}+\frac{3}{2}$	c. $\frac{4}{7}+\frac{7}{4}$	e. $\frac{5}{8}+\frac{8}{5}$	g. $\frac{11}{13}+\frac{13}{11}$
b. $\frac{3}{5}+\frac{5}{3}$	d. $\frac{5}{6}+\frac{6}{5}$	f. $\frac{5}{9}+\frac{9}{5}$	h. $\frac{25}{36}+\frac{36}{25}$

18. **Challenge:**

How many pairs of positive integers (a, b) with $a \le b$ satisfy $\frac{1}{a}+\frac{1}{b}=\frac{1}{6}$?

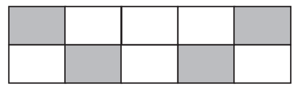
SECTION 4.R
CHAPTER REVIEW

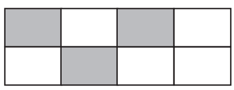
1. Find an equivalent fraction for each of the following:

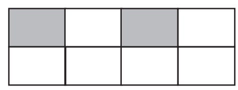
   a. $\frac{2}{5}$ b. $\frac{6}{10}$ c. $\frac{2}{3}$ d. $\frac{1}{4}$ e. $\frac{8}{12}$

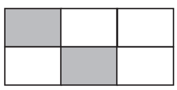
2. Label the fractional parts of the following shaded areas and order from least to greatest:

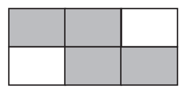
   a. 

   d. 

   b. 

   e. 

   c. 

   f. 

3. What fractions of the total number of these figures are squares, triangles, and circles? Order those fractions from greatest to least.

   △ ○ □ △ □ ○ △ ○ △ ○ △ ○ △ □ ○ △ □ ○ □ △ □

4. Using the number line below, give three fractions that are greater than $\frac{1}{2}$ and less than $\frac{4}{5}$.

5. Rewrite each of the following fractions in simplest form.

   a. $\frac{20}{25}$ c. $\frac{12}{36}$ e. $\frac{17}{68}$

   b. $\frac{45}{65}$ d. $\frac{16}{17}$ f. $\frac{18}{51}$

6. Answer the following using <, >, or =

   a. $\frac{4}{5}$ ____ $\frac{4}{9}$ c. $\frac{6}{10}$ ____ $\frac{5}{8}$ e. $\frac{8}{9}$ ____ $\frac{7}{8}$

   b. $\frac{4}{6}$ ____ $\frac{10}{15}$ d. $\frac{3}{8}$ ____ $\frac{4}{10}$ f. $\frac{36}{42}$ ____ $\frac{6}{7}$

Create a model for each of the following situations. Then answer each question.

7. If Isabel worked on her math homework for 20 minutes, what fraction of an hour did she work on her homework? Write this fraction in its simplest form.

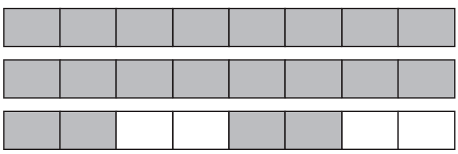
8. Your friend eats $\frac{1}{3}$ of a pizza and you eat $\frac{1}{2}$ of the same pizza. Who ate the most pizza? How much of the pizza is left over?

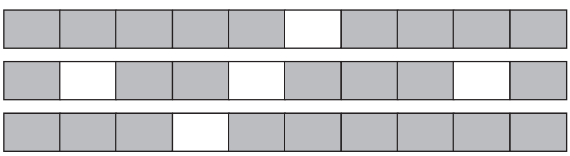
9. Joseph practiced the piano for $\frac{1}{4}$ of an hour and then cleaned the instrument for $\frac{1}{4}$ of an hour. How much time did he spend doing both activities?

10. The month of April has 30 days. If Pam worked $\frac{2}{3}$ of the number of days in April, how many days would that be?

11. Write the following shaded areas as an improper fraction and a mixed number.

a. 

b. 

c. 

12. Convert each improper fraction to a mixed number.

a. $\frac{12}{7}$ c. $\frac{55}{6}$

b. $\frac{23}{4}$ d. $\frac{10}{3}$

13. Convert each mixed number to an improper fraction.

a. $3\frac{2}{3}$ c. $8\frac{1}{8}$

b. $5\frac{4}{5}$ d. $12\frac{5}{6}$

14. Create a number line showing the following benchmark fractions: 0, $\frac{1}{2}$, and 1. Insert the approximate location of each given fraction by placing a point at the best location and naming it with the given letters.

$A = \frac{2}{11}$ $\quad B = \frac{3}{4}$ $\quad C = \frac{5}{9}$ $\quad D = \frac{7}{8}$ $\quad E = \frac{3}{10}$

15. Create a number line showing benchmark fractions by halves from 0 to 3. Insert the approximate location of each given fraction by placing a point at the best location and naming it with the given letters.

$A = 2\frac{1}{4}$ $\quad B = \frac{0}{8}$ $\quad C = \frac{3}{6}$ $\quad D = \frac{15}{8}$ $\quad E = \frac{7}{7}$

16. What common denominator can you use to add $\frac{2}{3}$ and $\frac{3}{5}$? Explain how you decided and show it visually as well.

17. Perform the following operations. Write your answer in simplest form and as a mixed fraction, if appropriate.

a. $\frac{2}{5} + \frac{3}{7}$

b. $1\frac{1}{2} - \frac{3}{8}$

c. $\frac{4 \cdot 5}{6} + \frac{5 \cdot 3}{8}$

d. $1\frac{1}{4} - 5\frac{3}{4}$

e. $4\frac{1}{3} - 2\frac{3}{4}$

f. $\frac{x}{p} - \frac{y}{p}$

DECIMAL AND PERCENT REPRESENTATIONS

5

SECTION 5.1
CONSTRUCTING DECIMALS

We have all been concerned about the cost of an item when we shop or go out to eat. For example, a cheeseburger might cost \$3.85 at a restaurant, a pair of jeans might cost \$18.97 at a store, or the entrance fee to an amusement park might cost \$27.00. All of these prices use decimal notation, a common way of writing numbers that include parts that are less than 1.

In the last example, \$27.00, we could have written \$27 instead. This means 2 tens and 7 ones. However, when we write \$27.00, the zeros to the right of 7 give us more information: that the price is exact and includes no cents.

Just as with the integers, a place value is one-tenth the place value of the digit to its immediate left. For example, in the number 43.26, the 3 is in the ones place, which is one-tenth the tens place occupied by the 4. Then 2 is in the tenths place which is one-tenth the ones place.

In an example with money, notice that the 8 in the cost of the \$3.85 cheeseburger is in the dimes place or the tenths-of-a-dollar place and 5 is in the pennies place or the hundredths place. It takes 10 dimes to make a dollar and it takes 10 pennies to make a dime. Therefore it takes 100 pennies to make a dollar.

EXPLORATION: LOCATING DECIMAL NUMBERS ON A NUMBER LINE.

If we think of the number 1 on the number line as \$1.00, where would we locate half a dollar or \$0.50? Because there are 10 dimes in a dollar, where would \$0.10 be located on the number line? \$0.20? \$0.30? Can you locate \$0.01, or more simply 0.01, on the number line, knowing that there are 10 pennies in a dime?

We know when we write the number 0.30 that there is another way that this decimal can be written. Thirty hundredths can be written as 0.3. How could you show the two numbers 0.3 and 0.30 are really equivalent to each other on the number line?

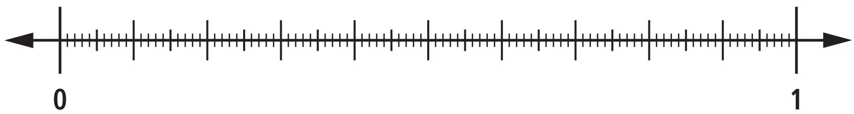
Use a number line like the one below to find the locations of the following decimal numbers. Notice that 0 and 1 are labeled on the number line.

a. 0.4

b. 0.27

c. 0.68

d. 0.7



Discuss how you approximated the specific location. What strategy did you use to determine which number is greater than or less than another? In general, what strategy can you use to compare decimal numbers?

Numbers can be compared on the number line. The smaller number appears to the left of the larger number. Another strategy in comparing decimals is to write the numbers so that they have the same number of place values. Compare the digits from left to right until there is a difference in the digits in the same place values. For example, 0.2 and 0.27 can be written as 0.20 and 0.27. When we compare the hundredths place, clearly 27 hundredths is more than 20 hundredths because 27 is greater than 20.

For each pair of numbers, determine which is greater. Justify your answer using the number line.

a. 0.68 and 0.7

b. 0.34 and 0.339

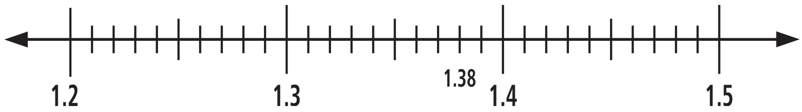
c. 0.268 and 0.271

Remember what you have learned about decimals. You know that the number line is one tool that can help you locate decimals, order them, and compare them. The place value chart is another tool that can help you do this. Let's take a look at the next example to understand how the place value chart may be useful.

EXAMPLE 1

In a science project, Jeremy measured the distance two cars traveled from a common starting point. Car A traveled 1.38 m and Car B traveled 1.4 m. Which car traveled farther?

SOLUTION



Since Car A is to the left of Car B on the number line, you can say that 1.38 < 1.4. Furthermore, by placing the lengths in a place value chart you can see that 38 hundredths is less than 40 hundredths.

If you place both numbers in a Place Value Chart you can clearly see that 1.38 is less than 1.40.

Place Value Chart				
Tens	Ones	.	Tenths	Hundredths
	1	.	3	8
	1	.	4	0

PROBLEM 1

For each pair of numbers, determine which is greater. Justify your answer using a number line. Double check your work on a Place Value Chart.

a 0.68 and 0.7

b. 0.34 and 0.339

c. 0.268 and 0.271

d. 1.12 and 1.02

e. 3.45 and 3.045

f. 2.133 and 2.10

Like whole numbers, decimals can also be rounded to a specified place value. Consider the number 18.625. If you were asked to round this to the nearest tenth, you begin by underlining the 6 in the tenths place, $18.\underline{6}25$. Then, you look to the right of the underlined digit to determine if you should round up or down. Rounding up allows you to increase the underlined digit by 1 while rounding down means the underlined digit remains the same. In our example, the 2 means that the underlined 6 will round down, or stay the same: 18.625 rounded to the tenth place is 18.6. An equivalent answer is

18.600 but the ending zeros are generally dropped when a decimal number ends in zeros.

To round decimals:

1. Find the place value you want to round to. Call this the specified digit (the underlined place in our previous example above). Look at the digit to the right of it.
2. If the digit to the right is less than 5, do not change the specified digit and drop all digits to the right of it.
3. If the digit to the right is 5 or greater, add one to the specified digit and drop all digits to the right of it.

EXAMPLE 2

Round 21.095 to the nearest hundredth.

SOLUTION

Underline the 9 in 21.0$\underline{9}$5. Since the number to its right, 5, tells the 9 to round up, the 9 becomes 10. Remember that in place value you can only have one digit per place, so you write the 0 in the hundredths place value and carry the 1 to the tenths place. Therefore, 21.095 rounded to the nearest hundredth is 21.10.

PROBLEM 2

Round the following number 127.398359 to the specified place value:

a. nearest ones
b. nearest tenths
c. nearest hundredths
d. nearest thousandths

PROBLEM 3

A 50-inch flat screen television is on sale for $719.29.

a. Round the price to the nearest dollar.
b. Round the price to the nearest hundred dollars.

When alphabetizing two or more words, you ignore beginning letters that are alike until the first different letter to determine which word comes before another. For example, concentrate, concert, and cone are in alphabetical order. Can you see why? Similarly, when ordering numbers, compare the digits starting with the largest place value. Ignore the digits that are alike until the first place value that shows a difference. This process is often easiest to see if you make a vertical column of the numbers you are ordering.

PROBLEM 4

Write the following numbers in order from least to greatest. Double check your work on a Place Value Chart.

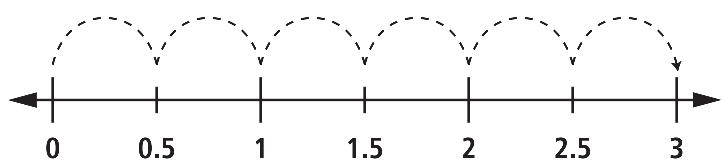
3.065, 3.6, 3.56, 3.605, 3.65

EXAMPLE 3

Sara spent \$3 on 6 chocolate bars. How much did each candy bar cost?

SOLUTION

This is a typical division problem where the cost of each candy bar is 3 ÷ 6. You might expect trouble because the divisor is greater than the dividend. Using the linear skip counting model, how long does each skip need to be to travel a distance of 3 units, or 3 dollars in this case, in 6 skips?



You can see that each jump is \$0.50 or half a dollar. This represents the fact that each candy bar costs \$0.50. If necessary, verify this using the calculator by computing 3 ÷ 6 or adding six skips 0.50 long. The long division method gives us the same result, because there is a decimal point before the 5. You might write the problem like step 1 on the next page. The divisor is greater than the dividend, so modify the long division process by placing a decimal point. Include two zeros to the right of the decimal point of the dividend because we are working with money. We know that \$3 = \$3.00 where

$0.00 represents no cents. Where does the decimal place appear in the quotient? Why does this make sense?

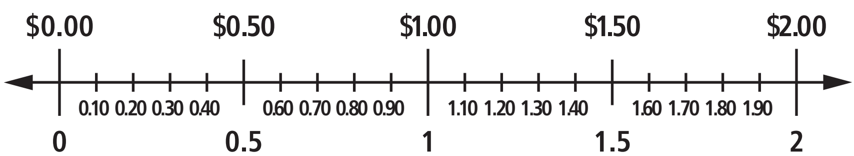
Step 1: $6\overline{)3}$

Step 2:

$$\begin{array}{r} 0.5 \\ 6\overline{)\ 3.00} \\ -\ 3.00 \\ \hline 0.00 \end{array}$$

Compute the following division problems by using an abbreviated number line from 0 to 2, like the one below. Find the quotient using the skip-counting method, then use the scaffolding method to verify your answer. Make sure the decimal point in the quotient makes sense in the context of the problem. Use the calculator to confirm your work, if necessary.

a. $1 ÷ 5 b. $1.60 ÷ 8 c. $1.20 ÷ 4



When long division involves two-digit numbers, the skip-counting model becomes more difficult. We can also use the reverse of the area model to understand long division. At this point knowing the multiplication facts and how to use them makes life much simpler.

Mr. Garza has some money in his pocket that he intends to divide equally among his four nephews. Use the area model and the scaffolding model to compute how much each nephew receives if he has

a. $26 in his pocket.

b. $27.40 in his pocket.

EXERCISES

1. Round the following numbers to the nearest tenths:

a. 0.25 b. 0.059 c. 4.531 d. 2.99

2. Round the following numbers to the nearest hundredths:

   a. 3.163 b. 210.152 c. 0.039

3. Round the following numbers to the nearest thousandths:

   a. 0.2531 b. 1.0099 c. 0.1327

4. Round the following numbers first to the nearest hundredths, then to the nearest tenths, and finally to the nearest ones:

   a. 0.78 b. 5.254 c. 10.949

5. Complete the table below:

Decimals	Words
0.07	
0.3	
	Ninety hundreths
	Three hundred and three hundredths
0.85	
0.603	
	One thousand and twenty-four ten thousandths

6. Compare the following pairs of numbers using <, >, or =

   a. 0.034 ____0.039
   b. 1.21 ____1.210
   c. 27.137____27.15
   d. 0.835 ____0.836
   e. 3.510 ____3.501
   f. 61.35____61.350
   g. 0.713 ____0.731
   h. 3.63 ____3.6317

7. Order the decimals from the least to the greatest

   a. 0.305, 0.035, 0.350, 0.530
   b. 25.1305, 25.1350, 25.1503, 25.0305

8. Diann and Terry walked to exercise. If Diann walked 2.26 miles and Terry walked 2.73 miles, who walked more miles? Joan then joined them the next evening. She walked 2.09 miles. Write the distances in order from greatest to least.

9. Students in Ms. Martin's class scored well on their math benchmarks. Their scores were: 93.4, 89.5, 87.3, 92.21, 93.53, and 90.01. Order the scores from least to greatest.

10. Divide each of the following.

    a. $2 \div 5$ b. $1.40 \div 7$ c. $15.24 \div 12$

11. In working each of the following exercises, be careful to scale your number line appropriately.

    a. Name two decimals between 0.1 and 0.3. Draw a number line and locate the four decimal numbers on it.

    b. Name four decimals between 0.29 and 0.307. Draw a number line and locate the six decimal numbers on it.

    c. Name two decimals between $^{-}0.11$ and $^{-}0.24$. Show whether $^{-}0.109$ is between $^{-}0.11$ and $^{-}0.24$. Draw a number line and locate the five decimal numbers on it.

Spiral Review

12. Joe is making a snack mix for a study group that is meeting at his house after school. He mixed $4\frac{1}{3}$ cups of cereal, $3\frac{1}{4}$ cups of nuts, and $2\frac{1}{3}$ cups of pretzels. How many total cups of ingredients are in his snack mix?

13. What is the value of the expression $8 + 7 \times 6^2 \div 2$?

14. **Ingenuity:**

    Consider all of the numbers that we can make by putting the digits 1, 2, 3, and 4 in some order, in the blanks below. Use each digit only once.

    _ . _ _ _

    How many of these numbers are greater than 2.5?

15. **Investigation:**

Consider the following rectangle, which is divided into tenths:

a. Copy the figure above, and shade $\frac{1}{2}$ of the figure. How many tenths are shaded?

b. Can we use what we found in part a to write $\frac{1}{2}$ as a decimal?

c. Copy the original figure again, and shade $\frac{1}{5}$ of it. How many tenths are shaded? Use this information to write $\frac{1}{5}$ as a decimal.

d. How could we write $\frac{1}{4}$ as a decimal? (Hint: Consider using a rectangle which is divided up into pieces smaller than tenths.)

5.2

SECTION 5.2
OPERATING WITH DECIMALS

EXPLORATION 1: ADDITION AND SUBTRACTION OF DECIMALS

Betty is about to take a trip. She fills her car with gas for \$29.90 and buys a map for \$3.49, a drink for \$1.09, and a pack of gum for \$0.99. Estimate the cost of her purchase before taxes. Is \$40.00 enough to pay for the purchase, excluding tax?

If you calculated \$30.00 + \$3.50 + \$1.00 + \$1.00 to get \$35.50 you had a good estimate of her cost. To get the exact cost, however, you must add \$29.90 + \$3.49 + \$1.09 + \$0.99 or take the estimated cost and subtract the excess of 10¢ + 1¢ + 1¢ from the estimated cost, then add in 9¢ for the underestimation of \$1.09. You overestimated by a total of 3¢ so you should subtract 3¢ from \$35.50 for the actual cost.

When you subtract 3¢ from \$35.50, you are really subtracting \$0.03 from \$35.50 to get \$35.47. As with addition, it is important to keep in mind the place value and subtract the hundredths from the hundredths, the tenths from the tenths, and so forth. You might have heard the phrase "line up the decimals." This vertical, stacking method assures that the place values also line up to do the calculation.

$$\begin{array}{r} 29.90 \\ 3.49 \\ 1.09 \\ \underline{+\ 0.99} \\ 35.47 \end{array}$$

Linear Model:

How do you use the number line to add decimal numbers? Compute the sums in parts a & b using the number line. Compute the sums in parts c & d using the stacking method.

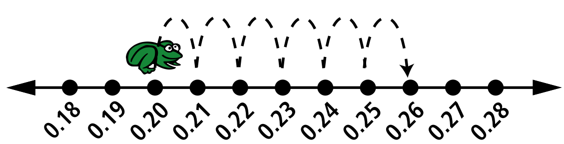
a. 0.2 + 0.06

c. 0.23 + 0.54

b. 0.38 + 0.47

d. 0.26 + 0.31

For example, 0.2 + 0.06 = 0.20 + 0.06.



Explain how the number line can help to estimate a sum before you calculate the actual total.

How do you use the number line to subtract decimals? Compute the following differences using the number line, and then subtract using the traditional stacking method.

a. 0.63 – 0. 47 b. 0.2 – 0.06

There are times when we need to write really big numbers but want to avoid writing too many zeros. For example, we write 2 million instead of 2,000,000 or 30 million instead of 30,000,000. If we write 700 thousand, we mean 700,000. Finally, if 1 million is 1,000,000 then 0.5 million is half of 1,000,000 or 500,000.

a. What is another way of writing 1.5 million?

b. What is another way of writing .9 million?

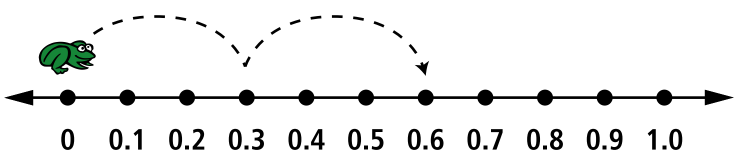
c. Write 8.9 billion as a whole number.

EXPLORATION 2: MULTIPLICATION OF DECIMALS

Use the linear model to show how to compute the following products:

a. $3 \cdot 2$ b. $(0.3) \cdot 2$ c. $(0.3) \cdot (0.2)$

The first product is simply 2 jumps of length 3. The second product is 2 jumps of length 0.3, as shown below:



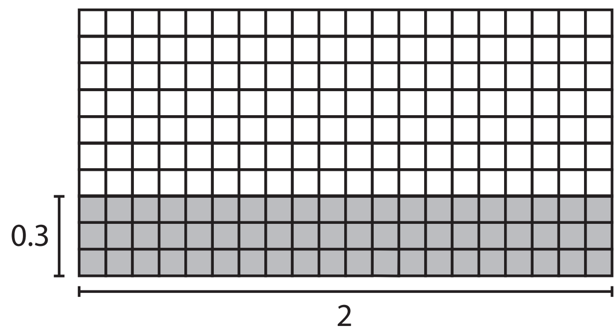
Two jumps of length 0.3 gives us the location of 0.6.

However, modeling the third product, (0.3)(0.2), is not clear. What do we mean by 0.2 jumps? For this product, the area model may be more helpful.

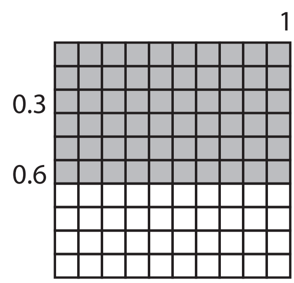
Using the area model for (3)(2), we find the area of a 3 by 2 rectangle. How do we use the area model for (0.3)(2)?

Draw a rectangle that has length 0.3 and width 2. When drawing this rectangle it is helpful to use grid paper and choose an appropriate scale. In this case, we need to measure both 0.3 and 2. Using a grid, assign each small square a length of 0.1.

Outline a 1 by 1 square using the 0.1 grid. How many 0.1 by 0.1 small squares are in a 1 by 1 square? Now outline a 0.3 by 2 rectangle on the grid. With a picture of the rectangle and its dimensions, you can see that the area is made up of 60 small squares, each with an area of $0.1 \times 0.1 = 0.01$. So the product of $(0.3)(2) = (60)(0.01)$.

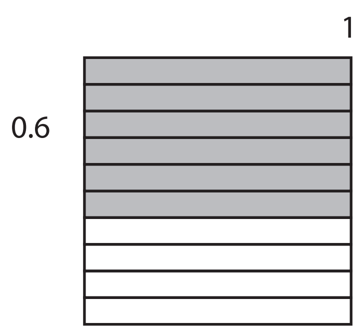


We can rearrange the model to look like the following:

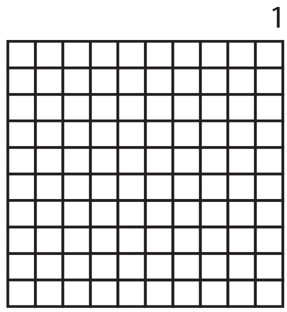


Thus, $(60)(0.01) = 0.60$ since there are 60 hundredths shaded above.

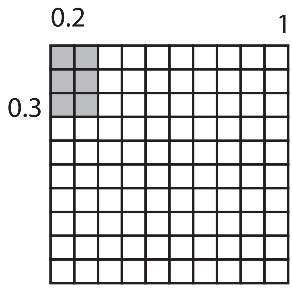
However, we can write 0.60 as 0.6, or 6 tenths, which is modeled below.



How do we compute the product (0.3)(0.2) using the area model? Consider the grid below. Each side of the square has length 1. Note that the length of each little square is 0.1. Use this grid to model the product (0.3)(0.2).



One way to show the product (0.3)(0.2) is to shade a rectangle within the grid that is 0.3 long (horizontally) and 0.2 wide (vertically).



The result is a small rectangle with 6 little squares. What is the area of each little square? Since the large square has area 1 and there are 100 little squares, the area of each little square is 0.01 or $\frac{1}{100}$. So the area of 6 little squares is 0.06 or $\frac{6}{100}$.

PROBLEM 1

Compute the following products using the grid.

a. (0.6)(0.7)

b. (0.5)(2)

c. (0.3)(0.8)

d. (0.9)(1.4)

PROBLEM 2

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

(3)(1) =	(0.3)(2) =	(0.6)(7) =
(3)(0.1) =	(0.3)(0.2) =	(0.6)(0.7) =
(3)(0.01) =	(0.3)(0.02) =	(0.6)(0.07) =
(3)(0.001) =	(0.3)(0.002) =	(0.6)(0.007) =
(0.3)(0.1) =	(0.03)(0.2) =	(0.06)(0.07) =

What patterns do you notice?

PROBLEM 3

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

(2.4)(3.1) =	(562)(7) =	(0.483)(27) =
(0.24)(3.1) =	(562)(0.7) =	(4.83)(27) =
(0.024)(3.1) =	(56.2)(0.7) =	(48.3)(27) =
(0.24)(0.31) =	(5.62)(0.7) =	(4.83)(2.7) =
(24)(0.31) =	(0.562)(7) =	(4.83)(0.27) =

a. What patterns do you notice?

b. How many decimal places does each factor have?

c. How many decimal places are in each product?

d. What is the connection between these two for each product?

One observation you may have made is that the products have the same digits as if you were doing the whole number multiplication. Another important observation is that the number of decimal places in the product is equal to the sum of the number of decimal places in the factors. For example,

0.2	(1 decimal place)	2	(no decimal places)
x 0.3	(1 decimal place)	x 0.3	(1 decimal place)
0.06	(2 decimal places)	0.6	(1 decimal place)

This pattern is useful when we multiply numbers with many decimal places. The linear model and even the area model become more difficult to use, but the pattern continues to be useful.

For example,

$$\begin{array}{rl} 0.2 & \text{(1 decimal place)} \\ \underline{\times\ 0.03} & \text{(2 decimal places)} \\ 0.006 & \text{(3 decimal places)} \end{array}$$

You may wish to use this method of multiplying the decimal numbers by first multiplying the factors as if they were whole numbers. After you find the whole number, include the decimal point in the correct place using the pattern we used above.

Division of Decimals

When you divide whole numbers, you may find the resulting quotient is no longer a whole number. Here is an example.

Lisa spent $4 on 8 identical candy bars. How much did each candy bar cost? Using a linear model, how long does each trip need to be in order to reach 4 using 8 total hops? You can see that each hop is $.50 or half a dollar. This represents the fact that each candy bar costs $0.50.

The long division method gives us the same result, because there is a decimal point before the 5. You could write the problem as we have below in Step 1.

Step 1: $8\overline{)4}$

The divisor is greater than the dividend, so modify the long division process as in step 2 below by placing a decimal point after the dividend. Include the two zeros in the dividend because we are working with money and we know that $4 = $4.00 where $0.00 represents zero cents.

Step 2:

$$\begin{array}{r} 0.50 \\ 8\overline{)4.00} \\ \underline{-4.00} \\ 0.00 \end{array}$$

PROBLEM 4

Adam bought 9 jawbreakers for $2.70. Use the long division process to determine the cost of each jawbreaker.

EXERCISES

1. Compute the following using the stacking method:

   a. 0.53 + 0.36
   b. 0.4 + 0.65
   c. 2.406 + 3.36
   d. 13.8 + 2.97 + 0.09
   e. 0.83 + 0.57
   f. 4.6 – 2.07
   g. 0.058 – 0.287
   h. 3.59 + 0.087
   i. 18.9 + 1.76
   j. 1.7 + 1.087
   k. 16.43 + 29
   l. 0.007 + 0.08 + 1.9

2. Determine which of the following pairs of numbers is closer together.

   a. 0.6 and 0.7 or 0.34 and 0.37
   b. 0.87 and 0.91 or 0.892 and 0.908
   c. 0.23 and 0.196 or 0.098 and 0.11

3. Valerie is making homemade tortillas from her grandmother's recipe. The recipe calls for 326 grams of salt. While experimenting with the recipe, she discovers that if she increases or decreases the recipe by 1.6 grams of salt, the tortillas will taste the same. Between what amounts of salt will the taste of the tortillas stay the same?

4. Erica planted a young tree last spring. During the next few months, the tree grew 1.745 meters. The height of the tree was 8.5 meters after this growth spurt. How tall was the tree when she planted it?

5. Mr. Trevino is conducting two different science experiments. He needs 0.789 grams of sodium for Experiment A and 2.15 grams for both Experiment A and B. How many grams of sodium does he need for Experiment B?

6. Normal body temperature is 98.6 ºF. One day Justin develops a fever and when the nurse takes his temperature, she finds it to be 101.8 ºF. What was the increase to Justin's body temperature due to his fever?

7. Pattie bought a 5 foot roll of red ribbon. She needed to make two bows. One bow required 2.3 feet of ribbon. The second bow is smaller and took only 1.25 feet of ribbon. How much ribbon will Pattie have left over for a third bow?

8. Andrew lives 1.48 miles from Christian. Christian lives 3.05 miles from Emilio. How far will Andrew travel if he drives from his house, picks up Christian and then drives to Emilio's?

9. Evan had \$239.57. He went to the store to buy some video games and a cell phone. He bought the first game for \$59.99 and the second game for \$49.99 (including taxes). How much money did he spend on the video games? If Evan wants to buy a cell phone for \$129.99, does he have enough money left?

10. Compute the following products using the area grid.

    a. (0.3)(0.2)
    b. (0.6)(0.5)
    c. (0.8)(0.7)
    d. (0.4)(0.9)

11. Compute the following products using the decimal place pattern.

    a. 0.03 x 0.2
    b. 3.6 x 0.7
    c. 11.8 x 2.6
    d. 5.09 x 2

12. Compute the following quotients using the long division method.

    a. $2 \div 5$
    b. $1 \div 4$
    c. $4.20 \div 7$
    d. $3.6 \div 9$
    e. $1.20 \div 12$

Spiral Review

13. Graph the following points on a coordinate grid.

    a. $(^-1,5)$
    b. $(0, 2\frac{1}{2})$
    c. $(^-3\frac{1}{2}, 0)$
    d. $(1\frac{1}{2}, {}^-1\frac{1}{2})$

14. Carol is making a border for her flower bed. She has a landscape timber that is $5\frac{7}{8}$ feet long. If she cuts off a piece that is $\frac{3}{4}$ yard long, represent the portion that is left of the original strip as a mixed fraction and an improper fraction.

15. **Ingenuity:**

Suppose we draw a number line and mark the points A = 0.72, B = 1.35, and C = 2.88. If X is a point on the number line, we use the symbols AX, BX, and CX to represent the distance from A to X, the distance from B to X, and the distance from C to X, respectively.

a. If X = 0.12, what is the sum $AX + BX + CX$?

b. If X = 2.49, what is the sum $AX + BX + CX$?

c. If X = 10, what is the sum $AX + BX + CX$?

d. If X can be any point on the number line, what is the smallest possible value of the sum $AX + BX + CX$?

16. **Investigation:**

In this Investigation, we will explore what happens when we add and subtract decimals that are close to integers.

a. What is the sum 2.45 + 0.98? Can you think of a way to find this without aligning the numbers vertically and using long addition? (Hint: It may be helpful to think of the problem in terms of money.)

b. Compute the following sums:

i. 4.83 + 6.96

ii. 10.99 + 4.99

iii. 54.22 + 199.99

c. What is the difference 5.16 – 0.97? Can you think of a way to find this without aligning the numbers vertically and using long subtraction?

d. Compute the following sums:

i. 8.24 – 2.95 – 79.99

ii. 14.41 – 8.92

iii. 266.17 – 79.99

SECTION 5.3
NUMBERS AS DECIMAL AND FRACTIONS

In the past, you have probably referred to one-half of a dollar as \$0.50 or 50 cents. One half is a fraction that is equal to 0.50, a decimal. We say that $\frac{1}{2}$ is **equivalent** to 0.50 in that they represent the same quantity though they are written differently. In this section, we will review how a fraction can be represented as a decimal number and how some decimals can be represented as fractions.

If you buy four apples for a dollar, how much does each apple cost? Dividing \$1 by 4 yields $\$1 \div 4 = \$0.25$, or 0.25 dollars. We can also say each apple costs a quarter or $\frac{1}{4}$ of a dollar. So $\frac{1}{4}$ and 0.25 are equal quantities or equivalent numbers.

Decimals can be read in terms of the place values that the digits occupy. For example, 0.47 is read "forty seven one hundredths." You know that the fractional representation $\frac{47}{100}$ is also read "forty seven one hundredths." You can also locate the numbers on the number line as the same point. The decimal form 0.47 and the fractional form $\frac{47}{100}$ actually represent the same value.

In converting a decimal to a fraction, we take advantage of the fact that we use the base ten system to write each decimal number. For example, the number 0.3 is called three tenths and so is equivalent to $\frac{3}{10}$. While the number 0.35 has 3 tenths and 5 hundredths, we usually read the number as 35 hundredths and is the same as the fractional form $\frac{35}{100}$.

PROBLEM 1

Write the fractional form of the following decimal numbers:

a. 0.7 b. 0.01 c. 0.216 d. 0.903 e. 5.4

Does the fraction $\frac{1}{5}$ have a decimal form? If we buy 5 bananas for \$1, we know that each banana costs $\$1 \div 5 = \$0.20$ or 20 cents. In other words, each banana costs $\frac{1}{5}$ of a dollar because it takes $5(\$0.20)$ to make a whole dollar. So $\frac{1}{5} = 1 \div 5 = 0.20$ or 20 hundredths. But the decimal 0.20, or twenty hundredths, has the same name as the fraction $\frac{20}{100}$. Does this mean the fraction $\frac{1}{5}$ is equal to $\frac{20}{100}$? These are equivalent fractions, so the decimal 0.20 and the fractions $\frac{1}{5}$ and $\frac{20}{100}$ all represent the same quantity.

The following property summarizes the relationship between division and the fractional notation that we just saw.

PROPERTY 5.1: FRACTIONS AND DIVISION
For any number m and nonzero number n the **fraction** $\frac{m}{n}$ is equivalent to the quotient $m \div n$.

Linear Model for Fractions Activity

On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now construct a number line from 0 to 1.

Materials: You will need a long strip of paper like a sentence strip or an 18-inch piece of adding machine paper.

1. Mark the left endpoint of the strip as 0 and the right endpoint as 1. You may fold this strip or use another strip to fold and transfer points to this master number line. Fold the strip end to end into two equal parts and mark the crease as the midpoint between 0 and 1. What fraction is this midpoint equivalent to? What decimal? Label the points on the number line as fractions above the line and as decimals below the number line.

2. Fold the strip again and use the creases to mark and label points on the master number line. Because the strip is now folded into 4 equal parts, we label the first point as $\frac{1}{4}$, which is equivalent to 0.25. Although the second point already has two labels, $\frac{1}{2}$ and 0.5 or 0.50, add the label $\frac{2}{4}$ above this point. The last crease will be labeled as $\frac{3}{4}$ or 0.75.

3. Repeat this method by folding the strip again into 8 equal parts, transferring the locations to the master number line and labeling the points with fractions and decimals. What is the decimal equivalent of $\frac{1}{8}$? Use this method to locate, mark, and label all the eighths on the number line.

4. Compare the number line with a typical foot ruler or yardstick.

5. Label these fractions and their decimal equivalents on the number line.

   a. $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$

   b. $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, \frac{10}{10}$

   c. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

   d. $\frac{1}{6}, \ldots, \frac{5}{6}, \frac{6}{6}$

   e. $\frac{1}{9}, \ldots, \frac{8}{9}, \frac{9}{9}$

6. Use your new number line to estimate the decimal form of the following fractions.

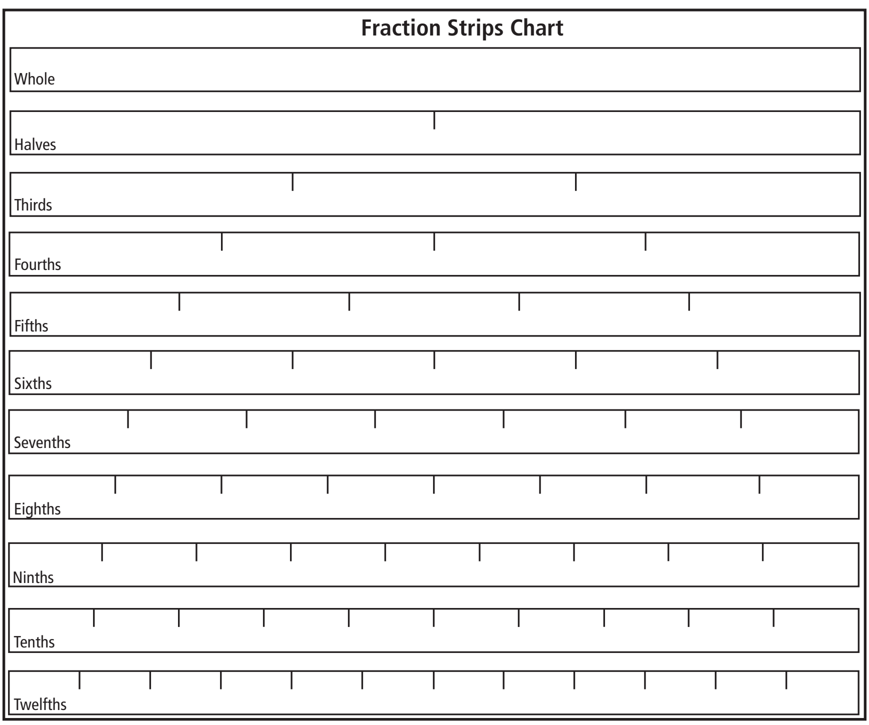
   a. $\frac{1}{20}$ $\frac{7}{20}$ $\frac{1}{40}$ $\frac{3}{40}$

   b. $\frac{1}{12}$ $\frac{5}{12}$ $\frac{1}{24}$ $\frac{4}{5}$

   c. $\frac{1}{16}$ $\frac{5}{16}$ $\frac{1}{32}$ $\frac{5}{32}$

   d. $\frac{1}{25}$ $\frac{2}{25}$ $\frac{1}{50}$ $\frac{5}{50}$

The rows of the Fraction Strips Chart below represent the result of folding the whole into equal parts to represent fractions as we did in the Linear Model Activity. The computer that created it allows the folds to be very accurate. For example, using the Fraction Strips Chart, determine which fraction is greater: $\frac{2}{5}$ or $\frac{3}{7}$.



Convert the following fractions to decimal form. You may verify your answer by dividing with a calculator, if necessary.

a. $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$

b. $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$

c. $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{5}{8}$

d. $\frac{1}{3}$, $\frac{1}{30}$, $\frac{1}{9}$, $\frac{1}{90}$

Now we ask, "What decimal is equivalent to $\frac{1}{3}$?" We could also ask what is $\frac{1}{3}$ of a dollar? We can use our new rule to see that $\frac{1}{3}$ is equivalent to the quotient of $1 \div 3$. The quotient is a repeating decimal, 0.33333... which can be written as $0.\overline{3}$. The bar over the 3 tells us that the digit 3 repeats without end. So $\frac{1}{3}$ of a dollar is $\$0.3\overline{3}$, and we cannot practically divide \$1 into 3 equal parts with our present set of coins, so we often approximate to \$ 0.33. There are other fractions that equal repeating decimals such as:

$$2 \div 3 = 0.6666... = 0.\overline{6} \approx 0.67$$

$$1 \div 6 = 0.1666... = 0.1\overline{6} \approx 0.167$$

Find other fractions that have repeating decimals.

PROBLEM 2

Complete the missing parts in the table below.

Fractions	Decimals	Words
	0.5	Five tenths
	0.35	
$\frac{3}{4}$		
		Six and 3 one hundredths
$8\frac{3}{12}$		

EXERCISES

1. Order from largest to smallest: 0.20, $\frac{1}{3}$, $\frac{3}{4}$, 0.125
2. Order from smallest to largest: $\frac{2}{8}$, $\frac{1}{5}$, 0.01, 0.1
3. Following are the final running times of 5 runners: 7.321, 7.4, 7.001, 6.2, 6.100. Write the times in order from fastest to slowest.
4. Convert each fraction to an equivalent decimal. (Hint: use what you have learned about simplifying fractions before converting to decimal form.) Explain the pattern you notice in each set of fractions.

   a. $\frac{9}{20}$, $\frac{45}{100}$

   b. $\frac{13}{25}$, $\frac{52}{100}$

   c. $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$

   d. $\frac{9}{12}$, $\frac{18}{24}$, $\frac{24}{32}$

5. Convert each unit fraction to an equivalent decimal. Write the decimals in a vertical list so that you can compare them.

   $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{7}$ $\frac{1}{8}$ $\frac{1}{9}$ $\frac{1}{10}$ $\frac{1}{11}$ $\frac{1}{12}$

6. Complete the table.

Simplest Form	Fractions in Tenths, Hundredths or Thousandths	Decimal Form
$\frac{3}{5}$		
$\frac{7}{10}$		
$\frac{9}{20}$		
$\frac{1}{8}$		
$\frac{7}{25}$		

7. Convert each decimal to an equivalent fraction. Simplify if needed.

a. 0.6	d. 0.05	g. 0.28	j. 0.96
b. 0.7	e. 0.25	h. 0.55	k. 0.075
c. 0.16	f. 0.125	i. 0.48	l. 0.625

8. Determine whether a decimal or fraction representation is more appropriate in the following situation. Explain your answer.

   a. Renee orders a meal at a restaurant. What best represents the cost of the meal, \$14.25 or \$$14\frac{1}{4}$?

   b. Billy orders a hamburger at a fast food restaurant. What best represents the weight of the hamburger, $\frac{1}{3}$ lb or 0.333… lbs?

   c. A large pizza is divided into 12 slices. Danny and Marie eat 9 slices from a large supreme pizza. What best represents the portion of the pizza eaten, 0.75 or $\frac{9}{12}$?

9. Use the Fraction Strips Chart to discover and represent three fractions that are greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. Explain how you can tell that a fraction is between $\frac{1}{4}$ and $\frac{1}{2}$ without using a Fraction Strips Chart.

10. Micah measured the diameter of the button on his shirt. The diameter was between 1.5 and 1.6 centimeters. Name three possible lengths of the diameter of his button.

11. The table below shows the distance Sergio jumped in the long jump contest at the school's track meet. What was the total distance Sergio jumped? What was the difference between the longest jump and the shortest jump?

Jump	Distance Jumped (feet)
1	$20\frac{1}{4}$ ft.
2	$18\frac{1}{2}$ ft.
3	19 ft.
4	$21\frac{3}{4}$ ft

12. **Ingenuity:**

    The decimal 0.2727272727... is equivalent to the fraction $\frac{3}{11}$. What is the fraction equivalent of the decimal 0.7727272727...?

13. **Investigation:**

    Janalyn has a piece of licorice that she wants to divide equally among her three sons Cody, Ross, and Trent.

    a. Suppose Janalyn divides the licorice stick into tenths. How many tenths will each son get? How many tenths will be left over?

    b. Since she doesn't want to waste any licorice, Janalyn then divides any tenths that remain into hundredths. How many will each son get? How many will be left over?

    c. Janalyn then divides any remaining hundredths into thousandths, and divides these up evenly. How thousandths will each son get? How many will be left over?

    d. What will happen if Janalyn continues this process forever? How much licorice will each son get?

    e. How can we use our findings from parts a through d to find a decimal equivalent for $\frac{1}{3}$?

    f. How could we find a decimal equivalent for $\frac{1}{9}$?

SECTION 5.4
FRACTIONS, DECIMALS, AND PERCENTS

EXPLORATION 1

Take a survey and gather data from your class with the question, "What is your favorite ___________?"

Represent your data using fractions, decimals, and percents.

You have learned that fractions can be written as equivalent fractions. For example, the fraction $\frac{3}{4}$ can be written as the equivalent fraction $\frac{75}{100}$. This equivalent fraction can also be represented by the decimal 0.75. In some instances, this number can then be converted to 75 percent, 75%. The word **percent** means "out of a hundred" in Latin.

Decimals can be converted to fractions by reading the decimal form. For example 0.75 is read "seventy-five one hundredths" which in fractional form is $\frac{75}{100}$. This in turn says 75 out of 100 or 75%. Notice how three different forms, the decimal, fraction, and percent are all referring to the same quantity.

Complete the table by finding the decimal and percent.

Fraction	Decimal	Percent
$\frac{3}{4} = \frac{75}{100}$	0.75	$(0.75)(100) = 75\%$
$\frac{12}{25} = \frac{48}{100}$	0.48	$(0.48)(100) = 48\%$
$\frac{4}{5}$		
$\frac{9}{15}$		
$\frac{4}{32}$		
$\frac{7}{10}$		
$\frac{1}{20}$		
$\frac{5}{8}$		

What ways did you use to rewrite the fraction to a decimal? One way is to rewrite the given fraction equivalently so that the denominator is a power of 10 such as 10, 100, and 1000. Then rewrite this fraction as a decimal and then a percent.

EXAMPLE 1

How do you convert a fraction like $\frac{3}{5}$ into a decimal and a percent?

SOLUTION

$\frac{3}{5} = \frac{3 \cdot 20}{5 \cdot 20} = \frac{60}{100} = 0.60 = 60\%$. Another way to convert a fraction to a decimal is to perform the division that is $\frac{3}{5} = 3 \div 5 = 0.60 = 60\%$.

Similarly, you can reverse the pattern of converting percents to decimals by dividing the percent by 100. For example 75% is equivalent to $75 \div 100 = \frac{75}{100} = 0.75$. Even if the percent includes a decimal part, simply divide by 100 to get its decimal equivalent. For example, 6.48% is equivalent to $6.48 \div 100 = 0.0648$.

The following table includes percents that are called benchmarks because they correspond to fractions that are very familiar to you. We look at 10% as an example to show how the number can be written in fractional and decimal forms as well as relate to visual models of various kinds.

Complete the table and create corresponding visual models for each.

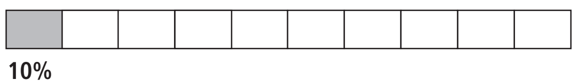
Fraction	Decimal	Percent
		1%
$\frac{1}{10}$	0.1	10%
		25%
		$33\frac{1}{3}\%$
		40%
		50%

5.4

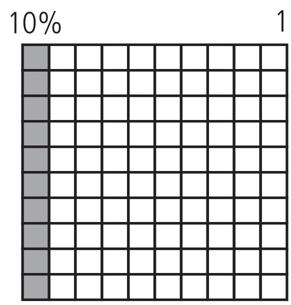
		$66\frac{2}{3}\%$
		75%
		90%
		100%

Number line model: 10% on the number line

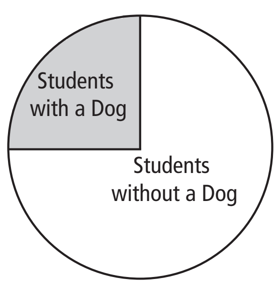
Fraction strip model: If the long rectangle is 100% then 10% is shaded.



10 x 10 grid model:



How do percents arise in real world problems? Suppose that in a survey of a class of 24 students concerning family pets, 6 had a dog. The fraction of the class with a dog was $\frac{6}{24}$ or $\frac{1}{4}$ of the class. Convert either of these fractions to a decimal. A pie graph is a visual representation of this class.



The portion of the pie graph labeled "Students with a Dog" is 25% of the whole class. What percent of the class does not have a pet dog? What do you notice about the percent of the class with a dog and the percent of the class without a dog? What does that percentage show about the part of the class involved in this situation?

PROBLEM 1

From the 10 x 10 grid model that you created for the benchmarks, what is the fraction pattern you notice about the multiples of 10%? 25%? $33\frac{1}{3}$%?

EXAMPLE 2

A class of 25 students has 12 girls. What percent of the class is girls?

SOLUTION

If 12 of the 25 students are girls, the fraction of girls to the total number of students is $\frac{12}{25}$. We can convert this fraction to a decimal and then to a percent. Or we can find an equivalent fraction for $\frac{12}{25}$ that has a denominator of 100. The fraction $\frac{12}{25}$ is equivalent to $\frac{48}{100}$, to the decimal 0.48, and to 48%.

EXAMPLE 3

Amy was shooting hoops in her backyard. She made 9 baskets out of 15 attempts. What percent of her shots did she make?

SOLUTION

The fraction of shots made to shots taken is 9 to 15 or $\frac{9}{15}$. To convert this fraction to a decimal, divide 9 by 15 to get $\frac{9}{15} = 0.6 = 0.60 = \frac{60}{100} = 60\%$, the percent of baskets made. It is a little surprising that the fraction $\frac{9}{15}$ is so nice. Many times a fraction is not so easily transformed into hundredths. We could have simplified $\frac{9}{15} = \frac{3}{5}$ and this equivalent fraction may have seemed more friendly.

EXAMPLE 4

A parking lot has 8 cars parked and 3 of them are red. What percent of the cars in the parking lot are red?

SOLUTION

The fraction of the red cars to the total number of cars is $\frac{3}{8}$. To convert this fraction to a decimal, divide 3 by 8 to obtain 0.375. In our previous examples we had 0.75 = 75% and 0.48 = 48%. You can see that 0.05 = 5% and 0.5 = 50%. You might notice a pattern that the resulting percent involves moving the decimal place two times to the right. The decimal 0.375 must then be 37.5%. In other words, 37.5% of the cars in the parking lot are red.

PROBLEM 2

In a small bag of 32 pieces of mixed candy, there are 4 pieces of lemon candy. What percent is lemon?

PROBLEM 3

1. Penelope has an 8-ounce measuring cup. Penelope pours the following amounts of water into the cup: a) 4 ounces, b) 2 ounces, c) 6 ounces, and d) 1 ounce. Determine the percent fullness in each case.
2. Victoria has a measuring cup marked from 100 milliliters (ml) to 400 milliliters. Victoria pours the following amounts of water into the cup: a) 200 ml, b) 100 ml, c) 300 ml, and d) 120 ml. Determine the percent fullness in each case.

EXERCISES

1. Convert each decimal below to a percent and a fraction. Use the 10×10 grid model to represent the numbers in parts e, f, and g.

a. 0.20	c. 0.75	e. 0.23	g. 0.03	i. 0.995
b. 0.25	d. 0.43	f. 0.7	h. 0.005	j. 0.499

2. Convert each fraction below first to a decimal, and then to a percent. Use the number line to represent the numbers in b, c, and e.

   a. $\frac{1}{4}$ e. $\frac{11}{20}$ i. $\frac{1}{250}$

   b. $\frac{3}{5}$ f. $\frac{3}{25}$ j. $\frac{1}{500}$

   c. $\frac{5}{8}$ g. $\frac{1}{200}$ k. $\frac{1}{12}$

   d. $\frac{15}{40}$ h. $\frac{1}{6}$ l. $\frac{1}{16}$

3. Convert each percent to a decimal and then to a fraction in simplest form. Use the fraction strip to represent the numbers in a, b, and c.

   a. 30% d. 2% g. 7% j. 57%

   b. 25% e. 85% h. 24% k. 28%

   c. 40% f. 325% i. 0.6%

4. Dirk Nowitzki made 72 out of 80 free throws in his first 10 games. What percent of his free throws did he make?

5. Lori completed 4 pages of a 20 page report. What percent of the report does she still have left to complete?

6. At Genius Middle School, there are 125 girls out of 200 students in the 6^{th} grade. What fraction of Genius Middle School 6^{th} grade students are girls? Simplify your answer, if possible. Write this as a decimal. What percent of Genius Middle School 6^{th} grade students are girls?

7. Twelve out of 48 cars were being repaired because of transmission problems.

   a. What fraction, decimal, and percent of the cars were being repaired for transmission problems?

   b. What fraction, decimal, and percent of the cars did not have transmission problems?

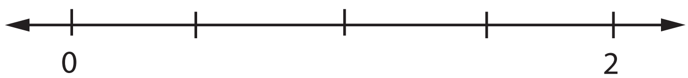
8. Ms. Garza had 24 students during 7th period. $\frac{1}{4}$ of the students forgot their books at home. What percent of the students brought their books to class?

9. Compare the following pairs of numbers. Place the correct relationship, <, >, or = between them.

a. $\frac{1}{4}$	0.27	c. $\frac{1}{3}$	0.30
b. $\frac{3}{5}$	0.59	d. $\frac{3}{4}$	0.79

10. Place the following numbers in order on the number line and label them.

a. 125%	c. 1.5	e. 0.25
b. $\frac{3}{4}$	d. 190%	f. $\frac{1}{2}$



11. a. Create a number line between 0 and 1. Place and label each number at the appropriate place on the number line: $\frac{3}{4}$, 0.25, $\frac{7}{8}$, 0.05.

    b. Create a number line between 0 and 2. Place and label each number at the appropriate place on the number line: 1.6, $\frac{5}{4}$, $1\frac{8}{10}$, $\frac{6}{8}$.

    c. Create a number line between $^{-}1$ and 2. Place each number at the appropriate place on the number line: $^{-}\frac{2}{8}$, 1.2, $^{-}0.75$, $\frac{2}{8}$

12. A survey at Kealing Middle School showed that on Monday, 20% of the students wore white shirts, 17% wore blue shirts, and 55% wore black shirts. The remaining students wore other colored shirts. What percent of the students wore shirts other than white, blue, or black? Represent the value as a fraction, decimal, and percent.

13. Order from largest to smallest: 0.20, $^{-}\frac{1}{2}$, 0.125, $^{-}\frac{3}{4}$

14. Order from smallest to largest: $\frac{2}{8}$, $\frac{1}{5}$, 0.01, 0.1

15. For each of the following, create a number line scaled by halves. Place the following numbers in their appropriate places on the number line. Be as accurate as possible with the spacing.

a. $\frac{1}{5}$, $\frac{3}{10}$, 0.1, 1, $\frac{1}{4}$	c. 0.65, $\frac{3}{5}$, $\frac{2}{3}$, 0.60
b. $\frac{3}{4}$, 0.70, $\frac{1}{2}$, $\frac{5}{6}$, 0.56	d. 0.4, $^{-}\frac{3}{6}$, 0.05, $^{-}0.75$

16. Lorenz and four runners have finishing times of 7.321, 7.4, 7.001, 6.2, and 6.100. Write the times in order from fastest to slowest.

17. Three-fourths of the planet is covered with water. What percent is not covered with water?

18. Five out of 8 cars are red. What percent are not red?

19. There are 80 students in the classroom. 50% of them ride the bus. Of the students that do not ride the bus, $\frac{1}{4}$ of them walk home. How many students walk home? Use a pictorial model.

Spiral Review:

20. Correctly place the following numbers on a number line.

$$^{-}3,\quad 2\frac{1}{4},\quad ^{-}3.5,\quad 5,\quad 5.25,\quad ^{-}1\frac{1}{2}$$

21. Mrs. Martinez has 60 bottles of blue paint, 45 rolls of tape, and 30 protractors. What is the greatest common factor Mrs. Martinez can use to divide the supplies into equal groups?

22. **Ingenuity:**

    At Pascal Middle School, 62% of the students have dogs and 44% of the students have cats. If 23% of the students have neither dogs nor cats, what percent of the students have both dogs and cats?

23. **Investigation:**

    Answer each of the following questions. For each question, first make a guess without doing any calculations, and explain your guess. Then check your answer by hand.

    a. In Mr. Byrd's algebra class, there are 10 boys and 20 girls. Six of the boys are in the jazz band, and eight of the girls are in the jazz band. Which is greater: the percentage of the boys who are in the jazz band, or the percentage of girls who are in the jazz band?

    b. In the Honor Society at Fourier Middle School, there are 50 eighth graders and 10 seventh graders. Of these students, 36 eighth graders and 6 seventh graders are taking honors classes. Which is greater: the percentage of the eighth graders who are taking honors classes, or the percentage of the seventh graders who are taking honors classes?

c. At a certain school, there are 500 sixth graders and 400 seventh graders. Of these students, 355 sixth and 324 seventh graders live north of Main Street. Which is greater: the percentage of the sixth graders who live north of Main Street, or the percentage of the seventh graders who live north of Main Street?

SECTION 5.R
CHAPTER REVIEW

1. Complete the following table.

Decimals	Words
0.8	
	sixteen hundredths
0.06	
	eight and three tenths
0.208	
3.9	
	one hundred and six hundredths

2. Round each decimal to the given place values:

a. 2.059 (hundredths)

b. 0.648 (ones)

c. 0.1386 (thousandths)

d. 315.183 (tenths)

e. 1.099 (hundredths)

f. 19.99 (tenths)

3. Compare the following pairs of numbers using, <, >, or =.

a. 0.076, 0.7

b. 3.410, 3.41

c. 0.098, 1

d. 0.923, 0.932

e. 61.75, 61.570

f. 0.065, 0.65

4. Compute the sum or difference of each:

a. 35.18 + 16.9

b. 0.078 + 1.75

c. 2.03 – 1.96

d. 0.104 + 0.87

e. 6 – 3.864

f. 0.15 + 2 + 2.505

g. 36.98 + 16.083

h. 98.6 – 73.217

5. Compute the product:

   a. (4.6) (0.002) b. (74.01) (3.8) c. (0.93) (5.1)

6. Compute the quotient:

   a. $1 \div 6$ c. $5 \div 8$

   b. $1.2 \div 4$ d. $1.2 \div 0.4$

7. Complete the following table

Simplest form	Fractions in tenths, hundredths, or thousandths	Decimal form	Percent form
$\frac{3}{5}$			
	$\frac{18}{100}$		
		1.6	
			45%
	$2\frac{8}{10}$		
		0.06	
$1\frac{3}{4}$			
			110%
$\frac{7}{25}$			

8. Convert each fraction to an equivalent decimal. Use equivalent fractions or division. Then write the answer as a percent.

   a. $\frac{3}{5}$ c. $\frac{9}{15}$ e. $\frac{11}{20}$ g. $\frac{6}{24}$

   b. $\frac{4}{25}$ d. $\frac{36}{50}$ f. $\frac{1}{4}$ h. $\frac{28}{35}$

5.R

9. Convert each decimal to an equivalent fraction. Simplify the fraction if needed.

a. 0.26	c. 0.98	e. 12.500
b. 0.015	d. 3.55	f. 0.008

10. Create a number line from $^-2$ to 2, counting by halves. Place the following numbers at the appropriate location on the number line.

$$-\frac{3}{4},\quad 1.6,\quad \frac{12}{20},\quad -\frac{5}{5},\quad {}^-0.25,\quad 0.05,\quad \frac{24}{12}$$

For the following word problems, write an expression that can be used to solve the problem. Then solve the problem and write your answer in a complete sentence.

11. At the local zoo, 15% of the money to feed the animals is spent on the monkeys, 25% is spent on the lions, 28% is spent on the elephants, 14% is spent on the giraffes, and the remaining 18% is spent on the other animals. What fraction, in simplest form, of the money is spent on food for the elephants?

12. Marci was paid $245.85 for books she sold online. Everest was paid $315. How much more money did Everest make than Marci?

13. Allison completed 16 of the 20 pages of her science project over the weekend. What percent of her project has she completed?

14. Uncle Harry ate $\frac{1}{6}$ of the cherry pie and Uncle Lloyd ate $\frac{1}{8}$ of the same pie. What decimal amount of the cherry pie did Uncle Lloyd eat?

15. The Hardware Store had a 25 foot spool of cable wire. They sold 12.5 feet in the morning and 11.75 feet in the afternoon. How much cable wire was left on the spool at the end of the day?

16. Four cars were pushed across a table during an experiment. Car A traveled 3.35 feet, Car B traveled 3.85 feet, Car C traveled 3.5 feet, and Car D traveled 3.58 feet. Place the cars' distances in order from least to greatest.

17. Karla lives 5.6 miles away from the doctor's office. The pharmacy is 2.85 miles away from the doctor's office. If Karla goes from her house to the doctor's office and then to the pharmacy, how far will she have to travel to go back home from the pharmacy following the same route?

18. A ribbon of length 5.6 feet is cut into 8 equal pieces. What is the length of each piece?

19. Jeremy averaged 7.2 minutes a mile in a 6.3 mile race. How many minutes did Jeremy take to complete the race?

EQUATIONS, INEQUALITIES, AND FUNCTIONS

6

SECTION 6.1
PATTERNS AND SEQUENCES

We have many different ways to describe any given number. For example, a number can be even or odd, prime or composite, and a power of just one number or a product of many different numbers. Think of other ways to classify numbers and examples for each.

A list of numbers can have interesting characteristics. For example, what **pattern** do you notice in the following list of numbers?

1, 3, 5, 7, 9, 11, 13, 15, ...

One pattern is that the numbers are positive odd numbers. Another observation is that the next number in the list is two larger than the number before it. You might even notice that the first number in the list is 1, the second number in the list is 3, the third number in the list is 5, and that the number in the list is one less than twice its location in the list.

If you want the number following 15 on the list, you would probably agree that 17 is reasonable. We could also say that the 9th number or **term** in the list is 1 less than twice 9. We could write the numerical expression as $2 \cdot 9 - 1 = 18 - 1 = 17$, which is indeed two more than 15. To summarize, we say that a number in any position, n, in this **sequence** or number list must be $2 \cdot n - 1$ or $2n - 1$. We also say n is the term number.

EXPLORATION 1

Consider the following five lists of numbers, or sequences. Write all the patterns that you observe. Explain your observations. Predict what the next term is. Explain why you predict this term. Use the process that you used to find the 20th term. Do this without writing all the terms between the next term you predicted and the 20th.

a. 3, 6, 9, 12, 15, 18, ...

b. 6, 12, 18, 24, ...

c. 2, 6, 10, 14, 18, 22, ...

d. 2, 6, 18, 54, 162, ...

e. 1, 4, 9, 16, 25, ...

Another way to write a sequence is by using a table. The table below for Exploration 1 would look like this:

Position	Term
1	3
2	6
3	9
4	12
5	15
6	18

One observation is that the terms are increasing by 3. We say there is a **constant difference**, 3, between the adjacent terms. Lists of numbers with a constant difference are called **arithmetic sequences**. Are any of the other sequences in the exploration above arithmetic sequences? Identify the constant differences.

Another observation that you may have noticed is "horizontally" across the table. Is there a pattern that relates the position of the number with the term in the list? The pattern should relate 1 to 3, 2 to 6, 3 to 9, 4 to 12, and so on.

We can describe the pattern between the position, n, and the corresponding term as $3n$. Check to see if this expression for the nth term works. The 20th term should be $3 \cdot 20 = 60$. Does this check with your work in Exploration 1?

PROBLEM 1

Use tables to describe the number sequences below. Include the next three terms in the sequence. Write an expression for what you think will be the nth term.

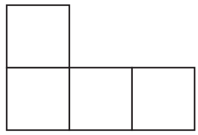
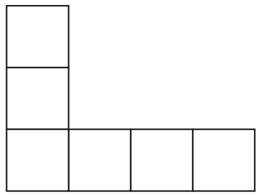
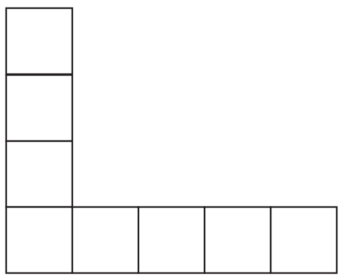
a. 10, 100, 1000, 10000, ...

b. 7, 12, 17, 22, 27, 32, ...

c. 3, 7, 11, 15, …

d. 1.2, 2.2, 3.2, 4.2, 5.2, ...

Some patterns may involve shapes that change in a predictable manner. The following example involves shapes created by squares. A pattern is observable both as a shape and as a number sequence.

Position	Figure	Term: Number of blocks
1		4
2		6
3		8

If the pattern continues, determine how many small squares would be in the 4th figure, and how many would be in the 5th figure. What pattern do you observe? Write an expression for the number of small squares in the nth figure.

EXERCISES

1. For each of the following sequences, write the next two terms.

a. 4, 8, 12, 16, …

b. 8, 12, 16, 20, …

c. 2, 4, 6, 8, …

d. 6, 8, 10, 12, ...

e. 5, 7, 9, 11, ...

f. 7, 14, 21, 28, ...

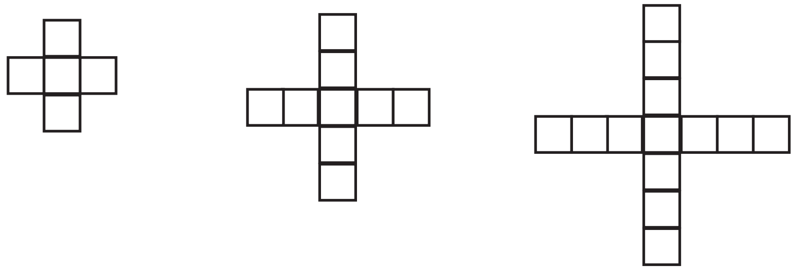
g. 1, 5, 9, 13, …

h. 4, 9, 14, 19, …

i. 1 , 6, 11, 16, ...

j. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

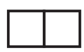
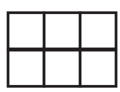
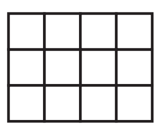
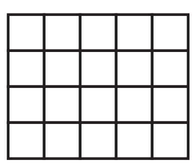
2. Observe the pattern below. Draw the next two figures of the pattern. Then create a list of numbers that describes the sequence of the figures.



Numerical Sequence: ______________________________

Describe the pattern that you created in the numerical sequence.

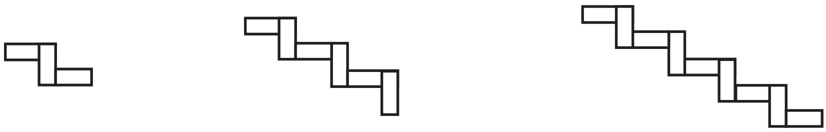
3.

Position	Figure	Term: Number of Unit Squares
1		2
2		6
3		12
4		20

If the pattern continues, determine how many unit squares would be in the 5th figure and in the 6th figure. What pattern do you observe?

Write an expression for the number of unit squares in the nth term.

4. Observe the pattern below. Draw the next two figures of the pattern. Then create a list of numbers that describes the sequence of figures..



Numerical Sequence: ____

Describe the pattern that you created in the numerical sequence.

5. Complete the table from the information below.

Bill is three years older than Rick.

Rick's Age	Numerical Process	Bill's Age
1		
2		
3		
4		
5		
x		y

6. Complete the table from the information below.

Decreasing Pam's age by 5 years will give her brother Rudy's age.

Pam's Age	Numerical Process	Rudy's Age
6		
7		
8		
9		
10		
x		y

7. Molly is making bracelets to sell at her booth at the school carnival. The table below shows the relationship between the number of bracelets she made and the total number of crystals she used to make the bracelets. Write an equation to find y, the number of crystals needed, in terms of n, the number of bracelets she made.

Number of Bracelets (n)	Number of Crystals (y)
1	12
2	24
3	36
4	48
n	y

Spiral Review:

8. Stephanie's heart beats 8 times per 10 seconds while she is resting. How many times would Stephanie's heart beat during 2 minutes of rest?

9. Emmie can run 100 meters in 23 seconds. If she competes in the 400-meter race, how many seconds will it take Emmie to run the race? Did she run the race in under 2 minutes?

Ingenuity:

10. Find the value of the following expression:

$$300 - 299 + 298 - 297 + 296 - \ldots + 4 - 3 + 2 - 1$$

Investigation:

11. The ***Fibonacci sequence*** is a well-known sequence of numbers that begins with the following terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

a. What is the pattern here? What are the next five terms of the sequence?

b. Suppose we pick a positive integer N, and add up the first N terms of the Fibonacci sequence. For example, if we pick $N = 4$, then the sum of the first N

terms is 1 + 1 + 2 + 3 = 7. Try doing this for other values of ***N***, and make a table of the sums you get. Do you see a pattern?

c. Suppose we pick one of the terms of the sequence and square it. Then we take the product of the terms on the left and right of that term. For example, if we choose 8 and square it, we get $8^2 = 64$. If we take the product of the two neighboring terms, we get $5 \times 13 = 65$. Try this for several terms of the sequence. What do you notice?

SECTION 6.2
EQUATIONS

Let's begin with the sentence "A number is 3 more than 7." You could figure out what this number is with relative ease, but how can you write this mathematically? You may recall that you can use numbers, variables, and operations to form expressions. We can now combine these expressions to form a mathematical sentence called an equation. An **equation** is a math sentence with an equality sign, =, that relates two expressions.

EXAMPLE 1

Translate the sentence "A number is 24 more than 17" into an equation. What is the difference between the expressions in your answer and the equation? Explain verbally, numerically, and algebraically.

SOLUTION 1

Step 1: We give the unknown number a name, N, and write "N = the number." N is a variable. It represents the number we are trying to find.

Step 2: We translate the sentence into an equation.

A number	is	24 more than 17
N	$=$	$17 + 24$

So the equation form of the sentence is $N = 17 + 24$.

Since $17 + 24 = 41$, we can conclude that $N = 41$. N now represents a known quantity, 41, instead of an unknown quantity. Therefore, we say that we have **solved** the equation for N.

Step 3: The expressions that you used may be numerical such as 17+24, or algebraic such as N. Numerical expressions only contain numbers, whereas algebraic expressions may include both variables and numbers.

The equation is the statement that these two expressions are equal, $N = 17+24$. Every equation has an equality sign, =.

PROBLEM 1

Translate the sentences below to equations where x is a number.

a. x less than 10 equals 8.

b. 10 less than x equals 8.

EXAMPLE 2

What number is twice as large as six? Use the number line to also illustrate the solution.

SOLUTION 2

Step 1: We define a variable to represent our number. Let T be a number that is twice as large as six.

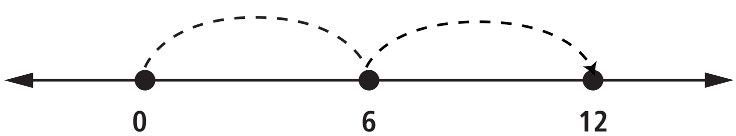
Step 2: The statement "A number is twice as large as six" translates as

$$T = 2 \times 6 = 2 \cdot 6 = (2)(6).$$

When we put the symbols 2 and 6 next to each other with parentheses around each, it is understood that we mean to multiply them. The small dot is also a symbol for multiplication. So $(2)(6) = 2 \cdot 6 = 2 \times 6$. We usually do not use the symbol $\times$, however, since it could be confused with a variable x.

Step 3: We know $2 \cdot 6 = 12$, so $T = 12$.

Step 4: Check. Is 12 a number that is twice as large as 6? Yes.



PROBLEM 2

Pat purchased a dozen doughnuts. The number of milk cartons Terry purchased is twice the number of doughnuts Pat purchased. How many milk cartons did Terry purchase?

Using variables to model problems is the beginning of learning algebra. Algebra enables us to translate problems into mathematical expressions and equations. We then use mathematics to solve for the unknowns, which provides solutions to our problems.

EXAMPLE 3

Translate the sentence "A number is 2 less than four times 10" into an equation and solve for the unknown variable. Does your answer make sense?

SOLUTION 3

We use the 4-Step process.

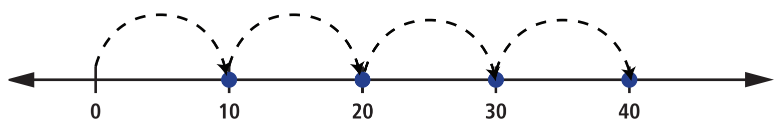
Step 1: Let's call our unknown number N.

Step 2: $N = (4)(10) - 2$ is our equation for the statement above.

Step 3: The right side is equal to 38. So N = 38. We have solved the problem! Now let's check our answer.

Step 4: Is 38 equal to 2 less than 4 times 10? Four times 10 equals 40, and 2 less than 40 is 38, so our answer is correct.

Notice on the number line below that 4 times 10 is 40.



Then 2 less would mean backing up from 40 to 38, which is our solution as shown below.

EXPLORATION 1: CHARTING THE PROCESS

We have seen how we can use numbers and variables to translate problems into equations. Consider the problem, "Jeremy is 9 years old. In how many years will Jeremy be 15 years old?"

How might you begin this problem? Did you define a variable? If so, how did you use this variable?

Here is a step-by-step approach. Do your steps resemble the following?

Step 1: Define your variable

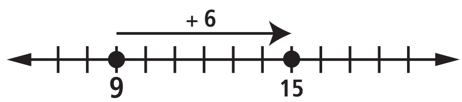
Y = the number of years it takes for Jeremy to reach 15.

Step 2: Translate the problem into an equation

We know that 15 is *Y* more than 9, so we write $15 = 9 + Y$, an equation with one variable, *Y*.

Step 3: Solve for the unknown variable

If you look on the number line, you'll notice you have to move right 6 units to go from 9 to 15. So $Y = 6$.



Step 4: Check your answer

Substitute $Y = 6$ into the original equation to see that $15 = 9 + 6$.

Using the 4-step process discussed above, create a poster showing how the steps should be used to solve a problem. Begin by writing the problem and continue by showing how each step helps you solve the problem.

PROBLEM 3

Write the following statements in equation form. Let *N* = the number.

a. A number is twice as large as 55.

b. A number is 12 more than 25.

c. 9 more than a number is 16.

d. 6 less than a number is 46.

This is a **balance scale**:

When we put a weight on one side of the scale, we must place the same weight on the other side in order for the scale to be balanced. If a scale is balanced and equal weights are added or subtracted from both sides of the scale, the scale will remain balanced.

In much the same way, an equation is a statement that two expressions are equal. We can think of the expressions on each side of the equality sign as representing the weight placed on each side of a balanced scale. When we add or subtract the same amount from each side of the equation, the equation will remain **balanced**.

PROBLEM 4

If Wesley finds 5 more marbles, he will have the same number of marbles as John. John has 11 marbles. How many marbles does Wesley have?

The rule we are using to solve these problems is called the **subtraction property of equality**, because in each, we are subtracting the same number (removing the same number of blocks) from both sides of an equation. The new equation we obtain is said to be **equivalent** to the original equation because the two equations have the same solution: any value for a variable that makes one of the equations balance will make the other balance as well.

DEFINITION 6.1: SUBTRACTION PROPERTY OF EQUALITIY
If $A = B$ then $A - C = B - C$.

Remember the sequence 1.2, 2.2, 3.2, 4.2, ... from the previous section. We saw that the term in a particular position, n, is $n + 0.2$. We can write this as the term = position + 0.2 or $t = n + 0.2$, where we let the variables t = term and n = position. According to this pattern or "rule," the 20th term would be 20.2 and the 200th term would be 200.2.

Suppose you were given the term 187.2. Can you determine which position this number is in the list? Remember that $t = n + 0.2$ and 187.2 is the term or symbolically, $t = 187.2$. If we rewrite the equation as $187.2 = n + 0.2$, how can we determine n? One way is to use the subtraction property. We have $187.2 = n + 0.2$, but we want n, so consider subtracting 0.2 from both sides of the equation. Then $187.2 - 0.2 = n + 0.2 - 0.2$ and $187 = n$.

Finding the value that makes an equation true is referred to as **solving an equation**. Solving equations is a very important part of doing algebra, and the subtraction property is an important tool for solving equations.

EXPLORATION 2

The total cost of a meal for three people is \$51. If the three people agree to split the cost equally, what would each person's cost be? Write two equations that could be used to model the problem. You do not have to solve the problem. Use x to represent the cost each person pays.

You may have found that one equation is $3 \cdot x = 51$. Another equation could be written as $x = \frac{51}{3}$. You may recall this important connection between multiplication and division. We will talk more about this when we multiply by fractions in the next chapter.

We can relate the idea above with the sequence 3, 6, 9, 12, 15, ... Suppose we ask the question, "what position is 51 in this list?" We write this question mathematically as $3 \cdot x = 51$, where we let x represent the position the term 51 occupies in the list. From above, $3 \cdot x = 51$ is equivalent to $x = \frac{51}{3} = 17$. We conclude that 51 occupies the 17th position in the list above.

EXAMPLE 4

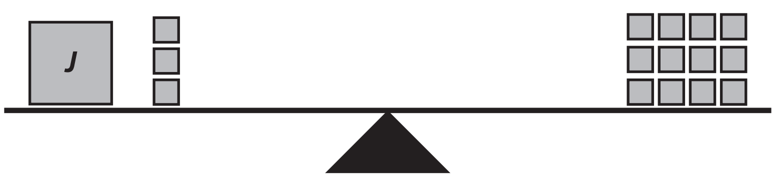
If Jeremy were three years older, he would be the same age as his twelve-year-old sister. What is Jeremy's age?

SOLUTION 4

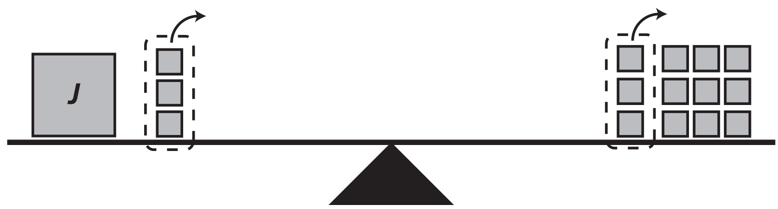
We let J be Jeremy's age, and translate the sentence into an equation as follows:

Jeremy's age	three years older	same age as	twelve-year-old sister
J	$+\ 3$	$=$	12

Now we have the equation $J + 3 = 12$. The unknown is J, Jeremy's age. Visually, this sentence says that $J + 3$ is the same as 12, which we can show on a balance scale:



In order to solve this equation for J, we must find what balances J. To do this, we remove three "blocks" from each side of the balance scale:



This is what we have left:

We can express this algebraically as follows:

$$\begin{aligned} J + 3 &= 12 \\ (J + 3) - 3 &= 12 - 3 \\ J + (3 - 3) &= 9 \\ J + 0 &= 9 \\ J &= 9 \end{aligned}$$

Because we have now solved for J, we can go back and check the solution. Substituting $J = 9$ into the original equation $J + 3 = 12$ gives us $9 + 3 = 12$, which is correct. Jeremy's age is 9.

EXERCISES

Try the following exercises using the four-step process. When you "solve" your problem, you should not only find the answer, but also show the way you got your answer, which is just as important.

1. Write an expression that represents each number below. Which of these expressions are numerical, and which are algebraic?

   a. 3 more than 17

   b. 5 less than 16

   c. Twice as much as 65

   d. 6 less than a number

2. Write the following statements in equation form. In each case, let ***N*** = the number. Explain verbally what are the numerical expressions, the algebraic expressions, and the equations for each part.

   a. A number is 3 more than 38.

   b. A number is 5 less than 16.

   c. A number is twice as large as 65.

   d. 6 less than a number is 16.

   e. A number is 4 more than twice the number 15.

   f. 7 less than a number is 40.

   g. A number is 7 less than twice the number 12.

   h. 6 more than a number is 22.

   i. A number is 5 less than three times 24.

   j. A number is double 19.

   k. A number is 5 more than twice the number 18.

   l. 8 more than a number is 13.

Do steps 1 and 2 of our four-step process for Exercises 3–6.

3. Jake has $65. How much more does he need if he wants to have $97?

4. If Lori will be 19 in 8 years, how old is Lori now?

5. Mark is 12 years old. In how many years will Mark be 21?

6. Sean has \$72 and lends George \$29. How much does he have left?

Write a story problem for the equations in Exercises 7 and 8:

7. $x + 9 = 27$

8. $81 - x = 17$

9. Solve the following equations.

   a. $p + 4 = 213$

   b. $17 + q = 1000$

   c. $\frac{1}{2} + r = 5$

   d. $t + \frac{2}{5} = \frac{7}{10}$

   e. $n + 0.18 = 0.32$

   f. $y + 0.6 = 1.8$

   g. $m + 5.382 = 6.7$

   h. $x + 0.03 = 0.51$

Use the 4-step process you learned to solve the following problems. Remember to use the correct labels (ºC, ºF, years, cards, etc.) when defining your variable and in your solution.

10. Mike has a certain number of baseball cards. Let m represent the number of cards that Mike has. Jill has 3 more cards than Mike, and Ramon has twice as many cards as Jill. How many cards does Ramon have if Mike has:

    a. 4 cards?
    b. 10 cards?
    c. x cards?

11. Sophia has two nephews, Juan and Ted. Ted is five years younger than Juan. Sophia is three times as old as Ted. If Juan is 12 years old, then what is Sophia's age?

12. Gloria has two nieces, Sara and Jane. Sara is three years younger than Jane. Gloria is twice as old as Sara. If Jane is 9 years old, how old is Gloria?

13. The lowest and highest points in North America are Death Valley in California and Denali in Alaska, respectively. Death Valley is below sea level, in fact, 282 feet below sea level! On a number line, we represent this elevation by $^{-}282$. Denali is 20,320 feet above sea level. There is a big difference between the two elevations. Use D to represent the height we must climb to go from the elevation of Death Valley to the elevation of Denali. The equation that models this situation is $^{-}282 + D = 20{,}320$. Solve for D.

Spiral Review:

In Exercises 14 and 15, define a variable, set up an equation, solve, and check your work. Remember to include your units of measurement, such as ˚F, ˚C, feet, inches, or years.

14. In the morning it was a cool 58 ˚F. By mid-afternoon the temperature had reached 87 ˚F. What was the increase in temperature from morning to mid-afternoon?

15. On a cold day in Canada, the temperature was $^{-}5$ ˚C at 6:00 a.m. How many degrees must it warm up to reach 8 ˚C?

Ingenuity:

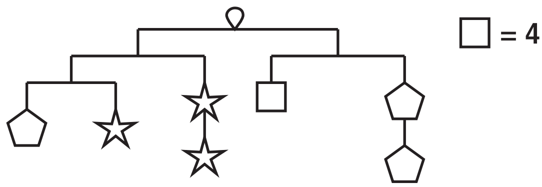
16. The Banneker Middle School band took a field trip to an amusement park to celebrate their success at a recent contest. They rented three vans to take students to the park. It turned out that the vans were not enough, so two students had to ride in a teacher's car. If each van had the same number of students, and a total of 41 students went to the park, how many students rode in each van?

Investigation:

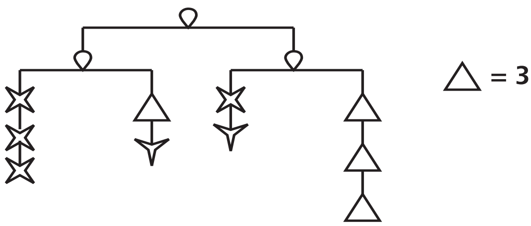
17. A mobile is a type of hanging sculpture in which several objects are suspended in balanced equilibrium. In the mathematical mobiles below, each shape has an associated weight, and for each horizontal beam, the total weight hanging on one side is equal to the total weight hanging on the other (we assume that the wire has no weight). For example, in the mobile on the left, the two circles together weigh as much as the rectangle: the rectangle weighs twice as much as a circle. In the mobile on the right, the circle has the same weight as the hexagon, and the rectangle has twice the weight of the hexagon.

The rectangle has weight 4, the hexagon and circle each have weight 2. We can write this symbolically as ▭ = 4, ⬡ = ○ = 2. Based on the weights given at the side of each problem and the mobile balance property, deduce each shape's weight.

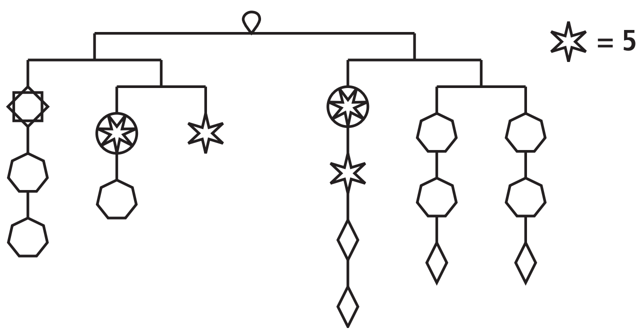
a.



b.



c.



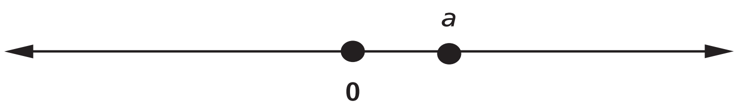
SECTION 6.3
EQUATIONS AND INEQUALITIES ON NUMBER LINES

In Section 6.2, we used the balance scale to solve equations such as "$x + 3 = 5$." We can also explore equations using a number line. We begin by investigating how to visualize expressions on a number line.

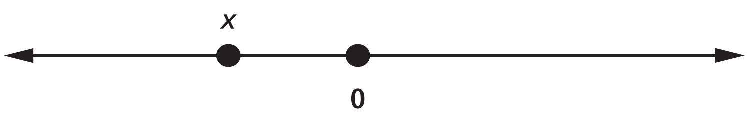
EXPLORATION 1

Suppose a and x are numbers located on the number line as seen below. Locate and label the points that represent the indicated numbers. Use string to act out how you determine your answer.

1. Plot points that represent each of the following: $2a$, $3a$, ^{-}a, $^{-}2a$, $^{-}3a$



2. Plot points that represents each of the following: $2x$, $3x$, ^{-}x, $^{-}2x$, $^{-}3x$



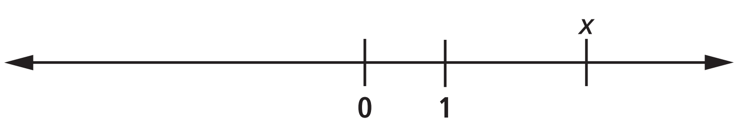
3. Compare the results from parts 1 and 2. What do you notice?

EXPLORATION 2

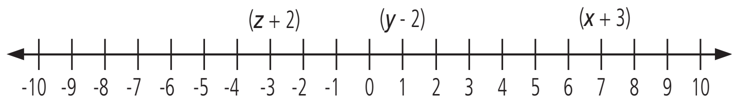
1. Suppose x is a number that is located on the number line as seen below. Locate and label the points that represent the indicated expressions. The numbers 0 and 1 are also labeled.

   Plot a point that represents each of the following expressions:

   $$x + 1,\quad x + 2,\quad x + 3,\quad x - 1,\quad x - 2$$



2. Suppose we know the location of each of the expressions as indicated on the number line below. Find the locations for x, y, and z. Explain how you locate each of these points on the number line.



In Part 1 in this exploration we used the location of a variable on the number line to locate expressions on the same number line. In Part 2, we were given the location of an expression, such as $x + 3 = 7$, and used it to find the location of the variable x on the number line. We see that $x = 4$. In other words, we solved the equation using the number line.

PROBLEM 1

Use the number line to solve each of the following equations:

a. $x + 3 = 5$ b. $y + 5 = 2$ c. $z - 4 = 2$

Discuss how solving these equations on the number line compares with the balance scale method.

Recall that an equation is a statement that two expressions are equivalent. A statement that one expression is always less than (or greater than) another is called an **inequality**.

EXAMPLE 1

1. The number of apples consumed, A, is more than twice the number of bananas, B.
2. Jack's age, J, is less than 40 years.

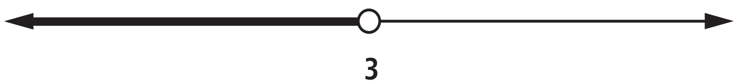
SOLUTION

1. $A > 2B$.
2. $J < 40$. Sometimes an inequality is a statement of comparison between two quantities, such as, $4 < 7$. But we can also use an inequality to describe a condition that a variable satisfies, as in Jack's age, J, is less than 40. So we say $J < 40$.

We can use a number line to represent all the possible numbers that satisfy an inequality. For example, suppose S is the set of all numbers less than 3. Another way to describe this set is:

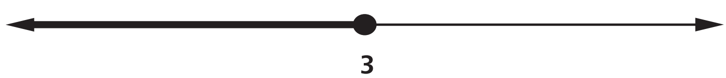
"S is the set of all numbers x so that $x < 3$."

We use the inequality $x < 3$ to define the set S. We can represent this set S on the number line below.



Notice that the part of the number line to the left of 3 represents the set S. This means that each number to the left of 3 is in S and every number in S is located on the line to the left of 3. Note that the point representing 3 is not filled in to indicate that 3 does not satisfy the condition that $x < 3$.

We write $x \leq 3$, read numbers x less than or equal to 3, to mean $x = 3$ is included. The number line representation looks like this:



EXAMPLE 2

Draw a number line and represent the set T of all numbers x such that $^{-}2 \leq x$.

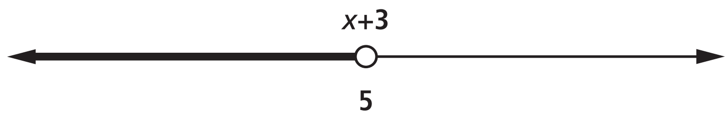
SOLUTION

Notice that the point at $^{-}2$ is filled in to indicate that $^{-}2$ does satisfy the condition that $^{-}2 \leq x$.

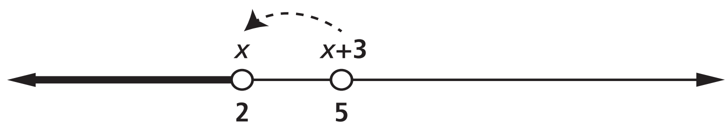
In solving an equation, such as $x + 3 = 5$, we want to find all numbers x that satisfy this statement. Since the only solution is $x = 2$, we say that the solution set is $\{2\}$.

If we start with an inequality, such as $x + 3 < 5$, we can ask:
What numbers x satisfy this inequality?

We can represent the inequality on the number line as shown below.



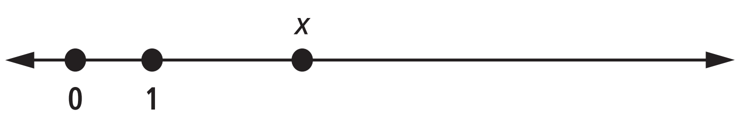
The shaded part of the number shows where the expression $x + 3$ could be located on the number line. We draw a new number line to represent where the variable x can be.



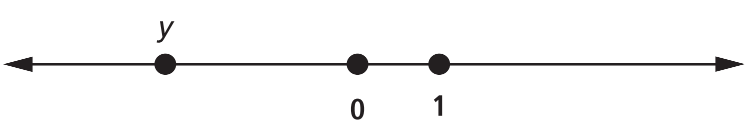
The number line tells us that x must be less than 2, or $x < 2$. Check to see if any number less than 2 makes the original inequality, $x + 3 < 5$ true. We say that we have solved the inequality $x + 3 < 5$ and that our solution is $x < 2$, as you can see on our number line.

EXERCISES

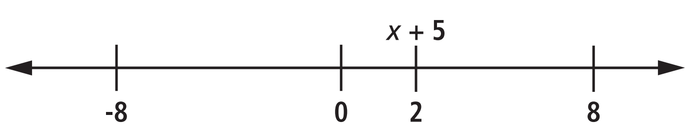
1. Plot a point that represents each expression: $2x$, $2x + 1$, $2x – 1$, $3x – 1$



2. Plot a point that represents each expression: $y + 1$, $y – 1$, $2y$, $2y + 1$, $2y – 1$, $2y + 8$



3. Solve the equation $x + 5 = 2$ using the number line.



4. Use a number line to solve each of the following equations:
   a. $x + 4 = 6$
   b. $x + 2 = 7$
   c. $x - 4 = 2$
   d. $x + 6 = {}^{-}2$
   e. $x - 5 = 2$
   f. $x - 4 = {}^{-}6$
   g. $x + 4 = 4$
5. Draw a number line and represent the set of all numbers x such that $x < 5$.
6. Draw a number line and represent the set of all numbers x such that $x < {}^{-}3$.
7. Draw a number line and represent the set of all numbers x such that $x > 1$.
8. Draw a number line and represent the set of all numbers x such that ${}^{-}2 < x$.
9. Solve each of the following inequalities and graph their solution sets on the number line:
   a. $x + 3 < 2$
   b. $x - 3 \leq 2$
   c. $x + 5 < 6$
   d. $x + 5 < 2$
   e. $x + 3 > 6$
   f. $x - 4 \geq 2$
10. Graph the solution sets for each of the following inequalities. Show your work on the number line.
    a. $2 < x + 3 < 5$
    b. $0 < x - 3 < 2$
    c. $2 < x + 5 < 8$
    d. $0 < x + 4 < 3$
11. Write word problems that can be modeled using the following inequalities. Be sure to state what the variable stands for in each of your word problems, then solve the inequality. Check to see that it makes sense.

    a. $x + 5 < 13$ b. $x - 7 > 2$ c. $x - 3 < {}^{-}1$

12. Investigation:

Use a number line to solve each of the following equations:

a. $2x + 1 = 5$ b. $2x + 3 = 9$ c. $2x - 3 = 5$

SECTION 6.4
FUNCTIONS

In our daily lives, we often encounter situations where we receive a set of instructions and then perform certain tasks based on those instructions. In mathematics, this is the role of a function. A **function** is a rule that assigns a unique **output** value to each number in a set of **input** values. Let's explore this.

EXPLORATION 1

Sarah builds model airplanes. Sarah makes two airplanes each day. How many airplanes will she make in 4 days? 10 days? Organize the information to reveal a pattern in the number of airplanes she makes in a given number of days.

How did you organize the information? Do you see a pattern in the number of airplanes Sarah can make in a given number of days?

One way to organize such information is to build a table like the one below. Notice that the first column is the number of days and the second column is the total number of airplanes that Sarah can make in the corresponding number of days.

Days	Total Number of Planes
0	
1	
2	
3	
5	
10	
x	

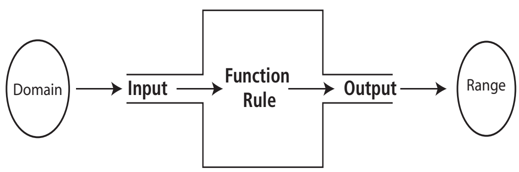
What do you notice about this table? Why is this a good way to organize the information?

This is an example of a function. There is a rule, or function, to determine how many planes Sarah has produced based on the number of days she has worked. You can think of a function as a machine with inputs and outputs. The input is the number of days Sarah has worked. The output is the number of planes produced.

Think of a number to input into the machine. Are there some restrictions on the number you can use? Does it make sense to input 7? $^-7$? 0.5? Explain your reasoning. Now use the number you chose for the input and determine the output corresponding to your number using this function.

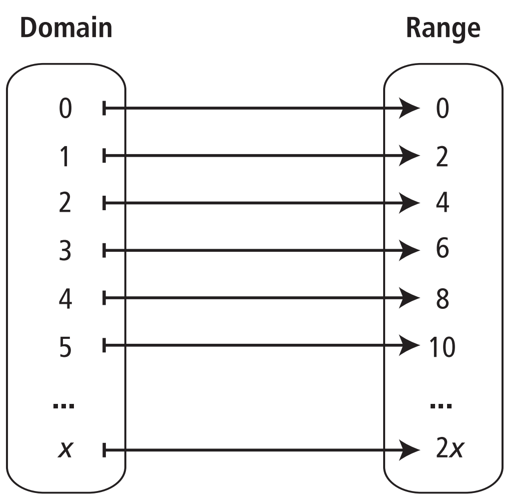
Now that we have an idea of what a function does, let's go ahead and make a more formal definition.

DEFINITION 6.2: FUNCTION
A **function** is a rule which assigns to each member of a set of inputs, called the **domain,** exactly one member of a set of outputs, called the **range**.



For example, consider Sarah's function from Exploration 1 again. The domain is the set of nonnegative integers 0, 1, 2, 3, … and the range is the set of even nonnegative integers 0, 2, 4, 6… as pictured below.

In general, the domain = set of inputs and the range = set of outputs.

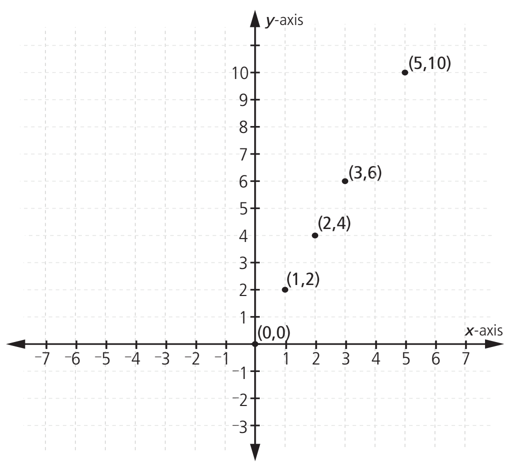


We let the variable x represent the number of days Sarah works.

Notice that a function produces input and output pairs of numbers.

Input	Function Rule	Output
x	Multiply input by 2 to get the output $y = 2(x) = 2x$	y
0	$y = 2(0)$	0
1	$y = 2(1)$	2
2	$y = 2(2)$	4
3	$y = 2(3)$	6
x	$y = 2(x)$	$2x$

Let's call Sarah's function F. From the table and the picture above, we can see that (0, 0), (1, 2), (2, 4), (3, 6), and (5, 10) are some of the pairs that belong to the function F, where the x or first-coordinate is the input and the y or second-coordinate is the output. In other words, the ordered pairs are of the form (input, output) or (x, y). Each pair of numbers can be thought of as a point on the coordinate system, so we can also talk about the **graph** of a function. The graph of the function is the visual representation of the function.



In mathematics, a **notation** is a technical system of symbols used to represent unique objects. We can write "The function F pairs the number 1 with 2" using the function notation as "$F(1) = 2$." We read this as "F of 1 equals 2." This means F sends the input 1 to the output 2. Similarly, because the function F pairs the number 2 in our domain with the number 4 in the range to give us the pair (2, 4), we write "$F(2) = 4$."

So $F(x)$ = the number of planes that can be produced in x days. We can express this rule in general as $F(x) = 2x$.

We sometimes express this rule as the equation $y = 2x$. Note that if the rule for the function F is $y = 2x$, then we say the point (x,y) belongs to the graph of F. Since the value of an output y depends on the value of its corresponding input x, we call y the **dependent variable** and we call x the **independent variable**.

EXPLORATION 2

Consider the following tables of numbers that describe a function.

Function f

Input	Function Rule	Output
x		y
1		13
2		14
5		17
7		19
8		20

Function g

Input	Function Rule	Output
x		y
0		0
1		3
2		6
3		9
4		12

1. Describe in words all the patterns that you observe between the input and the output of the two functions above.

2. Use your pattern rules to determine the output if the input is 10 in each of the functions above.

3. Write an expression for the output (dependent variable) y in terms of the input (independent variable) x.

4. Extend the tables for functions f and g by selecting inputs that are rational numbers that are not whole numbers. For example, select numbers like $x = \frac{1}{2}, {}^{-}12.5, {}^{-}11\frac{1}{2}$ for the inputs, then determine the corresponding outputs.

5. Plot the points for each of these functions on a coordinate system.

In Exploration 2, the table of inputs and outputs for functions f and g determine a pattern that can be expressed using independent and dependent variables. In function f, $y = x + 12$. In function g, $y = 3x$.

EXPLORATION 3

1. Make a table for each of the following functions. Plot these points on a coordinate system. The points from each table lie on a line. Draw each line in a different color. What do they have in common? How are they different?

   a. $f(x) = x$
   b. $g(x) = 2x$
   c. $h(x) = 3x$
   d. $j(x) = 4x$

2. Make a table for each of the following functions. Plot these points on another coordinate system. The points from each table lie on a line. Draw each line in a different color. What do they have in common? How are they different?

   a. $f(x) = x + 1$
   b. $g(x) = x + 2$
   c. $h(x) = x + 3$
   d. $j(x) = x + 4$

3. Look at these two groups of graphs. What do they have in common? How are they different?

EXPLORATION 4

The functions below have rules for the input, x, and the output, y. Make a table to indicate the outputs for the inputs 0, 1, 2, 3, 4, 10, 15. Make sure your table includes a column that shows how you got the output.

1. The function rule for $\boldsymbol{R}$ is given by the equation $y = 4x + 7$.

2. The function rule for $\boldsymbol{S}$ is given by the equation $y = 4$.

3. Use the $\boldsymbol{R}$ function to determine the input if the output is 55.

EXERCISES

1. Consider the following patterns of inputs and outputs. Write a rule for each that gives the output in terms of the input x. Plot the points for each of these functions on a coordinate plane.

a.

Input	Output
2	7
3	8
4	9
5	10
6	11

b.

Input	Output
1	3
2	3
3	3
4	3
5	3

c.

Input	Output
0	1
1	6
2	11
3	16
4	21
7	36
10	51

2. Consider the function machine with the following data where $y = x + 10$.

Input	Rule: $y = x + 1\ 0$	Output
0	0 + 10	10
1	1 + 10	11

Copy the input/output table and extend it to show the outputs for the following inputs:

a. $x = 3$

b. $x = 5$

c. $x = 7$

d. $x = 2.5$

e. $x = {}^{-}\frac{1}{2}$

f. $x = {}^{-}11\frac{3}{4}$

Plot the points for this function on a coordinate plane.

3. Consider the function machine with the following data where $y = 2x - 3$.

Input	Rule: $y = 2x - 3$	Output
2	$2 \cdot 2 - 3$	1
3	$2 \cdot 3 - 3$	3
5	$2 \cdot 5 - 3$	7

Copy the input/output table and extend it to show the outputs for the following inputs:

a. $x = 7$

b. $x = 9$

c. $x = 11$

d. $x = {}^{-}3.5$

e. $x = \frac{1}{2}$

f. $x = 2\frac{1}{4}$

Plot the points for this function on a coordinate plane.

4. Given the function where the output y is given by $x^3 + 1$, determine the output for the given inputs.

Input	Rule: $x^3 + 1$	Output
0		
1		
2		
3		
4		
5		
6		
7		
8		

5. a. Use the function rule $y = x + 1.5$ to make a table with inputs 0, 1, 2, 3, 4, and 5.

b. Use the function rule $y = 1.5x$ to make a table with inputs 0, 1, 2, 3, 4, and 5.

c. Plot the points from both of these tables and sketch the two lines.

d. Discuss what you observe from your graphs and tables. How are the functions $y = x$, $y = ax$ and $y = x + a$ similar or different numerically in your table?

Graphically? How does the constant a affect each graph?

e. Work this same problem with the rules $y = x + 3$ and $y = 3x$. Compare each of these new graphs with your previous solutions.

6. Suppose a function is given by the rule $y = x + 4$. Fill in the inputs that would yield the given outputs in the table below. In other words, what input x would produce each of the outputs listed in the following table?

Input	Output
	0
	2
	4
	6
	8

7. Miranda sells necklaces for \$25 each. Let n (input) represent the number of necklaces she sells and y (output) represent the amount of money she earns from selling n number of necklaces. Create an input/output table to reflect the following information. Then write a rule (function) that explains the relationship in terms of y.

$n = 4$ $n = 7$ $n = 11$ $n = 13$ $n = 15$.

8. Monica is going to sell lemonade. She spent \$10.00 for materials to get started. It then cost her \$3.00 a gallon for each gallon of lemonade she made.

a. Complete the input/output table to determine y, the cost of making x gallons of lemonade.

x = # of gallons	y = cost of x gallons
0	
1	
2	
3	
5	
10	
x	

b. Write an equation in terms of y that represents the cost of making x gallons of lemonade.

9. Consider the function given by the rule $y = 3x + 1$. Compute the outputs corresponding to the following inputs:

a. 0

b. 1

c. 2

d. $^{-}3$

e. 5

f. 30

g. We can think of the above as ordered pairs (x, y). Copy and complete the table below using the function as given.

x	y	$y = 3x + 1$
1	4	$3(1) + 1$
	7	

h. Graph the ordered pairs from parts a through g, and then connect the points to graph the function.

10. Terry has 11 M&M's. He eats them very slowly; in fact, he takes 1 minute to eat each one.

a. Make a table for the number of M&M's Terry has left after x minutes. Use zero for the starting time.

b. Graph the points from part a.

c. Find the function M that gives the number of M&M's, y, Terry has left after x minutes.

d. When will Terry run out of M&M's?

e. What are the domain and range for the function M?

11. Ingenuity:

Write a formula of a function with the following table of values:

Input	Output
$^-3$	5
$^-2$	4
$^-1$	3
0	2
1	3
2	4
3	5
4	6
5	7

12. **Investigation:**

In this Investigation, we will take a look at the function $F(x) = x^2 + 2x$. This function is called a quadratic function because it involves the square of the input and does not involve any higher power of the input.

a. Make a table of inputs and outputs for the function $F(x)$. What do you notice from looking at the table?

b. Graph the points from part a. What do you notice from looking at the graph?

c. Based on the table and/or the graph, can you find any solutions to the equation $x^2 + 2x = 0$?

SECTION 6.R
CHAPTER REVIEW

In problems 1–5, translate the sentence into an equation and solve for the unknown variable. Does your answer make sense?

1. A number is 4 less than 11.
2. A number is 3 more than twice 15.
3. A number is double the number 19.
4. A number is 15 less than 16.
5. A number is 3 more than a number that is half of 40.

In problems 6–15, solve the equation. Use either the balance scale or the number line to show how you got the answer.

6. $3 + t = 12$
7. $p - 16 = 15$
8. $y = 20 + 283$
9. $63 + m = 81$
10. $28 + d = 32$
11. $9 = a + 8$
12. $76 = m - 75$
13. $x = 82 - 50$
14. $54 - a = 25$
15. $b - 17 = 40$
16. Delores was born September 4, 1991. How old was she on the 4th of July in 2006? How old was she at Christmas of that same year?
17. The great Renaissance artist Michelangelo was born in 1475. If Michelangelo were still alive, how old would he be on his birthday this year? Write a mathematical equation, solve, and check.

18. The tallest mountain in the world, with an elevation of 8848 meters above sea level, is Mount Everest, a part of the Himalayan Mountains near Nepal and Tibet. The next highest peak, at an elevation of 8611 meters above sea level, is K2, also a part of the Himalayas. How much taller is Mount Everest than K2?

19. Marianne had \$43 in bills before she went out to eat. After paying for dinner, she found she had a \$10 bill and two \$1 bills. Approximately how much did she spend?

In problems 20 and 21, define a variable, set up an equation, solve, and check.

20. Valerie bought 8 books. She now has 23 books. How many books did she start out with?

21. In 24 years, Victor will be 42 years old. How old is Victor now?

22. $2 + a = 6$ and $b = 8 + a$. Solve for b and a. Use the number line to show your work.

In problems 23 and 24, write a rule for each that gives the output in terms of the input, x.

23.

Input	Output
2	6
3	11
4	16
6	26
10	46

24.

Input	Output
4	15
5	18
6	21
10	33
12	39

25. When Namiko was writing her new math book, she realized that she could finish 3 pages every hour. Let y = the number of pages Namiko completed. Let h = the number of hours she worked.

    a. Create an input/output table using the following inputs and find the corresponding outputs.

$h = 1$	$h = 3$	$h = 5$
$h = 2$	$h = 4$	$h = 10$

    b. Write an equation, in terms of y, to describe the relationship between the number of hours (h) and the total pages completed (y).

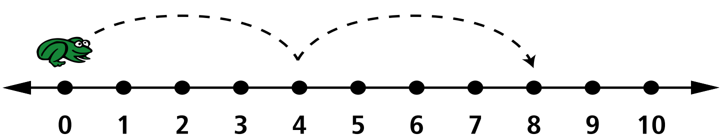
    c. Use the equation you created in part b to determine the number of hours Namiko will have to work to complete a 270-page book.

7 RATES, RATIOS, AND PROPORTIONS

SECTION 7.1
MULTIPLYING FRACTIONS

We used the linear model to understand addition and subtraction of fractions. Now let's use it to understand multiplication of fractions. However, as with multiplication of integers, the area model is also helpful in understanding multiplication of fractions. We will begin by first exploring the linear model.

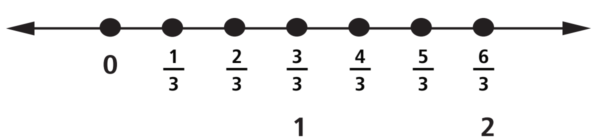
Mary trains at a cross-country facility that has a 4-mile trail. She usually runs one 4-mile lap but decides to go twice as far. She must run 8 miles because $4 \cdot 2 = 8$. Using the frog jump model, each jump represents a 4-mile lap and the number of jumps represents 2 laps.



If Mary runs half a lap, how far will she run? Show what this looks like on the number line. Now write Mary's distance as a multiplication problem.

PROBLEM 1

What would $\frac{1}{3}$ of 6 be? In other words, what is the product $\frac{1}{3} \cdot 6$? Remember that we multiply by using the first factor as the length of the jump and the second factor as the number of jumps. Illustrate the process on the number line below and represent the product in words.



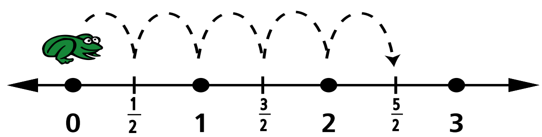
Notice that, $\frac{1}{3} \bullet 6 = \frac{1}{3} \bullet \frac{6}{1} = \frac{6}{3} = 2$. In general, when we multiply a fraction $\frac{1}{n}$ times a number m, or $\frac{1}{n} \cdot m$, then we have $\frac{m}{n}$. For example, in the area formula for a triangle, we can write both $A = \frac{1}{2}(bh)$ or as $A = \frac{(bh)}{2}$. Notice that $\frac{1}{n}$ times m, the product $\frac{m}{n}$ is smaller than m because it is a fraction of m.

EXAMPLE 1

Multiply $\frac{1}{2} \cdot 5$. Write your answer as both improper and mixed fractions.

SOLUTION

Notice that when we multiply $\frac{1}{2} \cdot 5$, we have:
$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \left(\frac{1}{2}\right)(5) = \frac{5}{2}$

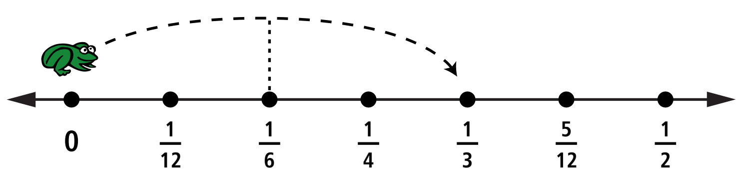


EXAMPLE 2

Jane has $\frac{1}{2}$ a yard of ribbon and needs to cut $\frac{1}{3}$ of its length. To do this, she finds out what $\frac{1}{3}$ of $\frac{1}{2}$ is. Use a number line to show how much ribbon she cuts.

SOLUTION

To find how much ribbon Jane cuts, use the linear model to calculate $\frac{1}{3} \cdot \frac{1}{2}$:



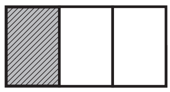
The arithmetic statement is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. If each jump is $\frac{1}{3}$ yard and the frog makes a $\frac{1}{2}$ of a jump, it travels $\frac{1}{6}$ of a yard.

PROBLEM 2

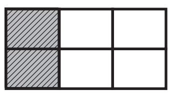
Show what $\frac{1}{4}$ of $\frac{1}{2}$ yard of ribbon is on an appropriate number line. Write the corresponding arithmetic statement for this and the amount in yards it equals.

What pattern do you see in these problems?

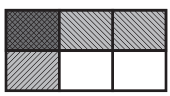
With the linear model it is important to be very exact when drawing the picture. To see the advantage of the area model, look at the problem in Example 2: $\frac{1}{3} \cdot \frac{1}{2}$. To begin the process using the area model, draw $\frac{1}{3}$ as a shaded part of the whole rectangle with area 1.



One way to represent $\frac{1}{2}$ of the shaded area is to cut the rectangle representing $\frac{1}{3}$ in half by cutting vertically. But this is the same process as the linear model. Instead, we cut the rectangle into 2 equal pieces by cutting horizontally.



One of the 2 pieces from the second cut is shaded to represent $\frac{1}{2}$ of the original $\frac{1}{3}$ rectangle. What part of the whole rectangle is the double-shaded area?



EXPLORATION 1

a. Translate $\frac{1}{2}$ of $\frac{1}{5}$ into a multiplication problem and draw the corresponding picture to find the product.

b. Translate $\frac{1}{4}$ of $\frac{1}{3}$ into a multiplication problem and draw the corresponding picture to find the product.

c. Make a conjecture about a rule for multiplying unit fractions:

$$\frac{1}{m} \cdot \frac{1}{n} = \frac{}{}$$

d. Explain why multiplication of unit fractions is commutative, that is

$$\frac{1}{m} \cdot \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{m}$$

EXPLORATION 2

What is the product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, and d are positive integers with b and d not zero? Use the area model to compute the following products.

a. What is $\frac{1}{2}$ of $\frac{2}{3}$? Use the area model to illustrate and find the answer.

b. What is $\frac{2}{5}$ of $\frac{2}{3}$? Use the area model to illustrate and find the answer.

c. What is $\frac{3}{5}$ of $\frac{4}{7}$? Use the area model to illustrate and find the answer.

d. Make a conjecture for the rule for multiplying two fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{\quad}{\quad}$$

Summarizing this pattern of multiplying fractions,

RULE 7.1: MULTIPLYING FRACTIONS
The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where b and d are non-zero, is $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Multiplying fractions can be very useful. For example, you know how to multiply $\frac{3}{2} \cdot 6$ to obtain the product 9. When we multiply by more than 1 of a number, such as $\frac{3}{2}$ of 6, the resulting product is more than $\frac{2}{2}$ or 1 times the 6. In fact, the product is $\frac{1}{2}$ of the 6 more than 6 because $\frac{3}{2} = \frac{2}{2} + \frac{1}{2}$. Multiplying $\frac{1}{2}$ of 6 yields 3, so the result is $6 + 3 = 9$, and there is a clear increase in the result. When you study ratios and rates in the next section, you will see that your skill in multiplying fractions will provide efficient ways to solve problems.

EXPLORATION 3

Multiply 24 by each fraction from the following 2 groups:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{5}{12}\right\} \text{ and } \left\{\frac{3}{2}, \frac{4}{3}, \frac{11}{6}, \frac{19}{12}\right\}$$

a. What do you notice about these products?

b. What do the products from each group have in common?

c. Make a rule about multiplying a number N by a fraction.

From Exploration 3, you can see that the product of two natural number factors is a natural number greater than or equal to the original natural numbers. However, when a number is multiplied by a fraction less than 1, the product will be less than the original number. Can you see why?

For example, $\frac{3}{4}$ of 12 is $\frac{(3 \cdot 12)}{(4 \cdot 1)}$ = 9, so 12 decreased to 9.

When a number is multiplied by a fraction greater than 1, the resulting product is greater than the original number so the number increased. For example, $\frac{3}{2}$ of 12 is $\frac{(3 \cdot 12)}{(2 \cdot 1)}$ = 18, and the product is greater than 12. In other words, 12 increased by a factor of $\frac{3}{2}$ is 18.

EXERCISES

1. Compute the following products and simplify if needed:

a. $\frac{1}{2} \cdot 10$	d. $\frac{1}{3} \cdot 18$	g. $16 \cdot \frac{3}{4}$	j. $\frac{4}{5} \cdot \frac{2}{3}$
b. $27 \cdot \frac{1}{3}$	e. $\frac{1}{7} \cdot 49$	h. $\frac{3}{8} \cdot \frac{1}{4}$	k. $\frac{2}{7} \cdot \frac{2}{9}$
c. $\frac{1}{5} \cdot 100$	f. $\frac{2}{3} \cdot 34$	i. $\frac{3}{5} \cdot 30$	l. $\frac{11}{12} \cdot \frac{2}{3}$

For each problem below, write an equation, solve the problem, and write your answer in a complete sentence.

2. Betty is riding her bike to the library which is $\frac{3}{5}$ of a mile from her house. She rides her bike $\frac{1}{2}$ of the distance before she gets a flat tire. What fraction of a mile did she bike before her bike broke down?

3. During a class discussion, Mr. Garza found out that $\frac{2}{3}$ of his students have younger brothers or sisters. If he has 24 students, how many students in his class have younger siblings?

4. Mr. Flores was setting up a rectangular window box to grow flowers under his kitchen window. It measures $\frac{3}{4}$ of a meter long and $\frac{3}{5}$ of a meter wide. What is the area of the rectangular box?

5. Madison had half of her birthday cake left over after her party. If she gives $\frac{3}{8}$ of the leftover cake to her friend, how much cake will her friend get?

6. In a recent school survey, $\frac{3}{5}$ of the students reported preferring soft drinks, $\frac{1}{4}$ preferred fruit juice, and the rest preferred drinking water.
   a. What fraction of the students preferred drinking water?
   b. Of those students choosing soft drinks, $\frac{2}{3}$ said they liked sugar-free soft drinks. What fraction of the students preferred sugar-free soft drinks?
   c. If the survey was conducted on 200 students, how many students would have selected each type of drink?
7. While conducting an experiment, Mrs. Ayala found that $\frac{4}{5}$ of her students were right handed and $\frac{1}{6}$ of the students were left handed. One student could use either hand to do many things. If there are 30 students in her class, how many of them are left handed?
8. Raymond wants to cover a window on a motorized child's car with window tinting film. The window measures $\frac{2}{3}$ of a foot in length and $\frac{3}{4}$ of a foot in height. How much tinting film will he use?
9. A pond at the fish hatchery contained 800 fish. They recently added $\frac{3}{5}$ the amount of the original number of fish to the pond. How many fish were added to the pond? What is the total number of fish in the pond now?

Spiral Review:

10. Shirley has $\frac{4}{5}$ of her homework finished for tomorrow. Which of the following is not equivalent to $\frac{4}{5}$?
    a. $\frac{8}{10}$ b. 0.80 c. $\frac{12}{18}$ d. 0.8
11. What is the prime factorization of 660?
12. **Ingenuity:**

    Randall made a large batch of cookies. $\frac{1}{3}$ of the cookies were sugar cookies, while the other $\frac{2}{3}$ were chocolate chip cookies. At a party, Randall's guests ate $\frac{7}{12}$ of the sugar cookies, along with most of the chocolate chip cookies. After the party, Randall noticed that he had only $\frac{1}{4}$ of his cookies left. What fraction of the chocolate chip cookies were eaten?

13. **Investigation:**

Solve each of the following problems. You may find it helpful to draw pictures.

a. Rikki has a bamboo pole of length $\frac{7}{2}$ feet that she wants to divide into $\frac{1}{2}$ foot pieces. How many pieces can she make?

b. Umberto has $\frac{15}{4}$ pounds of salt, and he wants to divide the salt into bags that contain $\frac{3}{4}$ pounds of salt each. How many bags can Umberto fill?

c. Bill has a $6\frac{1}{4}$-inch candy bar, which he divides into $1\frac{1}{4}$-inch pieces. How many pieces does he make?

7.2

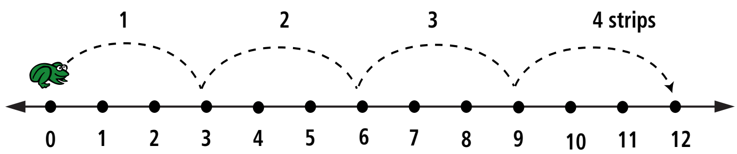
SECTION 7.2
DIVISION OF FRACTIONS

In order to better understand division of fractions, you can use linear and area models to represent the process of division, just as with whole numbers. Try the following exploration to see how you can solve and explain the process that you used. You may wish to use words as well as pictures to explain your thinking.

EXPLORATION 1

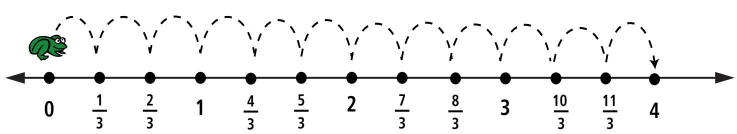
Melinda wants to cut a 4-yard fabric into $\frac{1}{3}$-yard strips. How many strips will she have? Explain how you reached your conclusion.

Just as with a whole number division problem, such as cutting a 12-yard fabric into 3-yard strips, you can use a linear model to show that in this case there are 4 strips possible.



That is, $12 \div 3 = 4$.

You can also use a linear model to represent the problem in Exploration 1.



We can see that $4 \div \frac{1}{3} = 12$.

PROBLEM 1

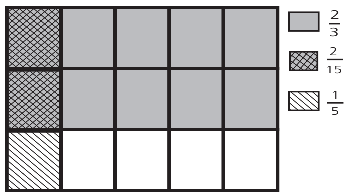
Liz has 4 pounds of jellybeans. She plans to make little party bags containing $\frac{1}{2}$ pound of jellybeans. How many party bags can she make?

EXPLORATION 2

Chuck has two-thirds of a pan of brownies and shares it evenly among his 5 friends. What fraction of the pan of brownies does each friend receive? Explain how you reached your conclusion.

One way to think about the problem in Exploration 2 is as a division of $\frac{2}{3} \div 5$. Another approach is to think of each of Chuck's 5 friends receiving $\frac{1}{5}$ of the brownies or as a multiplication problem, $\frac{2}{3} \cdot \frac{1}{5}$.

An area model of this problem can be represented as:



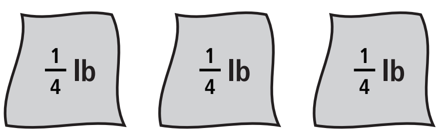
$\frac{2}{3} \div 5 = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}$ or the darker shaded region.

PROBLEM 2

Barbara has $\frac{1}{2}$ of a pan of brownies and shares it evenly among 6 friends. What part of the pan of brownies does each friend receive? Use the area model to show and explain how you reached your conclusion.

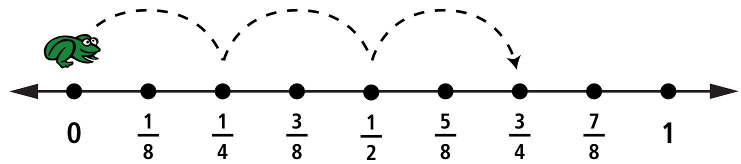
Look at a similar problem but with fractional quantities: How many $\frac{1}{4}$ pound bags does it take to pack $\frac{3}{4}$ pounds of sand? In other words, what is $\frac{3}{4} \div \frac{1}{4}$?

Using a repeated subtraction model, make 3 equal parts. With the first $\frac{1}{4}$ pound bag, $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ pounds are left. The second $\frac{1}{4}$ pound leaves $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so the third $\frac{1}{4}$ pound bag leaves no sand.

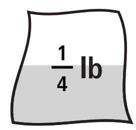


Writing this as a division problem, $\frac{3}{4} \div \frac{1}{4} = 3$. At first, it might be surprising that when dividing two fractions, the answer is an integer, especially when the integer is large compared to the fractions. What does 3 represent in this case?

Another way to think about this problem uses the missing factor method. What number times $\frac{1}{4}$ equals $\frac{3}{4}$? Starting at 0 on the number line, 3 jumps of length $\frac{1}{4}$ equals $\frac{3}{4}$. So, $\frac{3}{4} \div \frac{1}{4} = 3$.



Notice in the earlier example with bags of sand, the quantity of sand exceeded the bag size. A $\frac{3}{4}$ pound bag was being separated into smaller $\frac{1}{4}$ pound bags. The number of bags was $\frac{3}{4} \div \frac{1}{4} = 3$ bags. What if the initial quantity is less than the bag size, like having $\frac{1}{8}$ pound of sand and a bag that holds $\frac{1}{4}$ of a pound? What is $\frac{1}{8} \div \frac{1}{4}$?



It is impossible to use a "repeated-subtraction" model, because there is no way to fill even one $\frac{1}{4}$ pound bag with only $\frac{1}{8}$ pound of sand. You can see that with the $\frac{1}{8}$ pound, only $\frac{1}{2}$ of the $\frac{1}{4}$ pound bag is filled. Therefore, $\frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$.

Using the relationship between fractions and division, that $m \div n$ is the same as $\frac{m}{n}$, rewrite $\frac{1}{8} \div \frac{1}{4}$ as a big fraction, that is $\frac{\frac{1}{8}}{\frac{1}{4}}$. This looks pretty complicated, but luckily it can be simplified.

Writing a division problem such as $4 \div 8$ as a fraction $\frac{4}{8}$ does not appear complicated. However, working with a division problem with two fractions such as $\frac{1}{8} \div \frac{1}{4}$ can seem more difficult.

Recall that simplifying a fraction requires rewriting the fraction as an equivalent fraction. But instead of factoring the numerator and denominator and applying the equivalent fraction property, we create an equivalent fraction. Multiply by 1 but in an appropriate form to both the numerator and denominator. This process will convert the denominator to a very friendly product. Before we explore this process, remember that a non-zero number n divided by n is 1. We also saw from Exploration 2 that n divided by n is the same as multiplying n by $\frac{1}{n}$. We have the following result:

$n \div n = 1$ but we know that $n \div n = n \cdot \frac{1}{n}$ so $n \cdot \frac{1}{n} = 1$. We have a special name for $\frac{1}{n}$.

DEFINITION 7.1: MULTIPLICATIVE INVERSE
If a is any non-zero number, then the **multiplicative inverse** or **reciprocal** of a is the unit fraction $\frac{1}{a}$. The product $\frac{1}{a} \cdot a$ is $\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$

Note: In general, if x is a number and $x \neq 0$, then there exists a number, denoted as $\frac{1}{x}$, so that the product of x and $\frac{1}{x}$ is 1, that is, $x \cdot \frac{1}{x} = 1$. We call $\frac{1}{x}$ the reciprocal or multiplicative inverse of x.

EXPLORATION 3

Compute each of the following products:

a. $\frac{2}{3} \cdot \frac{3}{2}$ b. $\frac{4}{5} \cdot \frac{5}{4}$ c. $\frac{4}{7} \cdot \frac{7}{4}$ d. $\frac{5}{9} \cdot \frac{9}{5}$

What do you notice?

From the discussion on Exploration 3, we have the following rule:

DEFINITION 7.2: MULTIPLICATIVE IDENTITY
If each of a and b is not zero, then the product $\frac{a}{b} \cdot \frac{b}{a} = 1$. 1 is called the multiplicative identity.

We return to the division $\frac{1}{8} \div \frac{1}{4} = \frac{\frac{1}{8}}{\frac{1}{4}}$ expressed as a fraction. If we multiply the numerator and denominator by the reciprocal of $\frac{1}{4}$, namely by 4, then

$$\frac{1}{8} \div \frac{1}{4} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{\frac{1}{8} \bullet \frac{4}{1}}{\frac{1}{4} \bullet \frac{4}{1}} = \frac{\frac{4}{8}}{\frac{4}{4}} = \frac{\frac{4}{8}}{1} = \frac{4}{8} = \frac{1}{2}$$

In general, when the denominator of a fraction is a fraction, multiplying both the numerator and denominator by the reciprocal of the denominator produces a simpler fraction. Another approach to simplify complicated fractions uses the pattern that $m \div n = \frac{m}{n} = m \cdot \frac{1}{n}$. Using this pattern, rewrite $\frac{\frac{1}{8}}{\frac{1}{4}}$ as $\frac{1}{8} \cdot \frac{4}{1}$, since the reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$.

Then multiply to find the answer: $\frac{1}{8} \cdot \frac{4}{1} = \frac{4}{8} = \frac{1}{2}$.

PROBLEM 3

Find the multiplicative inverse or reciprocal of the following numbers.

a $\frac{1}{2}$ b. $\frac{3}{5}$ c. $\frac{8}{7}$ d. 1 e. 14

PROBLEM 4

Compute the following division of fractions using the stacking method from before.

a. $2 \div \frac{1}{4}$ b. $3 \div \frac{1}{4}$ c. $\frac{1}{2} \div \frac{1}{4}$ d. $\frac{1}{4} \div \frac{1}{2}$

PROBLEM 5

Valerie's bird feeder holds $\frac{5}{6}$ of a cup of birdseed. Valerie is filling the bird feeder with a scoop that holds $\frac{1}{6}$ of a cup. How many scoops of birdseed will Valerie put into the feeder? Use the numerical technique from above. Write your answer in simplest form.

EXAMPLE 1

Find the quotient of $\frac{2}{5} \div \frac{3}{7}$ by using the Stacking Method.

SOLUTION

We use Rule 7.2 to pick a fraction and then multiply it by the numerator and the denominator to obtain an equivalent fraction so that the denominator simplifies to be the value of 1. This transforms a division of fractions calculation into a multiplication of fractions calculation.

$$\frac{\frac{2}{5}}{\frac{3}{7}} = \frac{\frac{2}{5} \bullet \frac{7}{3}}{\frac{3}{7} \bullet \frac{7}{3}} = \frac{\frac{2}{5} \bullet \frac{7}{3}}{1} = \frac{2}{5} \bullet \frac{7}{3} = \frac{14}{15}$$

In summary, when dividing by a fraction, or simplifying a fraction whose denominator is a fraction, use one of the two following techniques:

Method 1: Write the division problem as a fraction and multiply the numerator and denominator of this fraction by the reciprocal of the denominator. This results in an equivalent fraction with denominator 1:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \bullet \frac{d}{c}}{\frac{c}{d} \bullet \frac{d}{c}} = \frac{\frac{a}{b} \bullet \frac{d}{c}}{1} = \frac{a}{b} \bullet \frac{d}{c} = \frac{ad}{bc}$$

Method 2: Or, because division by a number is equivalent to multiplication by the reciprocal, rewrite the division as multiplication:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Either approach will find the quotient. The major point is that division transforms into multiplication, not magically, but from a well-motivated reason using a deep understanding of fractions and how they work.

Remember that in $m \div n$, n cannot be zero.

EXERCISES

1. Use a visual model to compute the following quotients. Check your work by finding the product of the quotient and the divisor.

   a. $2 \div \frac{1}{4}$ c. $1 \div \frac{3}{4}$ e. $3 \div \frac{1}{6}$ g. $\frac{3}{5} \div \frac{2}{5}$

   b. $3 \div \frac{1}{4}$ d. $\frac{1}{3} \div \frac{1}{6}$ f. $\frac{1}{6} \div 3$ h. $\frac{2}{3} \div \frac{1}{2}$

2. Compute the following quotients by using the process developed in Method 1. Check your answer by using Method 2. Simplify your answer if needed.

   a. $\frac{\frac{1}{6}}{\frac{1}{3}}$ b. $\frac{\frac{1}{2}}{\frac{3}{4}}$ c. $\frac{\frac{2}{5}}{\frac{1}{2}}$ d $\frac{\frac{3}{10}}{\frac{1}{5}}$

3. Compute the following quotients by using either Method 1 or Method 2:

   a. $\frac{2}{3} \div \frac{2}{3}$ b. $\frac{\frac{3}{7}}{\frac{3}{7}}$ c. $\frac{3}{5} \div \frac{3}{5}$ d. $\frac{1}{\frac{1}{7}}$

4. Hugo has 14 meters of wire on a roll. He needs to cut the wire into $\frac{2}{5}$ meter lengths. If Hugo cuts the whole roll of wire, how many pieces of wire will Hugo have?

5. $\frac{7}{8}$ of a pound of pecans is packaged into $\frac{1}{4}$ pound bags. How many bags of pecans can be packaged?

6. August biked 10 miles in $\frac{1}{4}$ of an hour. How far can August bike in 1 hour?

7. A recipe calls for $\frac{3}{4}$ cup of flour and you have $\frac{1}{2}$ cup of flour. What part of the recipe can you make?

8. Jacob can fold an origami crane in $\frac{1}{12}$ of an hour. How many origami cranes can Jacob fold in $2\frac{1}{2}$ hours?

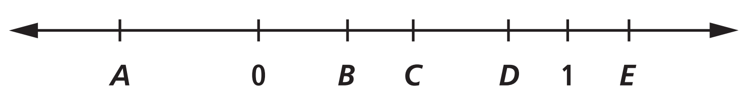
9. Stephanie is decorating costumes with $\frac{3}{5}$ yard of ribbon for each costume. How many costumes can Stephanie decorate if she has 3 yards of ribbon?

10. **Ingenuity:**

Two-thirds of Ms. Tate's sixth-grade students are boys. To make the number of boys and girls equal, 4 boys go to the other sixth-grade class, and 4 girls come from that class into Ms. Tate's class. Now one-half of her students are girls. How many students are in Ms. Tate's class?

11. **Investigation:**

Consider the following number line.



a. Which point best represents the sum of the fractions ***B*** and ***D***? Explain why you think so.

b. Which point best represents the difference, ***D*** – ***C***? Explain why you think so.

c. Which point best represents the product of the fractions ***C*** and ***D***? Explain why you think so.

d. Which point best represents the quotient ***D*** ÷ ***C***? Explain why you think so.

12. a. $\frac{2}{5} + \frac{5}{2}$
    b. $\frac{7}{5} - \frac{5}{7}$
    c. $\frac{18}{5} \cdot \frac{10}{9}$
    d. $\frac{4}{9} \div \frac{5}{6}$
13. a. $\frac{6}{25} + \frac{15}{8}$
    b. $\frac{6}{25} \div \frac{15}{8}$
    c. $\frac{8}{5} - \frac{10}{9}$
    d. $\frac{14}{9} \cdot \frac{15}{8}$
14. a. $\frac{1}{8} + \frac{1}{12}$
    b. $\frac{1}{8} - \frac{1}{12}$
    c. $\frac{1}{8} \div \frac{1}{12}$
    d. $\frac{1}{8} \cdot \frac{1}{12}$
15. a. $\frac{7}{15} + \frac{7}{10}$
    b. $\frac{17}{24} - \frac{5}{16}$
    c. $\frac{21}{20} \cdot \frac{25}{14}$
    d. $\frac{21}{16} \div \frac{35}{8}$
16. a. $4\frac{2}{5} + 2\frac{4}{5}$
    b. $7\frac{3}{4} - 4\frac{5}{8}$
    c. $\left(2\frac{3}{7}\right)\left(4\frac{1}{3}\right)$
    d. $\left(2\frac{3}{7}\right) \div \left(4\frac{1}{3}\right)$
17. a. $2\frac{5}{6} + 5\frac{3}{4}$
    b. $6\frac{2}{5} \div 3\frac{1}{5}$
    c. $\left(1\frac{3}{7}\right)\left(4\frac{1}{5}\right)$
    d. $4\frac{5}{12} - 2\frac{5}{8}$

7.3

SECTION 7.3
RATES AND RATIOS

Fractions are often used to compare quantities. For example, Kealing Middle School has 400 students, and 280 of them live within 2 miles of the campus. Simplifying, $\frac{280 \text{ students}}{400 \text{ students}} = \frac{7}{10}$. Notice that both units are "students" and the fraction simplifies. How can you interpret the meaning of the simplified fraction?

In Kealing Middle School, 7 out of every 10 students live within 2 miles of the school. Converting the fraction $\frac{7}{10}$ into a percent, 70% of the students live within 2 miles. The fractional form of this comparison is called a ratio. A **ratio** is a division comparison of two quantities with or without the same units.

Ratios can be written in the form of first one quantity, then a colon followed by a second quantity. Ratios can also be written using the word "to" in place of the colon as well as in fraction form. In this example, the unit of measure is the number of students.

Because there are 280 students who live within 2 miles, write

280 students : _______

Compared to these 280 students within 2 miles, there are 400 total students. So the ratio of students who live within 2 miles to total number of students is:

280 students who live within 2 miles : 400 total students

Notice that there is more than one way to write a ratio. A ratio that relates quantity x to quantity y can be written as:

1. x to y
2. $x : y$
3. $\frac{x}{y}$

Ignoring for a moment the units and using only the numbers, write the ratio as 280 : 400. Just as with fractions, we can simplify this to 7 : 10. Always remember what kinds of things are being compared. In this problem, what does the ratio 7 : 3 describe?

It is useful to observe that ratios may relate part to whole and in other instances ratios may relate part to part as in the example above with 7 : 10 and 7 : 3.

Rates are special ratios that compare different units. Suppose you earn 30 dollars for doing 5 hours of yard work and mowing the lawn. You know that $30 \div 5 = 6$ indicates how much money you earned per hour. Using fractions, this calculation looks like $\frac{30}{5} = 6$. However, it is usually helpful to write this problem using the units that describe each quantity. So the calculation becomes:

$$\frac{30 \text{ dollars}}{5 \text{ hours}} = \frac{6 \text{ dollars}}{1 \text{ hour}} = 6\,\frac{\text{dollars}}{\text{hours}}.$$

You read "$6\frac{\text{dollars}}{\text{hour}}$" as "six dollars per hour." The answer explains how many dollars you earned each hour. This quantity is an example of a rate. A **rate** is defined as a division comparison between two quantities, usually with two different units, like dollars and hours. What are some other rates that you have worked with or know about?

The simplified fractional answer in the example is called a **unit rate** because it represents a number or quantity per 1 unit, or hour in this case. The units may be written in fractional form, like $\frac{\text{dollars}}{\text{hour}}$, $\frac{\text{miles}}{\text{hour}}$, or $\frac{\text{miles}}{\text{gallon}}$, and are usually read as "dollars per hour," "miles per hour," or "miles per gallon."

PROBLEM 1

Diana sent 300 text messages in 5 hours. How many messages did she send each hour?

EXPLORATION 1

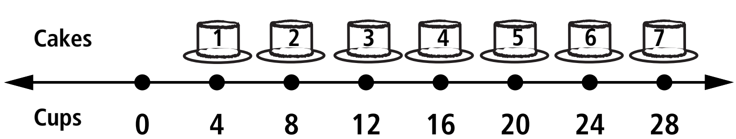
Juan drove 150 miles in 3 hours and used 5 gallons of gasoline. Make as many rates using these quantities and their units as possible. Explain what each unit fraction means.

EXAMPLE 1

Sandra's bakery uses 4 cups of flour per cake when she bakes. How many cups of flour will she use when she bakes 7 cakes for a customer?

SOLUTION

The unit rate for amount of flour per cake is given as $\frac{4 \text{ cups}}{1 \text{ cake}} = 4\,\frac{\text{cups}}{\text{cake}}$. To find how much flour is used to bake 7 cakes, we first use a visual representation.



This process can be written as follows:

$$4\,\frac{\text{cups}}{\text{cake}}\cdot(7\text{ cakes}) = 28\,\frac{\text{cups}}{\text{cake}}\cdot\text{cake} = 28\text{ cups}$$

We can view $\frac{\text{cups}}{\text{cakes}}\cdot\text{cakes} = \text{cups}$, just as when we multiply fractions such as $\frac{5}{3}\cdot 3 = 5$. The unit of "cake" in the numerator and denominator simplifies to 1 and the answer is in cups. This way of keeping track of the units is very useful in application problems.

EXAMPLE 2

Phil mixed one teaspoon of chocolate with one cup of milk, and Sam mixed three teaspoons of chocolate with one quart of milk.

a. Which mixture would taste more "chocolaty"? Justify your answer.

b. How much more chocolate per cup would the stronger mixture have?

SOLUTION

a. The ratio of chocolate to milk in Phil's mixture is 1 teaspoon of chocolate per 1 cup of milk. In order to compare this to Sam's mixture, we must use the same units. Convert the units from quarts to cups and multiply by the conversion factors $\left(\frac{1\text{ quart}}{2\text{ pints}}\right)\left(\frac{1\text{ pint}}{2\text{ cups}}\right) = \frac{1\text{ quart}}{4\text{ cups}}$. So the ratio of chocolate to milk in Sam's mixture is:

$$\left(\frac{3\text{ tsp chocolate}}{1\text{ quart milk}}\right)\left(\frac{1\text{ quart}}{2\text{ pints}}\right)\left(\frac{1\text{ pint}}{2\text{ cups}}\right) = \frac{3\text{ tsp chocolate}}{4\text{ cups milk}} = \frac{\frac{3}{4}\text{ tsp chocolate}}{\text{cup}}$$

We see that Phil's drink is more "chocolaty" because it has a greater ratio of chocolate to milk.

b. Phil's drink has $\frac{1}{4}$ teaspoon more chocolate per cup than Sam's.

EXAMPLE 3

Samantha and Georgeanne went bowling. Samantha made two strikes every seven times she tried, and Georgeanne made one strike every four times she tried.

a. Which bowler would be more likely to make a strike? Explain your answer by comparing the ratio of strikes to attempts for each bowler.

b. If each bowler tried 20 times, on average how many strikes would each be predicted to make?

SOLUTION

a. For Samantha the ratio is 2 strikes in 7 attempts, or 2:7 = $\frac{2}{7}$

For Georgeanne the ratio is 1 strike in 4 attempts, or 1:4= $\frac{1}{4}$

By converting these ratios to fractions with like denominators, we see that on average, Samantha is more likely to make a strike than Georgeanne, with a difference of 1 strike in 28 attempts:

$$\text{Samantha} = \frac{1}{4} = \left(\frac{1}{4} \bullet \frac{7}{7}\right) = \frac{7}{28}$$

$$\text{Georgeanne} = \frac{2}{7} = \left(\frac{2}{7} \bullet \frac{4}{4}\right) = \frac{8}{28}$$

b. To find how many strikes each bowler might make in 20 attempts, multiply:

Samantha: $\left(\frac{2 \text{ strikes}}{7 \text{ attempts}}\right)(20 \text{ attempts}) = \left(\frac{40}{7}\right)$ strikes, between 5 and 6 strikes.

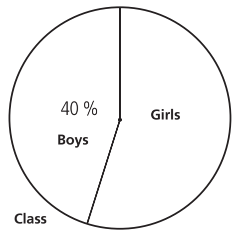
Georgeanne: $\left(\frac{1 \text{ strike}}{4 \text{ attempts}}\right)(20 \text{ attempts}) = 5$ strikes.

Another approach would be to let x = number of strikes Samantha would make in 20 attempts. Then $\frac{x}{20} = \frac{2}{7}$. Multiplying both sides of this equation by 20 to solve for x, $\frac{x}{20} \cdot 20 = x = \frac{2}{7} \cdot 20 = \frac{40}{7}$ strikes, which agrees with the results of part c.

PROBLEM 2

Norman rode his bike for $3\frac{1}{2}$ hours and traveled 56 miles. What was his average rate, or speed? Approximately how far did he travel in the first hour and a half?

When a comparison uses percents, it is based on the ratio of an amount to an original or base amount. For instance, "40% of the class is boys" can be thought of as the ratio of boys to the class. 40% boys is equivalent to $\frac{40}{100}$, the fractional part of the class that is boys. What percent of this class must be girls? How did you decide the percent girls must be 60%? One way to think of this is to consider the whole class as 100% and removing 40% leaves 60%. What then is the ratio of boys to girls? $\left(\frac{40}{60} = \frac{4}{6} = \frac{2}{3}\right)$ What is the ratio of girls to boys? $\left(\frac{60}{40} = \frac{6}{4} = \frac{3}{2}\right)$. In this class, the number of girls is 150% the number of boys. In this case, the number of boys is the original, or base, amount. In both cases comparing to the whole class, the original, or base, amount is 100%.

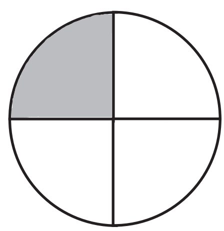


EXAMPLE 4

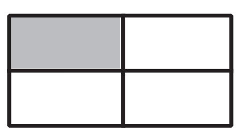
In a class of 40 students at Kealing Middle School, 25% of the students take band. How many students take band?

SOLUTION

One approach is to create a circle graph and cut it into four equal pieces, shown below. In this case, the whole circle represents forty students. Because 25% is the same as $\frac{1}{4}$ of the circle, then $\frac{1}{4}$ of 40 is 10. That means that 25% of the class of 40 is really 10 out of the 40 students in the class.



Another approach is to create an area model with 25% of it shaded. It is easy to see that this is $\frac{1}{4}$ of the whole area so represents 10 of the 40 students.



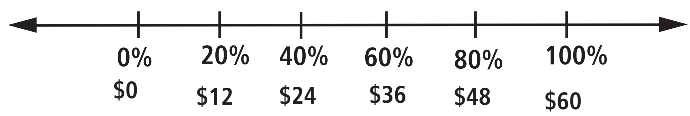
EXAMPLE 5

Frankie spent \$12 to see a movie. That amounted to 20% of her monthly allowance. What is Frankie's monthly allowance from her parents?

SOLUTION

$20\% = \frac{1}{5}$. Because \$12 is $\frac{1}{5}$ of Frankie's total allowance, the total must be 5 times \$12 because 5 times $\frac{1}{5}$ is $\frac{5}{5} = 1$ or all of the total allowance. $5 \cdot 12 = 60$, so Frankie's monthly allowance is \$60.

Another way to visualize the whole allowance from 20% of it is to look at a number line model to see five 20% increments to 100% with \$12 for each 20% summing to $5 \cdot 12 = 60$. This is the same answer as above.

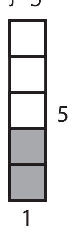


PROBLEM 3

Mr. Graham needs fifteen yards of fabric, which is 25% of a fabric bolt, to make curtains for his school's summer musical. He will use the other 75% of the bolt to make costumes. How many yards of material are in the bolt?

PROBLEM 4

The shaded part of the rectangular grid represents a five-gallon water jug that is 40% full. How many gallons are in the water jug?



EXPLORATION 2

Gloria works as a computer consultant for the Bayou Company and earns \$612 for working 36 hours. If she charges a fixed amount per hour, how much will she earn working 18 hours? 9 hours? 1 hour? 4 hours?

EXPLORATION 3

Ms. Jones starts filling a very large empty tank with water at the rate of 4 liters per minute.

a. How much water is in the tank after 1 minute? 2 minutes? 5 minutes?

b. Make a table with filling time as input in minutes and amount of tank filled as output in liters of water.

c. Draw a graph that includes at least 5 points.

d. Write an equation that relates how much water is in the tank after x minutes.

e. How much water will be in the tank after 42 minutes?

f. When will there be 244 liters in the tank?

g. When will there be 95 liters in the tank?

EXAMPLE 6

Lisa and Adam both jog every day. Lisa jogs an average of 2200 meters in 40 minutes and Adam jogs an average of 1500 meters in 30 minutes.

a. Who jogs faster? Explain.

b. On average, how far does Lisa jog in 12 minutes?

c. At the same rate, how much further would Lisa jog in 12 minutes than Adam?

d. Maintaining his pace, how long does it take Adam to jog 600 meters?

e. Find a rate that is exactly midway between the jogging rates of Lisa and Adam.

SOLUTION

a. Lisa jogs at a rate of $\frac{2200 \text{ meters}}{40 \text{ minutes}} = 55 \frac{\text{meters}}{\text{minute}}$. Adam jogs at a rate of $\frac{1500 \text{ meters}}{30 \text{ minutes}} = 50 \frac{\text{meters}}{\text{minute}}$. So Lisa jogs faster than Adam.

b. On average, Lisa would jog $(55 \frac{\text{meters}}{\text{minute}}) \cdot 12$ minutes = 660 meters.

c. Adam would jog $(50 \frac{\text{meters}}{\text{minute}}) \cdot 12$ minutes = 600 meters. So Lisa would go 60 meters further in 12 minutes than Adam.

d. Since Adam jogs 50 $\frac{\text{meters}}{\text{minute}}$, we let $\boldsymbol{x}$ = the number of minutes he takes to jog 600 meters. Then (50 $\frac{\text{meters}}{\text{minute}}$) · $\boldsymbol{x}$ minutes = 600 meters. 50 $\boldsymbol{x}$ = 600, so $\boldsymbol{x}$ = 12 minutes (which is what we already found above).

e. A rate midway between Lisa's rate and Adam's rate is 52.5 $\frac{\text{meters}}{\text{minute}}$. This is the average of the two rates, $\frac{(55 + 50)}{2} = 52.5$.

Example 6 is an example of the rate formula that assumes that something is traveling at a constant rate. The distance traveled is equal to the constant rate multiplied by the time traveled: $d = r \cdot t$ or $d = rt$. The constant rate is often written as the unit rate. Typical units for r are:

miles per hour $\left(\frac{\text{mi}}{\text{hr}}\right)$ = miles traveled in one hour

feet per second $\left(\frac{\text{ft}}{\text{sec}}\right)$ = feet traveled in one second

meters per minute $\left(\frac{\text{m}}{\text{min}}\right)$ = meters traveled in one minute

EXAMPLE 7

Class members are surveyed to find out what is their favorite type of drink, soda or lemonade. Forty percent of the class prefers soda, and sixty percent prefers lemonade. What is the ratio of students who prefer soda to students who prefer lemonade?

SOLUTION

If the class has T students, then the number of students who prefer soda is $0.40T$, and the number who prefer lemonade is $0.60T$. So the ratio is:

$$\frac{(0.40T)}{(0.60T)} = \frac{0.40}{0.60} = \frac{4}{6} = \frac{2}{3}.$$

Recall that this can also be written as 2:3 or 2 to 3.

EXAMPLE 8

Two quantities x and y are related as you can see in the following table:

x	y
1	$\frac{1}{2}$
2	1
3	$\frac{3}{2}$
4	2
5	$\frac{5}{2}$
6	3

a. Write an equation that gives y in terms of x.

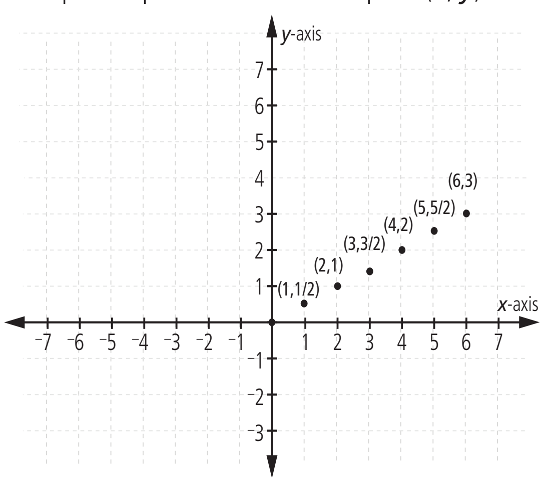
b. Graph the equation.

c. Examine the ratio of y to x using two different (x, y) pairs. What do you notice?

SOLUTION

a. The ratio of y to x is $\frac{1}{2}$ to 1. So $y = \frac{1}{2}x$.

b. The graph of the equation plots the six ordered pairs (x, y)



c. The ratio is $\frac{1}{2}$.

PROBLEM 5

The ratio of two quantities, p and q, is the same as in the equation $q = 3p$. Make a table with at least 5 entries that relate p and q.

EXERCISES

Answer the following questions using the fraction form of the ratios. Simplify your ratios if possible.

1. Carmen packed her suitcase with 6 shirts, 4 pairs of pants, and 3 belts.
   a. What is the ratio of shirts to pants?
   b. What is the ratio of pants to shirts?
   c. What is the ratio of pants to belts?
   d. What is the ratio of belts to pants?
   e. What is the ratio of shirts to belts?
   f. What is the ratio of belts to shirts?
2. Ms. Acosta decided to reward $\frac{2}{5}$ of her class with a free homework pass. There are 25 students in her class.
   a. How many students received a free homework pass?
   b. What ratio of students did not receive a free homework pass?
3. A bag contains red, blue, and green color tiles. It has 35% red tiles, 20% blue tiles, and 45% green tiles.
   a. What is the ratio of blue tiles to the total number of tiles?
   b. What is the ratio of green tiles to the total number of tiles?
   c. What is the ratio of blue tiles to red tiles?
   d. What is the ratio of red tiles to green tiles?
   e. What is the ratio of green tiles to blue tiles?
4. Sarah and Max each purchased donuts for their math team. Sarah bought three glazed donuts, two jelly-filled donuts, and three powdered donuts. Max purchased five glazed donuts, eight jelly-filled donuts, and six powdered donuts.
   a. What is the ratio of glazed donuts to total donuts for Max's team? For Sarah's team?

b. If one of Sarah's team members chose a donut at random from Sarah's donuts, and one of Max's team selected a donut at random from his donuts, explain who would be more likely to obtain a glazed donut.

5. a. What is the ratio of vowels to consonants in this question?

   b. Now examine the ratio in this statement as well.

   c. Conjecture what might be an expected ratio of vowels to consonants in most English sentences. Check with several other sentences and observe the ratios you find. Express the ratio of vowels : consonants.

   d. Is the ratio of vowels to consonants close to 1?

   e. What does it mean if the ratio is greater than 1? Less than 1?

   f. Based on the ratio you found in parts a, b, and c, if you read a book with 25,000 letters, about how many of these letters would you expect to be vowels?

6. Solve the following unit rates.

   a. A gallon of milk has 128 ounces. A cup holds 8 ounces. What is the number of cups per gallon?

   b. There are 12 eggs per dozen. A box holds 144 eggs. What is the number of dozens per box?

   c. You can buy 20 boxes of crayons for $60. What is the price per box?

7. Use the following table that gives the ratio of two quantities x to y:

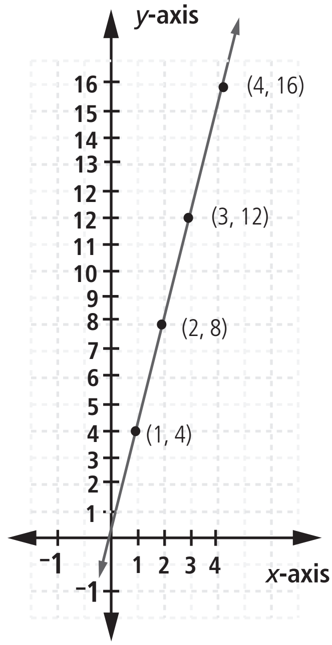
x	y
1	3
2	6
3	9
4	12
5	15

   a. Write an equation that relates x and y. Explain your answer.

   b. Graph the equation on a coordinate system.

   c. How are any two points (x, y) on the graph related? Explain your answer.

8. Use the graph below that relates two quantities ***x*** and ***y*** to write an equation that relates to the ratio of ***y*** to ***x***.



9. Fineas is riding his bike at 5 miles per hour.

   a. How many yards will he travel in one hour? (Hint: 1 mi. = 1760 yds)

   b. How many feet will he travel in one hour? (Hint 1 mi. = 5280 ft)

10. Luis eats 45 hot dogs in 5 hours. How many hot dogs will he eat in 3 hours?

11. Robert climbs 120 feet in one day. Assuming he climbs at a constant rate with no rest, how many feet does he climb in one hour? How many feet in 12 hours? (Hint: 1 day = 24 hrs)

12. Stephen swims 7 laps every 15 minutes. Complete the table to find the total number of laps he can swim in one hour. Represent this data on a graph.

Laps	7	14		
Time in minutes	15	30		

13. A store sells organic sugar at the rate of $2.50 per pound.

   a. How much does 2 pounds cost? 3 pounds? 6 pounds?

b. Make a table with input as pounds of sugar and outputs as cost in dollars.

c. Draw a graph using at least 5 points.

d. Write an equation for cost as a function of amount of sugar.

e. What is the cost of 24 pounds of sugar?

f. How much sugar can you buy for $100?

14. A floral shop uses 18 inches of ribbon to make each flower arrangement.

a. How much ribbon do you need to make 150 arrangements?

b. How many arrangements can you make with 4680 inches of ribbon?

Spiral Review:

15. Simplify the expression $5 + 3\,(11 - 2) \div 4^2$

16. The three baby robins in the nest outside Greg's window weigh 32.2 grams, 35.7 grams, and 29.2 grams. What is the combined weight of the three baby birds? What is the difference in weight between the largest and the smallest baby bird?

17. **Ingenuity:**

Sam, a snail, makes one lap around a circular track every five hours. Another snail, Sally, makes one lap around the same track every four hours. If Sam and Sally start at the same point on the track, how long does it take before Sally has completed one more lap than Sam?

18. **Investigation:**

A car travels 65 miles per hour on a highway. Based on this information answer the following questions:

a. How many meters does the car travel per hour? (Hint: 1 mi = about 1600 m.)

b. How many meters does the car travel per minute?

c. What is the car's speed in meters per second?

d. The speed of light is approximately 300 million meters per second. How many miles per hour is this?

SECTION 7.4
PROPORTIONS

When you look at a map of Texas, you know that the actual state is much larger than the map. For example, 1 inch can represent 50 miles, according to a scale designation on the map legend. That means that the ratio of the map distance to the actual distance is 1 inch to 50 miles. This ratio is written 1 inch : 50 miles or $\frac{1 \text{ in}}{50 \text{ mi}}$, as in Section 7.3.

Using this information, what actual distance does 2 inches on this map represent? This time, writing the information as a ratio of actual distance to the map distance, the fraction is $\frac{x \text{ mi}}{2 \text{ in}}$, where x is the actual distance represented by 2 inches on the map. Using the scale of 50 miles to 1 inch from the map, combine the two ratios in the equation $\frac{50 \text{ mi}}{1 \text{ in}} = \frac{x \text{ mi}}{2 \text{ in}}$. We will explore a way to solve equations like this for the unknown x later in this section.

DEFINITION 7.3: PROPORTION
A **proportion** is an equation of ratios in the form $\frac{a}{b} = \frac{c}{d}$, where b and d are not equal to zero.

In a proportion, each side of the equation is a ratio. Sometimes, a proportion can compare two different types of the same units, like inches to inches and miles to miles, as long as both ratios are equivalent as fractions: $\frac{x \text{ mi}}{2 \text{ in}} = \frac{50 \text{ mi}}{1 \text{ in}}$ or $\frac{2 \text{ in}}{1 \text{ in}} = \frac{x \text{ mi}}{50 \text{ mi}}$.

EXAMPLE 1

A colony of leaf cutter ants cuts up 4 leaves in 7 minutes. Now, write a ratio that corresponds to this relationship. Write a proportion that corresponds to the following relationship: How many leaves does the colony cut in 35 minutes?

SOLUTION

We first use a ratio of leaves to time, $\frac{4 \text{ leaves}}{7 \text{ min}}$. Then use the variable L to represent the number of leaves cut by the leaf cutter ant in 35 minutes. The second ratio looks like $\frac{L \text{ leaves}}{35 \text{ min}}$. The proportion that we obtain when the two ratios are set equal to each other looks like this: $\frac{4 \text{ leaves}}{7 \text{ min}} = \frac{L \text{ leaves}}{35 \text{ min}}$

We look at several ways that you can solve this problem.

Tabular Method:

Construct a table to record the time and the number of leaves cut.

Time in minutes	Number of leaves cut
0	0
7	4
14	8
21	12
28	16
35	20
42	24
49	28
56	32
63	36

From the table you can see that 35 minutes corresponds to 20 leaves cut by the leaf cutter ants.

Unit Rate Method:

Set up a proportion that compares the ratio of leaves to minutes. Because the ants cut 4 leaves in 7 minutes, using division, the ants must cut $\frac{4}{7}$ of a leaf in 1 minute. This is the unit rate or the number of leaves cut per minute. If the ants keep cutting at this rate, they will cut 35 times this number of leaves in 35 minutes. Call the number of leaves cut in 35 minutes x. Then:

$$x = \frac{4 \text{ leaves}}{7 \text{ minutes}} \cdot 35 \text{ min} = \frac{140}{7} \text{ leaves} = 20 \text{ leaves}$$

Proportion Method:

Set up a proportion by comparing amounts for the two different times. The ants cut

4 leaves in 7 minutes. How many leaves ***L*** will the ants cut in 35 minutes?

$$\frac{\boldsymbol{L}\text{ leaves}}{35\text{ min}} = \frac{4\text{ leaves}}{7\text{ minutes}}$$

To solve, multiply both sides of the equation by the denominator 35.

$$35\text{ min} \cdot \frac{\boldsymbol{L}\text{ leaves}}{35\text{ min}} = \frac{4\text{ leaves}}{7\text{ minutes}} \cdot 35\text{ min}$$

$$\boldsymbol{L} = \frac{4}{7} \cdot 35 = 20\text{ leaves.}$$

This proportion method involves the rate of change in the form of speed, the rate of leaves cut per unit time or minute. This is a rate of change like miles per hour or mph.

PROBLEM 1

Alberta likes to knit socks for her grandson's collection of toy aliens, but she has forgotten how many legs each alien has. She remembers knitting 24 socks for 3 aliens. Assuming that the aliens all have the same number of legs, how many socks should she knit for 5 aliens?

PROBLEM 2

In an old map, the map scale has become unreadable. We know that two locations are 120 miles apart and we use a ruler to determine that they are 3 inches apart on the map.

a. City A is 7 inches from mountain B on the map. Use proportion to calculate the distance between them.

b. What is the unit rate in miles per inch?

c. Use the unit rate of miles per inch from part b to calculate the distance in miles from city A to mountain B.

Here is another example that illustrates different approaches to solve proportional reasoning problems.

EXPLORATION 1

Materials: You will need a map of any region that contains a legend with the distance scale and a ruler or tape measure in the same unit system as the map.

Step 1: Find the legend in the map and write a ratio that relates the map measure to the actual measure.

Step 2: Use a measuring instrument to measure the straight-line distance between two major cities on the map.

Step 3: Determine the actual straight-line distance between the cities using proportions.

Step 4: Repeat Steps 2 and 3 with two other cities.

What are the actual straight-line distances between the cities that you chose?

EXAMPLE 2

Set up the following problem using a proportion.
3 bags of chips cost $2.79. How much do 7 bags of chips cost?

SOLUTION

In the chips problem, any one of these proportions will correctly determine x, the cost of 7 bags of chips. The possible proportions are indicated:

a. $\frac{\text{bags of chips}}{\text{bags of chips}} = \frac{\text{cost}}{\text{cost}}$ or $\frac{3}{7} = \frac{2.79}{x}$

b. $\frac{\text{bags of chips}}{\text{cost}} = \frac{\text{bags of chips}}{\text{cost}}$ or $\frac{3}{2.79} = \frac{7}{x}$

c. $\frac{\text{cost}}{\text{bags of chips}} = \frac{\text{cost}}{\text{bags of chips}}$ or $\frac{x}{7} = \frac{2.79}{3}$

d. $\frac{\text{cost}}{\text{cost}} = \frac{\text{bags of chips}}{\text{bags of chips}}$ or $\frac{x}{2.79} = \frac{7}{3}$.

Use one of the proportions to solve for x. Verify that the cost for 7 bags of chips is the same when each proportion is solved.

Setting up the correct proportion is often the hardest part of solving a proportion problem. Some proportions are easier to solve than others because of the way they are set up. Which of the proportions above was easiest to solve and why?

There are several ways in which you can solve the problem. If we use part c then $\frac{x}{7} = \frac{2.79}{3}$ and $x = \frac{2.79}{3} \cdot 7 = 6.51$. From our work, we see that 7 bags of chips cost $6.51.

EXAMPLE 3

John lives 3 miles from school, and takes his bicycle each day.

a. It takes him 40 minutes to get to school. What is his average rate in miles per hour?

b. If John biked twice as fast, how long would it take him to make the trip?

SOLUTION

a. John's average rate is 3 miles in 40 minutes. To convert miles per minute to miles per hour, use the fact that 60 minutes = 1 hour. Multiply by the factor $\frac{60 \text{ minutes}}{1 \text{ hour}}$ to obtain:

$$\frac{3 \text{ miles}}{40 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hour}} = \frac{\frac{30 \cdot 60}{40} \text{ miles}}{\text{hour}} = \frac{\frac{9}{2} \text{ miles}}{\text{hour}} = \frac{9}{2} \text{mph}$$

b. If John doubled his rate, he would go $2 \cdot \frac{9}{2} = 9$ mph. Remember that distance = rate · time. Let $\boldsymbol{t}$ = time in hours that it takes for John to make one trip. To find the time, set up 3 miles = (9 $\frac{\text{miles}}{\text{hour}}$) · $\boldsymbol{t}$ hours. Simplifying, $3 = 9\boldsymbol{t}$ and $\boldsymbol{t} = \frac{3}{9} = \frac{1}{3}$ hour.

Alternatively, convert the time to minutes:

$$\left(\frac{1}{3}\right)\text{hour} \cdot \left(\frac{60 \text{ minutes}}{1 \text{ hour}}\right) = \left(\frac{60}{3}\right) \text{ minutes} = 20 \text{ minutes.}$$

EXAMPLE 4

Sally walked up a hill at a rate of 2 miles per hour, then walked back down at a rate of 4 miles per hour.

a. Was her average rate of walking for the trip less than 3 miles per hour, equal to 3 miles per hour, or greater than 3 miles per hour?

b. What was her average rate of speed for the entire hike?

SOLUTION

a. Since Sally walked the same distance up as down, she spent twice as long walking up the hill as walking down the hill. Because Sally walked more time at the slower speed, her average rate for the entire walk should be closer to 2 miles per hour than to 4 miles per hour. Thus, her average rate is less than 3 miles per hour.

b. How can we calculate her average rate? Let D = distance up the hill. Then D is also the distance down the hill. If Sally spends t hours going up the hill, then she will spend $\frac{t}{2}$ hours doing down the hill.

Use the fact that distance = rate · time. Going up the hill, $D = 2t$. Going down the hill, $D = 4\left(\frac{t}{2}\right)$. For the entire trip, let r = average rate. Then $2D = r\left(t + \frac{t}{2}\right)$:

$$2D = r\left(\frac{3}{2}t\right).$$

We have $D = 2t$ so $2D = 4t$. We also have $2D = r\frac{3}{2}t$. We can eliminate t from both sides of the equation and we have:

$$4t = r\left(\frac{3}{2}t\right)$$
$$4 = r\left(\frac{3}{2}\right)$$
$$8 = 3r$$
$$r = \frac{8}{3} = 2\frac{2}{3} \text{ mph}$$

PROBLEM 3

a. Calculate the unit rate of cost per bag of chips in Example 2 and use it to calculate the cost of 7 bags of chips.

b. Write an equation of the cost function for this problem, using the number of bags of chips as the input.

c. Use the equation to calculate the cost of 20 bags of chips.

EXPLORATION 2: CONVERTING YOUR DOLLARS ACTIVITY

The currency in the United States is in dollars and cents. Do you know the currency of our neighbors in Mexico? Canada? What is the currency in England? France? Japan? China?

1. Find the currency for six different countries. You may need to use a reference or some outside source.
2. Determine what $1 US is worth in the currency of each of the six countries. You may need to use the Internet for the most current exchange rate.

3. Determine what \$50 is worth in each of the six currencies.

4. Using the currency of one of your countries, determine what 100 of that currency is worth in dollars.

EXPLORATION 3

1. Ben and Jerry are scooping ice cream into a large bowl. They started together, but Jerry scoops faster. While Ben scoops 4 ice cream scoops, Jerry scoops 12 ice cream scoops.

   a. When Ben has scooped 20 ice cream scoops, how many ice cream scoops has Jerry scooped?

   b. Write in words a relationship between the number of ice cream scoops Jerry scoops with the number of ice cream scoops Ben scoops.

   c. Write an equation for the number of ice cream scoops, J, that Jerry scoops in terms of the number of scoops, B, that Ben scoops.

   d. Write an equation for the number of ice cream scoops, B, that Ben scoops in terms of the number of scoops, J, that Jerry scoops.

2. On another day, Ben and Jerry are again scooping ice cream into a large bowl. They scoop equally fast, but Jerry started earlier. When Ben has scooped 4 ice cream scoops, Jerry has scooped 12 ice cream scoops.

   a. When Ben has scooped 20 ice cream scoops, how many ice cream scoops has Jerry scooped?

   b. Write in words a relationship between the number of ice cream scoops Jerry scoops with the number of ice cream scoops Ben scoops.

   c. Write an equation for the number of ice cream scoops, J, that Jerry scoops in terms of the number of scoops, B, that Ben scoops.

   d. Write an equation for the number of ice cream scoops, B, that Ben scoops in terms of the number of scoops, J, that Jerry scoops.

3. Compare the relationship found in problem 1 above between Jerry and Ben's scoops with the relationship found in problem 2. Describe the difference(s) in the two relationships.

EXERCISES

Solve the following proportions using ratio tables or by setting up equations.
Simplify your answers when possible.

1. Little Bobby was collecting seashells one afternoon. For every 6 shells he found, he would throw two back. After collecting shells for 1 hour, he had a total of 48 shells in his basket. How many shells did he throw back during that hour?
2. The candy store had a sale of lollipops last week. You could buy 8 lollipops for $2. At this price, how much would you pay for 36 lollipops?
3. A florist makes balloon bouquets with red, blue, and yellow balloons. The ratio of red to yellow balloons is 3:5 and the ratio of blue to yellow is 8:5. If a bouquet contains 42 red balloons:
    a. How many yellow balloons does each bouquet contain?
    b. How many blue balloons does each bouquet contain?
4. Solve the following equations.
    a. $\frac{3}{4} = \frac{x}{16}$ b. $\frac{x}{4} = \frac{36}{18}$ c. $\frac{7}{9} = \frac{42}{x}$ d. $\frac{35}{x} = \frac{105}{3}$
5. Certain types of vans hold 12 students. What is the minimum number of this type of van that is needed to take 78 students to the lake during summer camp?
6. If 6 large cakes can feed 150 guests, how many cakes will be needed to feed 100 guests?
7. Andrew makes 3 hits out of every 10 times he bats, and Sam makes 4 hits out of every 13 times he bats.
    a. Who is more likely to get a hit when at bat?
    b. If Sam bats 100 times, how many hits might he expect to get? If Andrew bats 100 times, how many hits might he expect to get?

c. If Andrew gets 84 hits this season, about how many times did he bat?

8. Marcia has a bag of purple and orange crystal beads. The ratio of purple beads to the total number of beads is 5 : 7. If she has 60 orange beads, how many total beads are in the bag?

9. Grandma Janie's recipe for sugar cookies calls for 4 cups of sugar for every 72 cookies. How much sugar is needed to make 180 cookies?

10. Josh's car gets 18 miles per gallon.

    a. Use a proportion to calculate how many gallons he needs to travel 126 miles.

    b. Write an equation for the distance traveled if he uses x gallons. Let d stand for the distance.

    c. How far can he travel if he has 6.5 gallons of gasoline?

    d. How many gallons does it take to travel 63 miles?

11. Melissa can type 144 words in 3 minutes. If she keeps the same speed, how many minutes will it take Melissa to type 240 words?

12. Diane has made 40 baskets out of 120 attempts. How many baskets can Diane's coach expect her to make if she attempts 30 baskets in a game?

13. Roxy purchased a book with 380 pages. If she reads 19 pages each day, how many days will it take her to read the entire book?

14. Emilio noticed that 12 seats out of every 75 had designs. If the entire arena has 900 seats, how many seats have designs?

15. Evan buys 6 movie tickets for $42.

    a. How many movie tickets can Evan buy with $133?

    b. How much does each ticket cost?

    c. Write an equation for the price P of x tickets.

    d. How many tickets can he buy with $100?

16. Porky the dachshund walks at a rate of $\frac{3}{4}$ mph. What fraction of an hour does it take Porky to walk $\frac{1}{8}$ of a mile?

17. Oliver walks at a speed of $3\frac{1}{5}$ mph. How long will it take him to walk 6 miles?

18. John can finish a job in 45 minutes. Working with Sue, they can finish the job in 30 minutes.

    a. Who works faster, John or Sue? Explain.

    b. How long would it take Sue to finish the job alone?

Spiral Review:

19. After one hour, 25% of 6th grade students at Emerson elementary had finished the STAAR test. What percent had not finished the test? Represent the answer as a decimal, fraction, and percent.

20. Brandy bought lunch for herself and for two of her friends. Brandy's lunch cost $6.53 and her friends' lunches cost $5.75 and $4.26. If tax was already included in the cost, what is the change Brandy received if she paid with a $20.00 bill?

21. **Ingenuity:**

    Mitchell is driving from Austin to San Antonio, a total distance of 80 miles. He leaves Austin at 11:53 a.m. and reaches San Antonio at 1:05 p.m. If Mitchell drives the same speed for the entire trip, at what time does he pass through San Marcos, which is 30 miles from Austin?

22. **Investigation:**

    In a standard deck of playing cards, there are 52 cards. These cards are divided equally into four suits: spades, hearts, diamonds, and clubs.

    a. How many of the cards in the deck are spades?

    b. Suppose we pull 20 cards from the deck without looking at them. About how many of these cards would you expect to be hearts? Explain.

    c. Suppose we pull a card from the deck without looking first, then look at the card and record the result. Suppose we repeat this process 280 times. About how many times should we expect to draw clubs? Explain your answer.

SECTION 7.R
CHAPTER REVIEW

1. Compute the following products. Simplify if needed.

   a. $\frac{1}{3} \cdot 33$ c. $\frac{2}{3} \cdot \frac{1}{4}$ e. $\frac{8}{12} \cdot \frac{3}{4}$

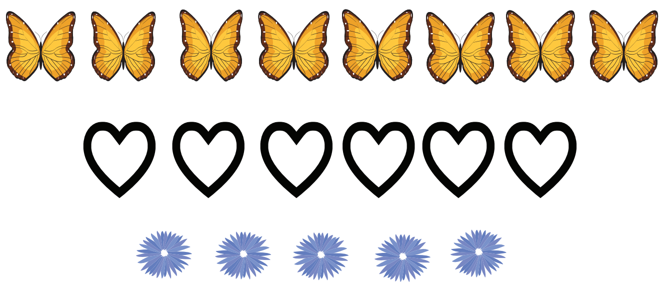
   b $\frac{5}{6} \cdot \frac{6}{8}$ d. $\frac{3}{5} \cdot 20$ f. $15 \cdot \frac{3}{8}$

2. Compute the following quotients.

   a. $4 \div \frac{1}{4}$ c. $\frac{1}{2} \div \frac{1}{4}$ e. $\frac{3}{4} \div \frac{1}{8}$

   b. $5 \div \frac{1}{4}$ d. $\frac{1}{4} \div \frac{1}{4}$ f. $3 \div \frac{1}{2}$

Use the picture to answer the following ratio problems. Write the ratios as fractions. Simplify if needed.



3. What is the ratio of flowers to hearts?

4. What is the ratio of butterflies to hearts?

5. What is the ratio of hearts and flowers to all?

6. What is the ratio of all to butterflies?

7. Solve the following proportions for x. (Hint: Use what you know about simplifying fractions to help you.)

   a. $\frac{2}{3} = \frac{x}{21}$ c. $\frac{6}{16} = \frac{9}{x}$ e. $\frac{x}{10} = \frac{20}{25}$

   b. $\frac{5}{x} = \frac{20}{48}$ d. $\frac{15}{27} = \frac{5}{x}$ f. $\frac{6}{x} = \frac{4}{14}$

Set up unit rates for the given data. Solve.

8. Twelve tables will seat 168 people. What is the number of people per table?

9. Six buses hold 330 students. What is the number of students per bus?

Solve the following word problems using what you know about ratios and proportions.

10. If there are 32 students and three-eights of them wear glasses, how many students wear glasses?

11. A survey shows that $\frac{2}{5}$ of girls like purple as their favorite color. $\frac{1}{3}$ of the girls who like purple prefer the lighter shades of purple. What fraction of girls like lighter shades of purple?

12. Mrs. Fields boxed a dozen cookies. The box contained 5 chocolate chip cookies, 3 peanut butter cookies, 2 oatmeal cookies, and 2 sugar cookies.

    a. What ratio of the cookies are peanut butter cookies?

    b. What ratio of the cookies are NOT oatmeal or sugar?

13. Sergio is in a hot dog eating contest. He ate 2 hot dogs in 3 minutes. If he continues eating at this rate, how many minutes will it take him to eat 8 hot dogs?

14. Lorianne checked the number of texts she sent in one week and found it to be 595. If she sent the same number of texts each day of the week, how many texts did she send each day?

15. Jerry's car averages 14 miles per gallon when he drives around the city. If he has 7 gallons of gas in his car, how many miles will he be able to travel around the city?

16. At the animal shelter, 6 workers equally care for 180 animals. How many animals can be cared for by 4 workers?

17. April found a box of red and blue marbles. The ratio of blue marbles to all the marbles is 4:7. If the box has 48 blue marbles, how many total marbles are in the box?

18. The latest book by my favorite author has 240 pages. If I read 15 pages each day, how many days will it take me to read the entire book?

19. Nicole scoops 9 cups of potting soil from a big bag using a $\frac{3}{4}$-cup scooper. How many times did she have to scoop to remove all 9 cups of soil from the bag?

MEASUREMENT

8

SECTION 8.1
LENGTH

What do you notice is common to the following questions?

- How far is your home from school?
- How tall are you?
- What is the record high temperature in San Marcos for today?
- How much water do you consume in one day?
- How much did you weigh when you were born?
- How much time until lunch?

You probably observed that they all involve measurements of some type.

Whether you are measuring distance, height, temperature, capacity, weight, or time there are several important concepts to keep in mind. First, you must determine what is being measured. For example, if you want to know how tall you were when you were born, you would recognize that you are referring to measuring a length and not weight, capacity, or time. Second, you must determine what unit or possibly units are appropriate for measuring the height of a baby. And finally, you must get the actual numerical value for the measurement in the unit chosen.

Throughout this chapter, we will look at two systems of measurement: customary units and metric units. Customary units are what we in the U.S. use most often though the rest of the world, with a few exceptions, uses metric units.

EXPLORATION 1

What are the customary units you use for lengths and distances? How do you measure length? Use an appropriate measuring device such as a ruler, meter stick, measuring tape, or other available devices to measure five different objects using five different units. Explain why you chose the particular unit.

Inches may be a good choice for the height of a newborn baby, but not as good a choice for a running race distance. Here are some useful conversions in customary units:

LENGTH

Customary

1 mile	=	1760 yards
1 mile	=	5280 feet
1 yard	=	3 feet
1 foot	=	12 inches

If a person runs 8 miles, what is this distance in yards? One way to approach this problem is to use the concept of proportions from the last chapter. Let x represent this distance in yards. One form of this proportion could look like this:

$$\frac{x}{8 \text{ miles}} = \frac{1760 \text{ yds}}{1 \text{ mile}}$$

Solving for x, we then have, $x = 8 \cdot \frac{1760}{1} = 8 \cdot 1760 = 14{,}080$ yards. Another way to think of this problem is to use the idea of unit rates and notice that there are1760 yards per mile or 1760 $\frac{\text{yds}}{\text{mile}}$. If there are 1760 yards in each mile then in 8 miles there must be $8 \cdot 1760 = 14{,}080$ yards. We see there are two ways to approach this problem of converting 8 miles to yards.

We introduce another powerful process for converting units called **dimensional analysis.** This process is based on equivalent forms of measurement. Recall that a fraction of the form $\frac{n}{n} = 1$, with n a non-zero number. You would easily recognize that $\frac{15}{15} = 1$. Now consider $\frac{1 \text{ foot}}{12 \text{ inches}}$. Because 1 foot = 12 inches, a non-zero quantity in one unit divided by an equal non-zero quantity in another unit must equal 1. What is important here is that the units must be indicated every time. Remember, we are not claiming that in general, $\frac{1}{12} = 1$. Units are important!

Create other 1's using the equivalent unit form from the chart above.

Some typical equivalents with the corresponding conversion rates equal to 1 are:

3 feet = 1 yard which means $\frac{3 \text{ ft}}{1 \text{ yd}} = 1 = \frac{1 \text{ yd}}{3 \text{ ft}}$ and

60 minutes = 1 hour means $\frac{60 \text{ min}}{1 \text{ hr}} = 1 = \frac{1 \text{ hr}}{60 \text{ min}}$

Rewriting a measurement from one unit to another is much easier using dimensional analysis.

EXAMPLE 1

Convert 31,680 inches to feet.

SOLUTION

Inches and feet are related by the equivalent forms 1 foot = 12 inches. Use this equivalence to write the number 1 in two ways, $\frac{1\text{ foot}}{12\text{ inches}}$ and $\frac{12\text{ inches}}{1\text{ foot}}$. The problem is to convert 31,680 inches to feet. Use the $\frac{1\text{ foot}}{12\text{ inches}}$ form of 1, where the numerator contains the feet unit to which you want to convert. Set up the problem: $31{,}680\text{ inches}\cdot\frac{1\text{ foot}}{12\text{ inches}}$.

Multiplying the numerical part of the fractions gives us $\frac{31680}{12} = 2640$.

What unit is this? Notice that we have $\text{inches}\cdot\frac{\text{feet}}{\text{inches}}$. Treating these expressions as we would fractions, $\frac{\text{inches}\cdot\text{feet}}{\text{inches}}$ simplifies to feet. The unit for 2640 is feet. Therefore, 31,680 inches = 2640 feet.

PROBLEM 1

Convert the following measurements. Make sure to set up equivalent units correctly.

a. 24 inches = _____ ft

b. 12 ft = _____ inches

c. 3 mi = _____ ft

d. 136 inches = _____ yards

e. 7040 yd = _____ mi

f. 18 yd = _____ ft

PROBLEM 2

Convert 2640 feet to miles.

Another system for measuring length is the metric system. This system is a base 10 or decimal system. Your knowledge of decimals will be very useful in the metric system. The base unit of length in the metric system is the meter. Prefixes are used with the base to create larger and smaller units.

Prefixes commonly used in order from larger units to smaller units are:

Prefix of Metric units

Kilo	Hecto	Deka	(base)	Deci	Centi	Milli
k	h	da		d	c	m

A given unit is 10 times larger than the one to its right. In the metric system, the base unit for length is given by **meter**. The table of units from larger units to smaller looks like the following:

Metric Units for Length

Kilometer	Hectometer	Dekameter	Meter	Decimeter	Centimeter	Millimeter
km	hm	dem	m	dm	cm	mm

The chart above also includes the abbreviation used for the units.

For example, 10 millimeters = 1 centimeter, 10 centimeters = 1 decimeter, 10 decimeters = 1 meter, and so on.

Here are some ways to connect metric units of measure to common references.

- A meter is a little longer than a yard.
- A decimeter is about the width of a hand.
- A centimeter is about the width across your fingernail.
- A millimeter is about the thickness of your fingernail.

You can think of a kilometer as about the length of 10 football fields. Use 1 km = 1000 m to convert 17 km to meters. By dimensional analysis, use $\frac{1000 \text{ m}}{1 \text{ km}} = 1$ and multiply this by 17 km.

$$17 \text{ km} \cdot \frac{1000 \text{ m}}{1 \text{ km}} = 17 \cdot 1000 \text{ m} = 17000 \text{ m}.$$

How does this differ from the problem to convert 17 m to kilometers?

Notice how this is related to the place value chart and converting from km to m.

Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths

The metric system is based on our decimal system. Can you see that just as our number system gets larger by a factor of 10 as our units increase, the metric system unit size increases by a factor of 10 as we move to the left on the prefixes?

Some useful metric conversions include:

LENGTH

Metric

1 kilometer	=	1000 meters
1 meter	=	100 centimeters
1 centimeter	=	10 millimeters

PROBLEM 3

Henry measured the length of his patio and found it to be 1200 cm long. He went to purchase an outdoor rug to cover the length of the patio. He noticed it was sold in meters. How long is his patio in meters?

PROBLEM 4

Convert the following metric measurements to the specified units. Make sure you set up equivalent units correctly:

a. 3250 cm = ______ m

b. 4000 m = ______cm

c. 140,000 mm = ______ km

d. 32 mm = ______ m

e. 5 km = ______cm

EXERCISES

1. Convert 12 meters to the indicated units using each of the following methods: i) proportions, ii) unit rate, and iii) dimensional analysis:

   a. decimeters

   b. centimeters

   c. millimeters

   d. kilometers

2. Convert 12 feet to the indicated units using each of the following methods: i) proportions, ii) unit rate, and iii) dimensional analysis:

   a. inches

   b. yards

   c. miles

3. Coach Rodriguez is 5 feet 11 inches tall. How tall is he in inches?

4. Kassandra walked 1 mile on Monday, 1 mile on Tuesday, and 2 miles on Wednesday. How many yards did she walk?

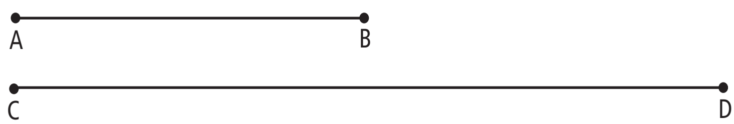
5. Luis lives 2538 meters away from Bob. What is this distance in kilometers?

6. The distance from one town to another is 8 kilometers. How many centimeters is this distance?

7. The length of a table is 138 centimeters. The length of a tablecloth is 1 meter. Will the tablecloth be long enough to cover the entire top of the table? Explain your reasoning.

8. Emily ran a total of 18,480 feet in one week. How many total miles is this?

9. Measure the lengths of the two line segments $\overline{AB}$ and $\overline{CD}$ using inches and then centimeters.



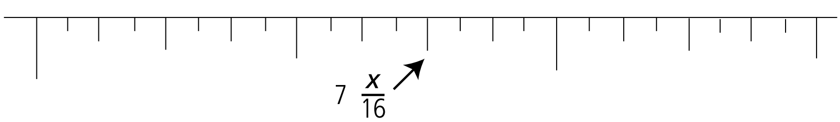
   a. Determine the ratio of $\overline{AB}$ to $\overline{CD}$ using customary units.

   b. Determine the ratio of $\overline{AB}$ to $\overline{CD}$ using metric units. Compare the two ratios. What do you notice? Explain any difference or similarities in the two ratios.

10. Convert 3400 mm to kilometers.

11. **Ingenuity:**

    Shown below is a small part of the edge of a ruler. If the line indicated below is the $7\frac{x}{16}$ inch mark on the ruler, what are the possible values of x?



12. **Investigation:**

    Look up the following units of length and put them in order from longest to shortest. Explain where each unit of length is (or was) used in real life.

    angstrom, furlong, league, light-year, micron, nautical mile, parsec

SECTION 8.2
CAPACITY AND VOLUME

The measure of the space in a container is called the **volume** of the container. We will study volumes of familiar shapes such as cubes more carefully in Chapter 9. You will see that cubic units are used to measure volume. In this section, we will explore the capacity of a container. **Capacity** refers to how much liquid a container can hold. Our examples are often in liquid form because it will easily conform to any shaped container. The customary units for capacity in decreasing order are gallons, quarts, pints, cups, and fluid ounces.

EXPLORATION 1

Examine the household items that you brought from home to determine which of the following customary units would be most useful for measuring their capacity:

Gallons, quarts, pints, cups, and fluid ounces.

Explain why you chose those units.

Record your prediction for the capacities of each item that you brought using two different units. Use a measuring cup and water to record and confirm your predictions.

We summarize the relationships among the customary units as follows:

CAPACITY

Customary

1 gallon	=	4 quarts (qt.)
1 quart	=	2 pints (pt.)
1 pint	=	2 cups
1 cup	=	8 fluid ounces (fl. oz.)

EXAMPLE 1

Isabele is making soup that requires 3 quarts of water. She only has a 1-cup measuring cup. How many cups of water will Isabele need in order to make this soup?

SOLUTION

Recall that quarts can be converted to pints using $\frac{2 \text{ pints}}{1 \text{ quart}}$. Pints can then be converted to cups using $\frac{2 \text{ cups}}{1 \text{ pint}}$. Using dimensional analysis, you can set up the problem as follows:

$$3 \text{ qt} \cdot \frac{2 \text{ pt}}{1 \text{ qt}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = (3 \cdot 2 \cdot 2) \text{ cups} = 12 \text{ cups}.$$

PROBLEM 1

Convert 100 fluid ounces to quarts.

EXAMPLE 2

Determine what fractional part

a. 1 quart is of 1 gallon

b. 1 pint is of 1 gallon

SOLUTION

a. 1 gallon = 4 quarts. 1 quart then must equal $\frac{1}{4}$ of a gallon.

b. 2 pints = 1 quart. Therefore, 1 pint = $\frac{1}{2}$ quart. 4 quarts = 1 gallon. Therefore, 1 quart = $\frac{1}{4}$ gallon. 1 pint = $\frac{1}{2}$ of $\frac{1}{4}$ = $\frac{1}{8}$ gallon. We can use dimensional analysis to see this:

$$1 \text{ pt} \cdot \frac{1 \text{ qt}}{2 \text{ pt}} \cdot \frac{1 \text{ gal}}{4 \text{ qt}} = \frac{1}{8} \text{ gal.}$$

The metric system uses **liters** as a base for capacity. Just as with length measures, the prefixes will create larger or smaller units based on liters.

Write three other units that are larger than liters and three other units that are smaller. The units with liter (L) as base:

Metric Units for Capacity

Kiloliter	Hectoliter	Dekaliter	Liter	Deciliter	Centiliter	Milliliter
kL	hL	daL	(base) L	dL	d	mL

The units that are used most often are the liter and milliliter. Notice that the relationship between the two units is 1 liter = 1000 milliliters, which can also be written as, 1 milliliter = $\frac{1}{1000}$ liter.

EXAMPLE 3

A camel drinks 20 liters of water a day. How many milliliters does this equal?

SOLUTION

1 liter is equal to 1000 milliliters. Using dimensional analysis, we can set up the problem as follows: $20 \text{ liter} \cdot \frac{1000 \text{ mL}}{1 \text{ liter}}$ and calculate that 20 liters must equal 20000 milliliters.

PROBLEM 2

Convert 350 ml to liters.

PROBLEM 3

Determine what fractional part

a. 1 cup is of 1 gallon
b. 1 milliliter is of 1 liter

EXERCISES

1. Convert each of the following:

   a. 3 liters = ______________milliliters

   b. 5340 mL = ______________liters

   c. 5 cups = ______________ fluid ounces

   d. 5 pints = _______________fluid ounces

   e. 5 quarts = ______________fluid ounces

   f. 5 gallons = _____________ fluid ounces

2. Jeremy measured the amount of sports drink in the cooler and found it contained 6500 mL. If the container originally held 12 L, how much sports drink had been dispensed?

3. A container can hold 10 gallons of water. There are 41 quarts of water in another container. Will 41 quarts of water fill the 10-gallon container? Explain.

4. 16 pints of paint were left over. How many quarts of paint is this?

5. How many quarts of ice cream are in an 8-gallon container, assuming it is full?

6. Jeremy's mom bought 6 gallons of iced tea for a party. She stored all the tea in pint-sized containers. How many containers would she need?

7. Mr. Reyna bought 12 gallons of punch for the Math Camp reception. How many cups will he be able to serve?

8. After a strong storm, Lisa found 7 liters of rain water had been trapped in an empty bucket. How many milliliters of rainwater did she have?

9. Amy is using a cheesecake recipe that calls for 3 cups of sour cream. She has 3 pints of sour cream. How many cheesecakes can she make? If Amy makes as many whole cheesecakes as possible with the 3 pints of sour cream, how much, if any, sour cream is left for the stroganoff?

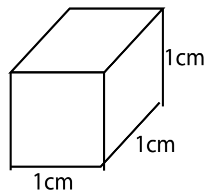
10. Nathan drinks 64 fluid ounces of water after playing badminton. Is this amount more or less than one gallon? How much more or less?

11. **Ingenuity:**

    Dee has two buckets; she knows that the buckets can hold exactly 3 gallons and 7 gallons of water, respectively. She takes the buckets to a stream and wants to measure exactly 5 gallons of water. The buckets do not have any intermediate markings that would allow Dee to know when she has filled a bucket with a certain amount of water. How could Dee put exactly 5 gallons of water in the 7-gallon bucket?

12. **Investigation:**

    One milliliter is equal to one cubic centimeter. That is, if we made a cube with dimensions 1 cm × 1 cm × 1 cm, as shown below, and filled it with water, it would take 1 mL of water to fill the cube.



a. Suppose we made a rectangular box with dimensions 1 dm × 1 cm × 1 cm. How many milliliters of water would it take to fill this box?

b. Suppose we made a rectangular box with dimensions 1 dm × 1 dm × 1 cm. How many milliliters of water would it take to fill this box?

c. Suppose we made a cube with dimensions 1 dm × 1 dm × 1 dm. How many milliliters of water would it take to fill this cube? How many liters is this?

8.3

SECTION 8.3
WEIGHT AND MASS

How heavy is an object? This question refers to the **weight** of an object. We often refer to the **mass** of an object in terms of weight but we should note that there is a scientific distinction. Mass is a measure of how much matter is in an object.

The customary units for weight in decreasing order are **tons**, **pounds**, and **ounces**.

EXPLORATION1

Examine the items that you brought from home and determine the mass or weight of each item. Feel how heavy each item is. Find two items in the classroom and make an educated guess about their weights.

Here are some useful relationships between customary units:

MASS AND WEIGHT

Customary

1 ton (T)	=	2000 pounds (lbs.)
1 pound (aviordupois)	=	16 ounces (oz.)

EXAMPLE 1

A female African elephant weighs approximately 7900 pounds. Convert the weight to tons using two different methods, for example the dimensional analysis method, unit rate method, or the proportions method.

SOLUTION

We use the equivalence of 1 ton = 2000 lbs and write 1 in the form of the fraction $\frac{1\text{ ton}}{2000\text{ lbs.}}$. The problem can then be solved using 7900 lbs $\cdot \frac{1\text{ ton}}{2000\text{ lbs.}} = \frac{7900}{2000}$ tons. Doing the division $\frac{79}{20} = 3.95$ tons or a little less than 4 tons.

The above uses the dimensional analysis method. In the unit rate method we see that 2000 pounds is equivalent to 1 ton or $\frac{1\text{ ton}}{2000\text{ lbs.}} = 0.0005 \frac{\text{ton}}{\text{pound}}$. Then in 7900 pounds, there must be $7900 \cdot 0.0005 = 3.95$ tons.

Finally, the proportion method introduces the variable x to represent the unknown number of tons 7900 pounds represents. The proportion can be written as

$$\frac{x}{7900 \text{ pounds}} = \frac{1 \text{ ton}}{2000 \text{ pounds}}$$

Solving for x, we have $x = 7900 \cdot \frac{1}{2000} = \frac{7900}{2000} = 3.95$ tons.

PROBLEM 1

Convert the weight of 7900 pounds to ounces.

The metric system uses **grams** as its base for mass measurements, often referred to as weight. The abbreviation for grams is simply the letter g. The prefixes will give us larger and smaller units. What units would be larger weight units? What units would be the smaller weight units?

In the table below are the metric units for weight with gram as base:

Metric unit for Weight

Kilogram	Hectogram	Dekagram	Gram	Decigram	Centigram	Milligram
kg	hg	Dg	base g	dg	cg	mg

Do you know what weighs 1 gram (g)? A cubic centimeter of water weighs 1 gram. A paper clip is approximately 1 gram. In other words, a gram weighs very little. 450 grams is approximately equal to one pound. 1 kilogram is approximately equal to 2.2 pounds.

EXAMPLE 2

A female Asian elephant weighs approximately 3000 kilograms. Convert the weight to grams.

SOLUTION

We use the equivalence 1 kilogram = 1000 grams and write 1 in the form of the fraction $\frac{1000 \text{ g}}{1 \text{ kg}}$, a unit rate, because we are converting to grams. The problem can be solved using $3000 \text{ kg} \cdot \frac{1000 \text{ g}}{1 \text{ kg}} = 3000 \cdot 1000 \text{ g} = 3{,}000{,}000 \text{ g}$.

PROBLEM 2

Convert 300 milligrams to grams.

EXERCISES

1. Give an example of an object that can be measured in each of the following units:

   a. grams
   b. liters
   c. millimeters
   d. kiloliters

2. Ms. Voigt's baby weighed 7 pounds and 12 ounces when the baby was born. What was the baby's weight in ounces?

3. A bag of potatoes weighs 8 kilograms. How many grams of potatoes would half the bag weigh?

4. Isaac is baking a cake that requires 550 grams of sugar. How many milligrams of sugar does the cake require?

5. Jacob bought a watermelon that weighed 9 pounds. How many ounces does this equal?

6. Stephen was carrying his violin case onto his flight to Oregon. The flight attendant said the case weighed 7 kilograms. How many grams does Stephen's violin case weigh?

7. Aaron buys 20 ounces of sliced baked ham from the deli counter. How many pounds of ham did Aaron buy?

8. Mr. Reyes can lift 72 kilograms of weight. Mr. Saenz can lift 68,790 grams of weight. Who can lift more weight, Mr. Saenz or Mr. Reyes? Explain your answer.

9. David can lift 200 pounds of weight.

   a. How many tons can he lift?
   b. How many ounces can he lift?

10. A young elephant weighs 3 tons. A car weighs 8000 pounds. Which weighs more: the car or the elephant? Explain your reasoning.

11. **Ingenuity:**

Roy has nine coins, one of which is counterfeit and slightly lighter than the others. Roy doesn't remember which coin is the counterfeit coin, but he has a balance scale that he can use to test the coins. Roy can put some of his coins on the scale, and it will tell him whether one side is lighter than the other. Is it possible for Roy to figure out which coin is the counterfeit coin by using his balance scale three or fewer times?

12. **Investigation:**

Recall that mass and weight are two different concepts. The mass of an object is the same no matter where the object is in the universe. The weight of an object, however, depends on how strong gravity is at the object's location. The following chart shows the weight of a one-kilogram mass on each of the eight planets in our solar system.

Planet	Weight of 1 kg
Mercury	0.83 lb.
Venus	2.00 lb.
Earth	2.20 lb.
Mars	0.83 lb.
Jupiter	5.21 lb.
Saturn	2.02 lb.
Uranus	1.96 lb.
Neptune	2.48 lb.

a. On which planets does one kilogram weigh the most? On which planets does one kilogram weigh the least? Why do you think this is?

b. How much does a 5-kilogram mass weigh on Earth? How much would it weigh if it were transported to Jupiter?

c. A man on Earth weighs 150 pounds. How much would he weigh if he took a trip to Mars?

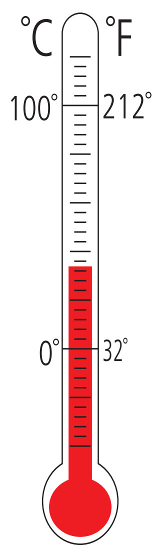
SECTION 8.4
TIME AND TEMPERATURE

Temperature

Does it make sense to say the temperature on a hot day in Texas is 32 degrees? Does 100 degrees make sense? Actually, the answer could be yes or no. How can that be? As you have seen in the previous sections, whenever you measure, you must be careful to always include the units along with the numerical value. For example, a pencil of length 12 inches is very different from a pencil of length 12 feet, so saying a pencil is of length 12 is not enough.

The two common units of temperature measure are the Fahrenheit and Celsius units. Fahrenheit is associated with the customary system while Celsius is associated with the metric system.

You did an investigation in Section 1.1 regarding the thermometer as part of a number line. Let us recall some important aspects of the two units. The freezing point of water is 32° Fahrenheit and 0° Celsius. The boiling point of water is 212° Fahrenheit and 100° Celsius.



Now, 32° in Fahrenheit is not a reasonable temperature on a hot day in Texas. Look on the thermometer above. Does 32° Celsius make sense for a hot day? 32° Celsius is approximately equal to 90° Fahrenheit, which is a reasonable temperature in a Texas summer.

What is a reasonable springtime temperature in Texas? What about a cold winter day in Chicago? What is the average body temperature of a healthy person? Give your temperature readings in degrees Fahrenheit and in degrees Celsius.

Time

What units of time are most familiar to you? What units of time would be appropriate for each of the following instances?

a. What was the runner's time in a 100 meter run?

b. How long is your summer vacation?

c. How much time until dinner?

d. How much time will pass before you turn 21 years old?

We do not specify a customary or metric system of time measurement. However, some countries use a 24-hour reading of time while other countries use a 12-hour reading and use a.m. and p.m. to distinguish the morning time from the afternoon time. Generally, a.m. time goes from 12:00 midnight until 12:00 noon and p.m. goes from 12:00 noon until 12:00 midnight. In this book, we will use the 12-hour clock.

Familiar time equivalences include:

Time

1 year	=	365 days
1 year	=	12 months
1 year	=	52 weeks
1 week	=	7 days
1 day	=	24 hours
1 hour	=	60 minutes
1 minute	=	60 seconds

PROBLEM 1

What fraction is:

a. One day of a week?

b. One day of a year?

c. One hour of a day?

d. One second of an hour?

EXAMPLE 1

12 hours and 20 minutes is equal to how many minutes?

SOLUTION

12 hours $\cdot \frac{60\ \boldsymbol{min}}{1\ \text{hour}}$ = 12 · 60 minutes = 720 minutes. So,12 hours = 720 minutes and 12 hours and 20 minutes = 720 minutes + 20 minutes = 740 minutes.

EXAMPLE 2

100 minutes is equal to how many hours?

SOLUTION

100 minutes $\cdot \frac{1\ \text{hour}}{60\ \boldsymbol{min}} = \frac{100}{60}$ hours = $\frac{5}{3}$ hours = $1\frac{2}{3}$ hours.

PROBLEM 2

a. 20 minutes is equal to what fraction of an hour?

b. 15 minutes is equal to what fraction of an hour?

c. 5 minutes is equal to what fraction of an hour?

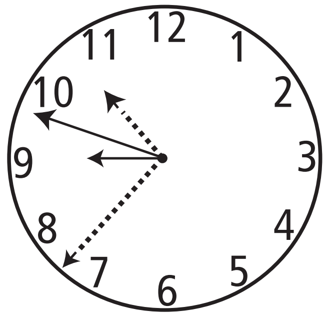
EXAMPLE 3

Ms. Lott's class begins math class at 9:47 a.m. The math classes are 50 minutes long. At what time will Ms. Lott's math class end?

SOLUTION

The time 9:47 is really 9 hours + 47 minutes. The elapsed time is 50 minutes so we add 50 minutes to this time. If we add the minutes, we have 47 + 50 = 97 minutes. 97 minutes = 60 minutes + 37 minutes = 1 hour + 37 minutes.

We can view 9 hours + (47 + 50) minutes = 9 hours + 1 hour + 37 minutes = 10 hours + 37 minutes or 10:37 a.m.



EXAMPLE 4

Mr. Mungia left McAllen, Texas, at 11:28 a.m. on a flight to Chicago, Illinois. His flight arrived in Chicago at 3:05 p.m. How long was his trip?

SOLUTION

One way to answer this question is to add on the time in easy increments. One hour into the trip, the time was 12:28 p.m. Two hours into the trip, the time was 1:28 p.m. Three hours into the trip, the time was 2:28 p.m. Adding another hour will go beyond Mr. Mungia's arrival time so we can add a half an hour or 30 minutes.

Adding 30 minutes to 2:28 p.m. will make the time 2:58 p.m. So far 3 hours and 30 minutes have elapsed. If we add the minutes from 2:58 p.m. to 3:05 p.m., you will see that 7 minutes have elapsed. Ms. Mr. Mungia's trip must have taken 3 hours and 37 minutes.

Another way to answer this question is to subtract the 11 hours and 28 minutes from noon or 12 hours. Then add 3 hours and 5 minutes to that answer. 12 hours – 11 hours and 28 minutes is the same as regrouping the 12 hours into (11 hours + 60 minutes) – (11 hours + 28 minutes) = 60 – 28 = 32 minutes. Add 3 hours and 5 minutes to 32 minutes and we have a total of 3 hours and 37 minutes.

EXERCISES

1. Add

a. 1 hour 38 minutes
<u>+ 3 hours 32 minutes</u>

b. 3 hours 51 minutes
<u>+ 0 hours 47 minutes</u>

2. Subtract

a.
  3 hrs 38 min
  - 1 hr 39 min

b.
  5 hrs 16 min
  - 3 hrs 41 min

3. On a hot summer day in McAllen, Texas, the temperature at 10:00 a.m. is 85° F. On a cold winter day in McAllen, the temperature at 10:00 a.m. is 67° F. Find the difference in the temperature.

4. Josephine left McAllen at 7:35 a.m. and flew to Houston. She then changed planes at 10:30 a.m. and flew to St. Louis. She arrived in St. Louis at 2:45 p.m. What was Josephine's total travel time?

5. Ms. Badgett leaves her house at 7:05 a.m. to go to school. She arrives at school at 7:25 a.m. to prepare for the day. She leaves work at 3:45 p.m. and goes home. If she arrives home at 4:05 p.m., how long has she been away from home that day?

6. A plane arrived in the Austin airport at 11:48 p.m. The same plane left the airport at 6:23 a.m. the next day. How long was the airplane on the ground at the airport?

7. Nama arrived in San Marcos at 2:14 p.m. She had traveled 3 hours and 30 minutes non-stop from Waco. At what time did she leave Waco?

8. A CD plays for 72 minutes. If each song ranges from 3 minutes to 5 minutes, what is the range of the number of songs that can be on the CD?

9. Mr. Reyna works 40 hours a week. He has three weeks of vacation in a year. How many hours does Mr. Reyna work in a year?

10. How many minutes are in one day? How many seconds are in one day?

11. **Ingenuity:**

    Now that you have learned about various units in this chapter, create a new unit of measurement of your own. Describe what it can be used to measure, and explain the process of converting it into other appropriate customary or metric units.

12. **Investigation:**

    The earth year lasts about 365.25 days; this is the time it takes for the earth to make a full revolution around the sun. Explain why, based on this knowledge, it is logical to have a leap year every fourth year that is one day longer than the typical year.

SECTION 8.R
CHAPTER REVIEW

1. Convert the following customary units of measure. Refer to the appropriate conversion charts.

a. 48 in. = _________ ft.	h. 16 lbs. = _________ oz.
b. 112 oz. = _________ lbs.	i. 3 oz. = _________ lbs.
c. 5 gal. = _________ qt.	j. 720 sec. = _________ min.
d. 18 pt. = _________ qt.	k. 7040 yd. = _________ mi.
e. 8 hrs. = _________ min.	l. 3 pts. = _________ fl.oz.
f. 7 T = _________ lbs.	m. 1500 lbs = _________ T
g. 75 in. = _________ ft.	n. 93 oz. = _________ lbs.

2. Convert the following metric units of measure. Refer to the conversion chart.

a. 250 cm = _________ mm	g. 16 kg = _________ g
b. 7 L = _________ mL	h. 28 m = _________ km
c. 14 km = _________ m	i. 254 mL = _________ L
d. 1800 mg = _________ g	j. 45 mm = _________ cm
e. 175 mm = _________ m	k. 128 mg = _________ kg
f. 16 g = _________ mg	l. 9 km = _________ mm

Solve the following conversions. Remember to label your answers carefully.

3. Frankie left her house at 11:30 a.m. She drove for 3 hrs and 15 min. At what time did Frankie arrive at her destination?

4. The average temperature for July in Texas is 97 ºF while the average temperature in Washington State in July is 69 ºF. What is the difference in the average temperatures of the two states?

5. Dr. Shima arrived in New York City at 3:53 p.m. His trip lasted 4 hrs and 20 min. At what time must Dr. Shima have begun his trip to New York City?

Solve the following problems involving unit conversions. Set up proportions whenever possible. Remember to label your answers carefully.

6. Mrs. Lowrance's baby weighed 8 lbs. 13 oz. when he was born. How many ounces did he weigh at birth?
7. The distance from the front door to the mailbox is 28 feet. What is the distance in inches?
8. A baby hippo weighs 1300 pounds. How many tons does the baby hippo weigh?
9. A pitcher of lemonade holds 4 gallons. How many pints are contained in the pitcher?
10. A recipe calls for 4 cups of flour. How many cups of flour will be needed to repeat this recipe 12 times?
11. Jon can lift 145 pounds. What is this weight in ounces?
12. The library is 1574 meters away from the corner store. If Julie wants to walk from the library to the corner store, how many kilometers will she walk?
13. The new dining room table is 8 feet long. If Mom buys 3 yards of fabric to make a table cloth, will she have enough fabric? What is the difference between the two amounts?
14. A bunch of grapes weighs 638 milligrams. How many grams does the bunch of grapes weigh?
15. Sandra walked 18 yards while Melissa walked 55 feet. Which girl walked farther?
16. For a wedding, the caterer made 68 quarts of gravy for the meal. He needs to store the gravy in 1-cup sized containers. How many containers of gravy will he have?

GEOMETRY

9

SECTION 9.1
MEASURING ANGLES

How would you answer the question, "What is an angle?"

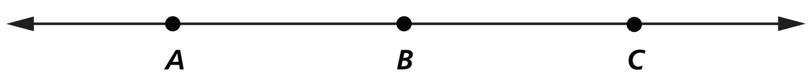
You have probably seen angles in many places in everyday life. Can you name a few of these places? In this section, you will learn what angles are, how to construct angles and how to measure the size of an angle.

ACTIVITY: LINES, LINE SEGMENTS, AND RAYS

Locate three points on the line below:

Label the left point A, the middle point B, and the right point C. Typically, two points on the line are used to identify a line. For example, if points A and B are used, then we use the notation, $\overleftrightarrow{AB}$. You can also use points B and C, in which case the notation is $\overleftrightarrow{BC}$.

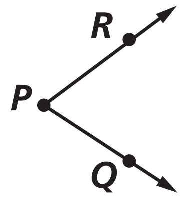
A **line segment** is a part of a line that includes two endpoints and all the points in between the endpoints. For example, the line segment with endpoints A and B is written $\overline{AB}$. Identify other line segments on this line that involve using a pair of the given points A, B, C.



A **ray** is also a part of a line that has a starting point and continues forever in only one direction. One ray on the line above is the ray that has starting point B and goes in the direction of C. We write $\overrightarrow{BC}$ for the ray BC. If we want to describe the ray starting at B that goes in the direction of A, what notation could you use? What other rays can you describe on the line above?

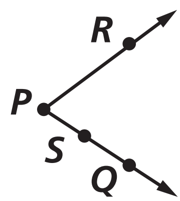
Notice that while line segments $\overline{AB}$ and $\overline{BA}$ describe the same line segment, the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are very different parts of the line.

To construct an angle, first draw two rays from a common point ***P***.



In the figure above there are two rays, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Both of these rays begin at point ***P***, and pass through the points ***Q*** and ***R*** respectively.

If another point ***S*** is between points ***P*** and ***Q***, then the ray $\overrightarrow{PS}$ is the same as ray $\overrightarrow{PQ}$. However, the ray $\overrightarrow{SP}$ is a different ray, because it begins at point ***S***. In fact, $\overrightarrow{SQ}$ is a third ray that is different from all of the rays mentioned so far.



We can now answer our original question: "What is an angle?"

DEFINITION 9.1: ANGLE
An **angle** is formed when two rays share a common vertex.

The common endpoint ***P*** on rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is called the **vertex** of the angle. In the diagram, these rays form an angle called angle ***QPR***, written $\angle QPR$. The symbol "$\angle$" is the math symbol for the word angle. To name an angle, you can do the following three steps in order:

1. Write the name of one of the non-vertex points on one of the rays.
2. Write the name of the vertex.
3. Write the name of a non-vertex point on the other ray.

There can be many ways to name the same angle because there are many choices of points on the two rays in steps 1 and 3. You could also label this angle $\angle RPQ$. As long as the middle point identifies the vertex of the angle, the order of the first and last points does not matter.

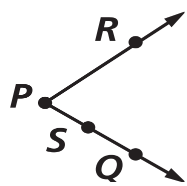
EXAMPLE 1

Draw an angle $\angle XYZ$. Identify the rays and identify the vertex of this angle.

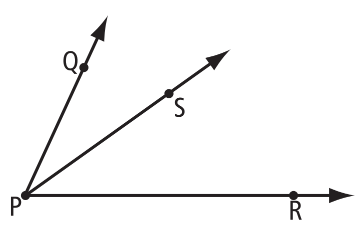
SOLUTION

$\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are the rays, and Y is the vertex of the angle. We use different letters to label different angles, like $\angle QPR$ or $\angle XYZ$.

Sometimes the single letter of the vertex is used to name an angle when it will not lead to any confusion. For example, in the figure below, $\angle QPR$ can be called $\angle P$ without any confusion.

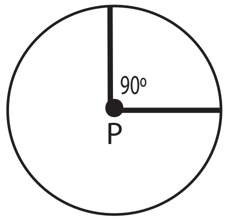


However, if more than 2 rays emanate from one common vertex, then it would not be clear to which angle we are referring. For example, in the figure below, we would not use a single letter notation for an angle and instead use the three letter notation.



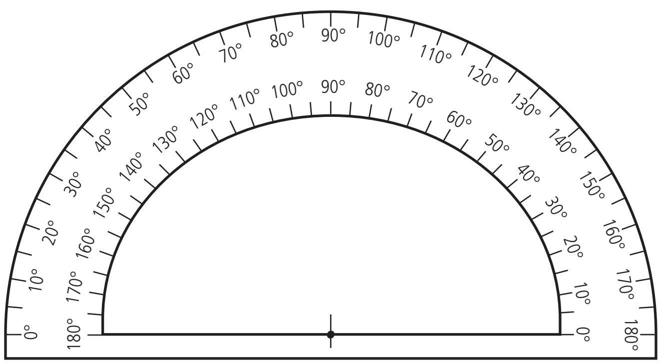
Once you understand the definition of an angle, the next step is to measure the size of the angle. Degrees are commonly used to measure angles. One **degree**, written 1°, is the angle formed by $\frac{1}{360}$ of a full revolution around a circle.

An angle that makes a full revolution has measure 360º. One fourth of a revolution would have measure 90º and could look like this:



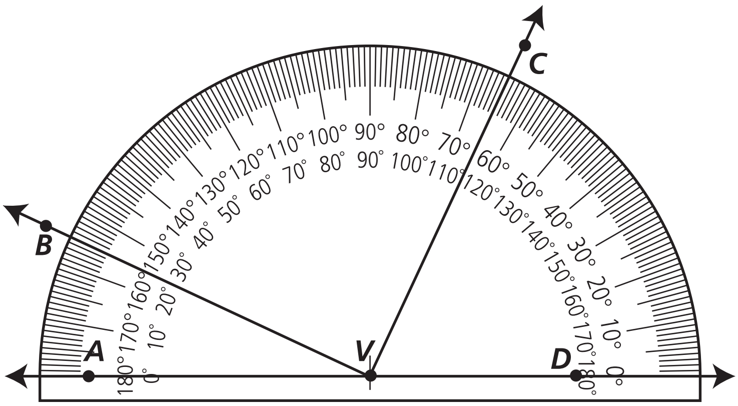
A **protractor** is an instrument used to measure angles. Most protractors use degree units to measure the angles.

Protractors have degree markings along the outside of the curved edge. To measure an angle, place the vertex at the center of the semi-circle so that one ray passes through 0° or 180° and the other ray passes through a mark on the curved edge. If necessary, extend the other ray so that it falls on a mark along the curved edge. The degree at this mark is the measure of the angle or its supplement, which we will define later in this section.



If two rays with a common endpoint form a straight line, the angle they form has a measure of 180 degrees, or 180°. This is called a **straight** angle. Angles that have a measure between 0° and 90° are called **acute** angles. Angles that have a measure greater than 90° but less than 180° are called **obtuse** angles.

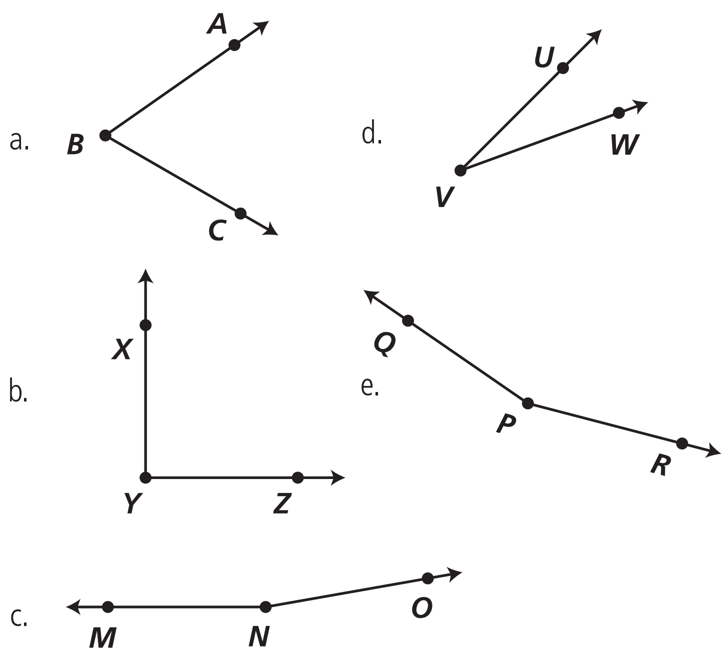
Angles that measure exactly 90º are called **right** angles. Using a protractor, construct and label a straight angle, a right angle, an acute angle, and an obtuse angle and indicate the measure of each angle.



Identify the straight angle, an acute angle, an obtuse angle, and a right angle from the figure above.

EXPLORATION 1

1. Measure each of the angles below with your protractor:



2. Consider the rays and points above. Name each angle in two ways.

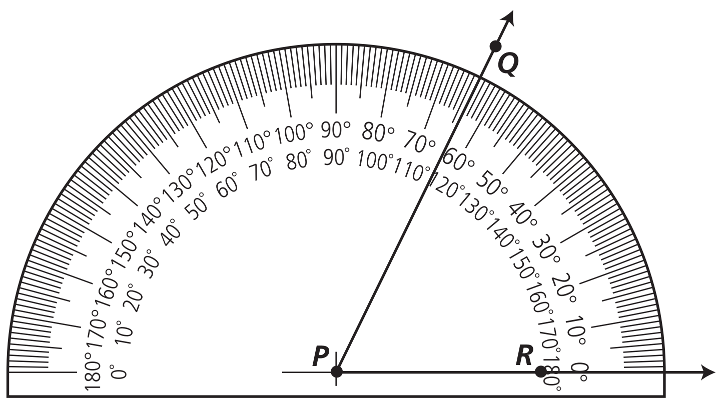
3. Classify each of the angles as acute, obtuse, right, or straight.

EXPLORATION 2

1. Divide a straight angle into two angles. Describe how you constructed it to another student or your teacher. Measure each angle using your protractor.
2. Use your protractor to draw rays making the following angles: 35°, 80°, and 100°.

How did you construct an angle? Here is one approach to construct an angle with a given measure such as 64°.

1. Draw an initial ray and label it $\overrightarrow{PR}$. The initial ray is usually, but not necessarily, horizontal.
2. Place the center of the semi-circle of the protractor on top of the point ***P***, with the ray passing through 0°. Label a point ***R*** on this ray.
3. Find the place along the curved edge of the protractor that corresponds to the degree measure you are constructing and mark it with a new point ***Q***.
4. Draw a line connecting the point ***P***, which is the vertex of the angle, to the new point ***Q*** with a straight edge to obtain an angle of 64°.



PROBLEM 1

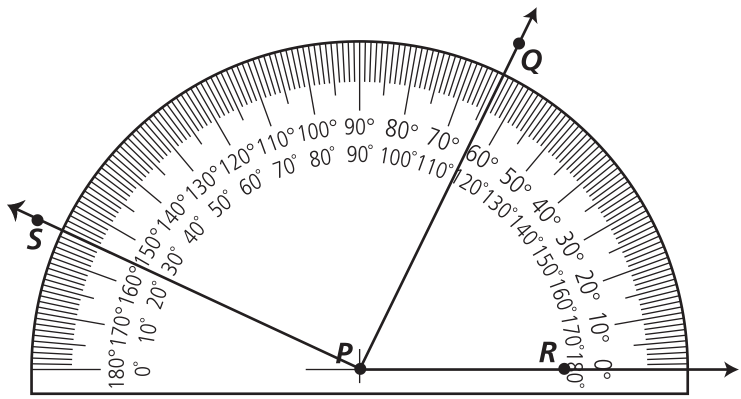
Use a protractor to construct an angle starting at 0° with measure 45°.

PROBLEM 2

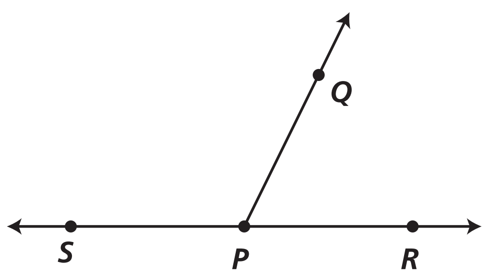
Use a protractor to construct an angle with one ray at the 20° mark and with measure 45°.

PROBLEM 3

Consider the rays on the protractor below. Measure the angles, $\angle QPS$ and $\angle SPR$.



You may wish to investigate on your own why there are 360° in a full circle. Who first used the number 360? Why didn't they choose 300 or 400 or 500 degrees to make a circle?



Notice that $\angle QPR$ and $\angle QPS$ divide the $\angle RPS$ above into two parts. The measure of each angle is a number and we use the notation for the measure of an angle such as an $\angle QPR$ as $m(\angle QPR)$. We can then add these numbers together to get the equations below:

$$m(\angle QPR) + m(\angle QPS) = m(\angle RPS)$$

$\overline{SR}$ is a line, so $m(\angle RPS) = 180°$. Then we have

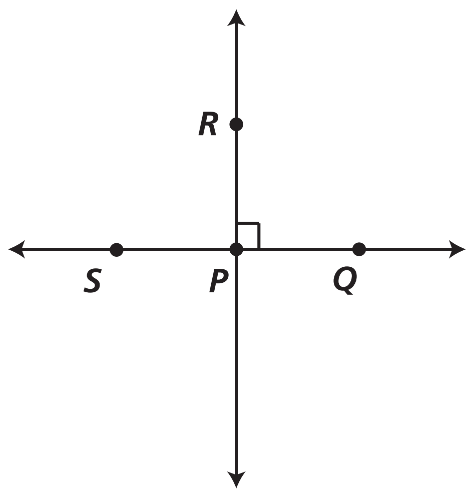
$$m(\angle QPR) + m(\angle QPS) = 180°.$$

DEFINITION 9.2: SUPPLEMENTARY
Two angles are **supplementary** if the sum of their measures is 180°.

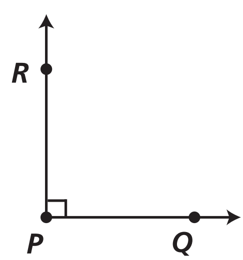
In the previous example, $\angle QPR$ and $\angle QPS$ are supplementary angles. This means $\angle QPS$ is the supplement of $\angle QPR$ and $\angle QPR$ is the supplement of $\angle QPS$.

Now divide a straight angle in half. Each angle formed is a right angle and measures 90° because $\frac{1}{2}$ of 180° is 90°. When two lines or line segments meet and form a right angle, they are **perpendicular** to each other.

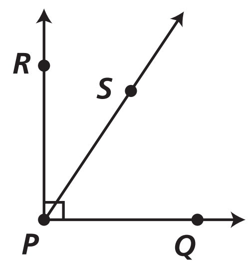
We often label the right angle with ∟ to indicate that it is 90°.



When two rays meet to form a right angle, they are perpendicular rays.



Next, divide the right angle $\angle RPQ$ into two parts using ray $\overrightarrow{PS}$.



Because $\angle RPS$ and $\angle SPQ$ divide $\angle RPQ$, $m(\angle RPS) + m(\angle SPQ) = m(\angle RPQ)$ and $m(\angle RPS) + m(\angle SPQ) = 90°$.

DEFINITION 9.3: COMPLEMENTARY
Two angles are **complementary** if the sum of their measures totals 90°.

In the example above, $\angle RPS$ and $\angle SPQ$ are complementary angles. This means $\angle SPQ$ is the complement of $\angle RPS$ and $\angle RPS$ is the complement of $\angle SPQ$.

EXAMPLE 2

1. One angle measures 34°. What is its complement? Check your work by creating and solving an equation.
2. One angle measures 34°. What is its supplement? Check your work by creating and solving an equation.
3. Two angles form an acute angle. If one angle is 34°, what are the possible measures of the second angle? Check your work by creating and solving an inequality.
4. Two angles form an obtuse angle. If one angle is 34°, what are the possible measures of the second angle? Check your work by creating and solving an inequality.

SOLUTION

1. Call C the measure of the complement to the 34° angle. Two complementary angles equal 90°. In our case, we have the equation, $C + 34° = 90°$. Solving the equation we have $C = 90° – 34°$ or $C = 56°$. The complement of the angle with measure 34° is an angle with measure 56°.

2. Call S the supplement of a 34° angle. Then $S + 34° = 180°$. Solving, $S = 180° – 34°$ or $S = 146°$.

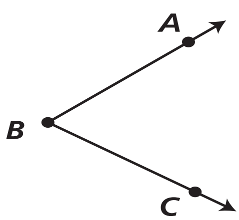
3. If two angles combine to form an acute angle, their sum must be less than 90°. Angle A plus 34° is less than 90°. Write the inequality as $A + 34° < 90°$. Solve to get $A < 90° – 34°$ which means $A < 56°$.

4. If two angles form an obtuse angle, the sum of their two measures must be less than 180° but greater than 90°. Call the angle B. The inequality is $90° < B + 34° < 180°$. Solve for B, which means isolate B algebraically, to get $90° – 34° < B < 180° – 34°$ or $56° < B < 146°$. In other words, only angles greater than 56° but less than 146° can be combined with a 34° angle to form an obtuse angle.

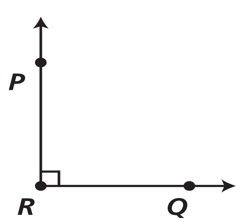
EXERCISES

1. Name each angle shown. Classify as acute, obtuse, right, or straight. Use a protractor to find the measure of each.

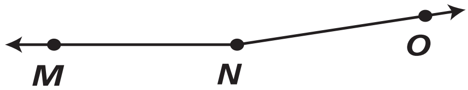
a.



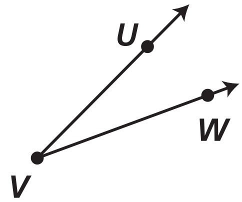
b.



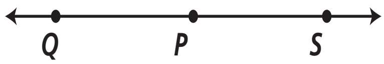
c.



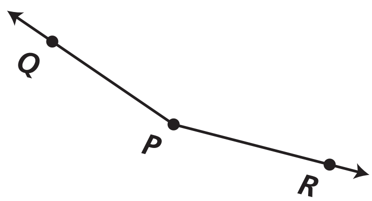
d.



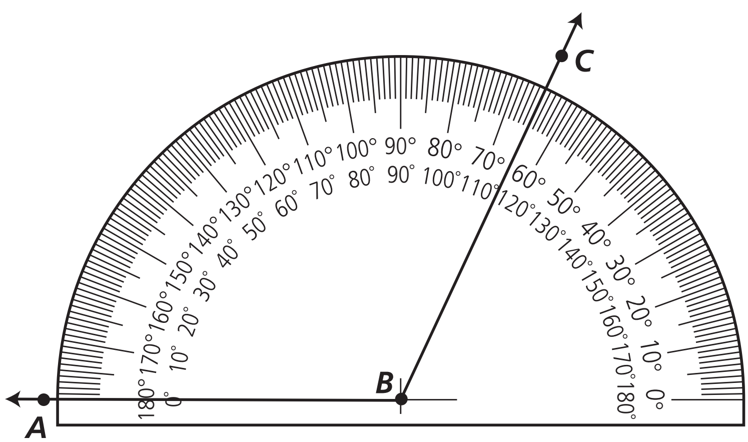
e.



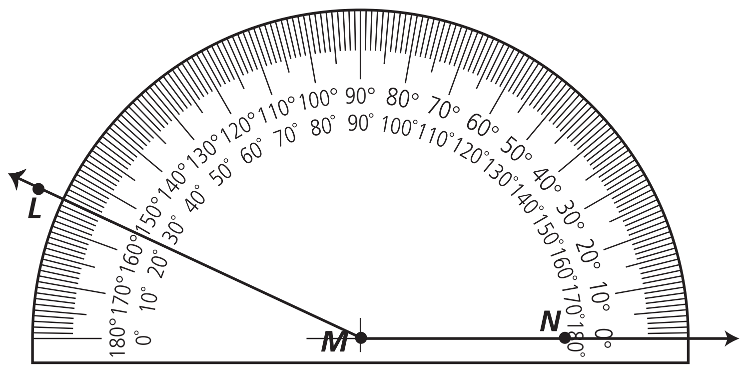
f.



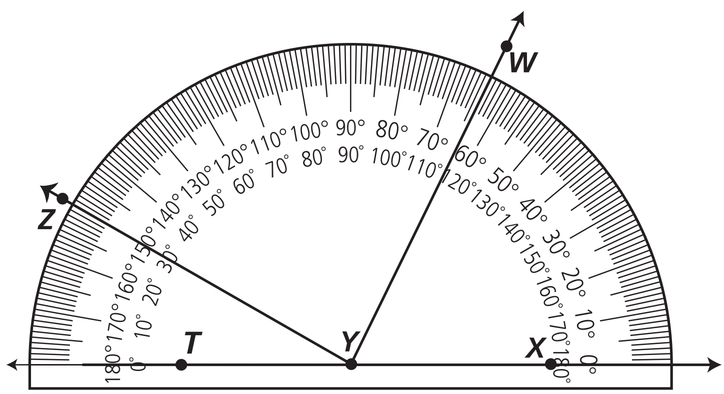
2. Classify angle *ABC* in the figure below as acute, obtuse, right, or straight. Use the protractor in the figure to state its measure.



3. Classify $\angle LMN$ in the figure below as acute, obtuse, right, or straight. Use the protractor in the figure to state its measure.



4. Name at least 2 acute and 2 obtuse angles. Give the measures of each of these angles. Are there any right or straight angles? If so, name those as well.



5. Draw each of the following type of angles using a ruler and a protractor. Write the measure of each of your angles. Remember to label your measurements with degree signs.

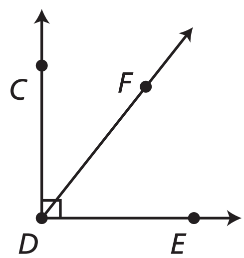
a. Straight angle
b. Obtuse angle
c. Acute angle
d. Right angle

6. Draw an angle with each of the following measures:

   a. 30° c. 77° e. 113°

   b. 120° d. 45°

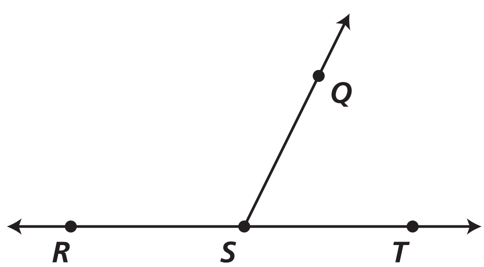
7. a. Using a protractor, give the measure of the angles below.



m(∠*CDF*) = _______ m(∠*FDE*) = _______ m(∠*CDE*) = ______

   b. What relationship do you notice about the three angles?

8. a. Using a protractor, give the measures of the angles below.



m(∠*RSQ*) = _______

m(∠*QST*) = _______

m(∠*RST*) = ______

   b. What relationship do you notice about the three angles?

9. Draw right angle ∠*STV* using a protractor and a ruler. Draw ray $\overrightarrow{TR}$ within the right angle.

   a. Name the two angles you created within your right angle.

   b. Give the measure of each of the angles you created.

   c. How do you describe the relationship of the two angles?

10. Draw a straight angle $\angle JKL$ using a protractor and a ruler. Draw ray $\overrightarrow{KP}$ within the straight angle.

    a. Name the two angles you created within your straight angle.

    b. Give the measure of each of the angles you created.

    c. How do you describe the relationship of the two angles?

11. The sum of the measure of three angles is 180 degrees. The measure of the largest of the three angles is the sum of the measures of the two smaller angles. What is the measure of the largest angle?

12. Start with an angle $A = 17°$.

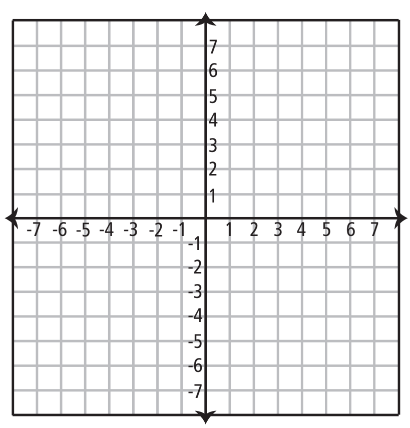
    a. What are the possible measures of angle B that could be combined with A to form an acute angle? Begin by writing an inequality that relates A and B with the condition.

    b. What are the possible measures of angle C that could be combined with angle A to form an obtuse angle? Begin by writing an inequality that relates A and C with the condition.

13. How many acute angles can you find with measures between 80° and 85°?

Spiral Review:

14. If point A is located at $(^-2, 3)$, find the following transformations of A.



Point ***B***: translate ***A*** 2 left and 3 down

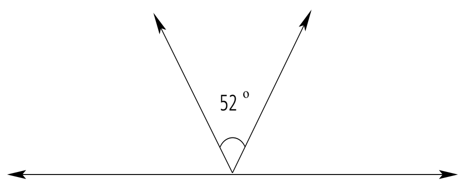
Point ***C***: reflect ***A*** over ***x***-axis

Point ***D***: reflect ***A*** over ***y***-axis

15. Krista left for school at 7:45 a.m. and returned home at 4:25 p.m. How much time elapsed between the time she left for school and returned home?

16. **Ingenuity:**

    In the figure below, two rays rest on a straight line. The two rays make an angle of 52º and rest on the line so that the angles between the line and the rays on both sides are equal. What is the angle between the line and each of the rays?



17. **Investigation:**

    Suppose we turn due north, and then make a sequence of clockwise turns until we face due north again. For example, if we turn 90º each time, we will face north after exactly 4 turns.

    a. How many turns will we make if we turn 30º each time?

    b. How many turns will we make if we turn 45º each time?

    c. How many turns will we make if we turn 80º each time?

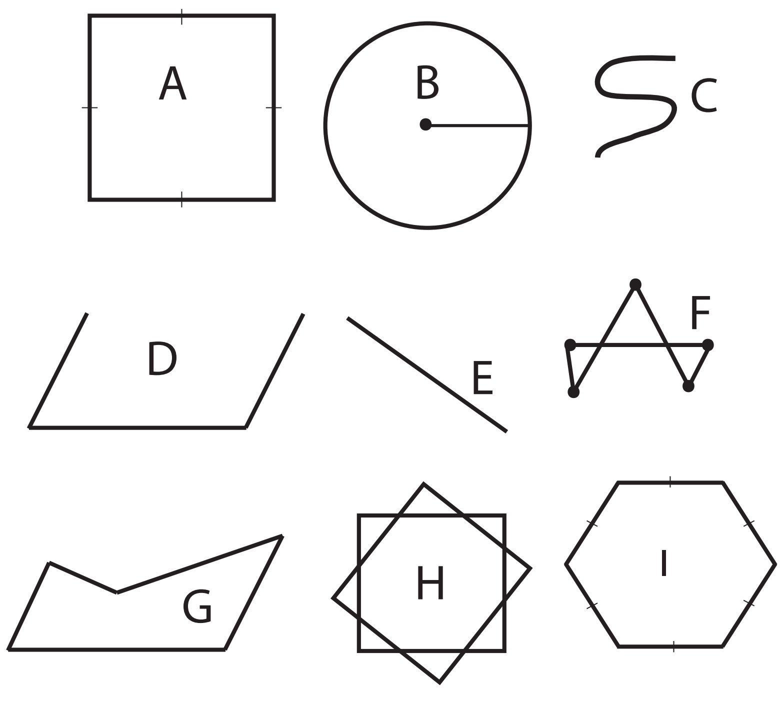
    d. How many turns will we make if we turn 108º each time?

    e. Can you find a way to answer questions like a through d without actually going through the sequence of turns and keeping track of the angles? Explain.

SECTION 9.2
TRIANGLES

EXPLORATION 1

Consider the following figures and describe them by features that you observe. Identify common features as well as differing features among the shapes.



DEFINITION 9.4: POLYGON
A **polygon** is a closed, plane figure formed by 3 or more line segments. Each line segment joins with exactly two other line segments, one at each end. The point where the two line segments meet is called a **vertex** of a polygon.

Classify the shapes in Exploration 1 as a polygon or not a polygon. We often note the number of sides a polygon has. Include the number of sides each of the polygons has in Exploration 1.

A triangle is one type of a polygon. Triangles are closed two-dimensional figures whose three sides are line segments. Triangles can be classified by different properties - their size, shapes, and angles. In Exploration 2, you may discover different properties of triangles by making measurements of the triangles' lengths and angles.

EXPLORATION 2

a. Draw a line segment that is five units long. Now draw two other line segments to complete a triangle. Repeat this process several times. What do you notice about the sum of the lengths of the other two sides relative to the length of the original side?

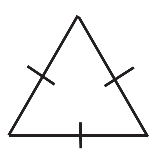
b. Make a triangle with two of the sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle having two sides of equal length.

c. Make a triangle with all three sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle with all three sides of equal length.

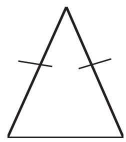
DEFINITION 9.5: EQULATERAL TRIANGLE

A triangle in which all three sides have the same length is called an **equilateral triangle**. This term comes from the Greek *equi*, meaning "the same," and Latin *latus*, meaning "side." You have probably discovered in the exploration that all the angles of an equilateral triangle have the same measure.

A way to indicate that lengths are equal in a given figure is using tick marks as indicated in an equilateral triangle as shown below.



DEFINITION 9.6: ISOSCELES TRIANGLE
A triangle with two sides of equal length is called an **isosceles** triangle.



Let's explore isosceles triangles a little more.

PROBLEM 1

Draw two different isosceles triangles with the two equal sides of length 4 inches. Measure each of the angles. What do you notice about their measures?

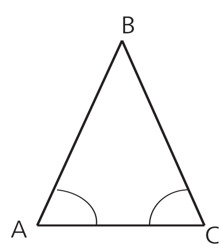
PROBLEM 2

Draw two different isosceles triangles with the two equal sides of length 2 inches. Measure each of the angles. What do you notice about their measures?

In each of the problems, did you notice that the angles opposite the equal sides are of equal measure? This is actually one of the properties of all isosceles triangles:

- The angles opposite the equal sides are always of equal measure.

Just as we use a tick mark for equal lengths, there is a mark that is often used to indicate angles of equal measure.



When two lengths or two angles have equal measure, then we say that they are **congruent** and have a special notation, $\cong$, to indicate congruence. For example, $\angle A \cong \angle C$. The angle measures are equal and in that case we write $m(\angle A) = m(\angle C)$.

Conversely, if two of the angles in a triangle are equal, then the sides opposite these equal angles will be equal and the triangle will be isosceles. These are properties that you will learn when you take a course in Geometry. Do you see why they might be true?

It is also possible that all three sides of a triangle have different lengths. This leads to the following definition.

DEFINITION 9.7 SCALENE TRIANGLE
A triangle with all three sides of different lengths is called a **scalene** triangle.

PROBLEM 3

Draw two scalene triangles. Measure each of the angles. What do you notice about their measures?

In a scalene triangle, the three angles will also have different measures, since if two of the angles were equal, then the sides opposite these equal angles would also be equal.

EXPLORATION 3

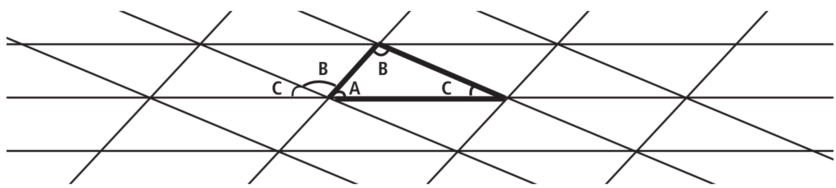
Draw a large triangle on a sheet of paper, using a straight edge. Color or label the three angles of the triangle with different colors. Carefully cut out the triangle. Next, cut the triangle into 3 pieces, each including one angle from the triangle. Put the angles together with the 3 vertices meeting. What is the sum of the three angles of the triangle? Compare your result with others.

In each case, the sum of the measures of the angles in the triangles appears to be 180°. This leads to a conjecture: "The sum of the measures of the angles in any triangle is 180°." This is a **conjecture**, because it is a statement we think might be true based on our observations, but we have not yet proved it is always true. Is there a way to give a convincing argument or proof that the sum of the measures of the angles in any triangle is 180°? To answer that, investigate some triangles.

EXPLORATION 4

Divide the class into groups. In each group, draw a small triangle and several copies of it on lined paper using a straight edge. Be as exact in your work as possible. Use these copies to tessellate the paper or plane. A **tessellation** or tiling of the plane with some shape is a way of covering the plane with that shape with no gaps. This tessellation can be used to show that the sum of the measures of the angles of any triangle adds up to 180°.

To do this, begin by putting your triangles together to make a series of equal four-sided figures whose opposite sides are parallel. Use these to cover the plane. Label the angles of one of the triangles ***A***, ***B,*** and ***C***. Place it in the middle of a sheet of paper and, using a straight edge, draw lines parallel to the three sides. Then draw three sets of equally-spaced parallel lines like the example.



This forms tessellated tiles in which each tiling piece is a triangle congruent to triangle ***ABC***. **Congruent** means that all the tiling pieces, or triangles, have exactly the same size and shape. Label each of the angles in the picture ***A, B,*** or ***C***. It is now easy to see something quite remarkable: the measures of angles ***A, B,*** and ***C*** sum to a straight angle. Explain why.

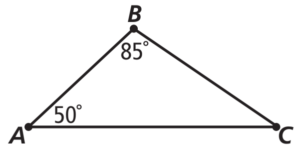
We can now state the **Triangle Sum Theorem**:

TRIANGLE SUM THEOREM
The sum of the measures of the angles in any triangle equals 180°.

The tessellation is a sketch of the proof that the sum of the measures of the angles in a triangle always adds up to 180°. How can knowing that the sum of 3 angles of a triangle is 180° help you figure out missing angles?

EXAMPLE 1

Find the measure of $\angle C$ in the following triangle.



SOLUTION

m ($\angle A$) + m ($\angle B$) + m ($\angle C$) = 180°

50° + 85° + m ($\angle C$) = 180°

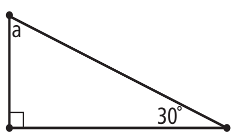
135° + m ($\angle C$) = 180°

m ($\angle C$) = 45°

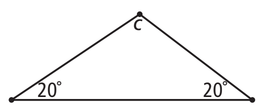
PROBLEM 3

Find the measure of the missing angles:

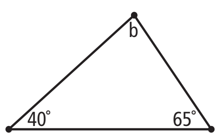
a.



c.



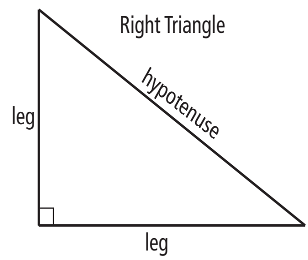
b.



In geometry, you will learn how to prove many interesting properties of geometric shapes using only the basic ideas above.

Much of the geometry that you study in middle school comes from the studies developed by Euclid several thousand years ago. He based the study of geometry on the foundations of axioms, postulates, and theorems. Part of the excitement of mathematics involves seeing new relationships based on simple ideas, like the sum of the angles in a triangle.

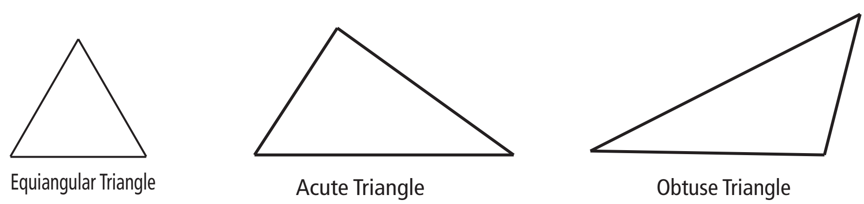
Triangles can also be categorized by their angles. One kind of triangle is a **right triangle**. A right triangle is a triangle with a right angle, which we know is an angle with measure 90°.



The longest side of a right triangle is called the **hypotenuse**. The right angle is **opposite** the hypotenuse. The two shorter sides are called the **legs** of the right triangle.

You will eventually learn a special theorem that relates the lengths of the legs of a right triangle to the length of the hypotenuse. This theorem, the **Pythagorean Theorem**, enables you to find the length of any side of a right triangle if you are given the lengths of the other two sides.

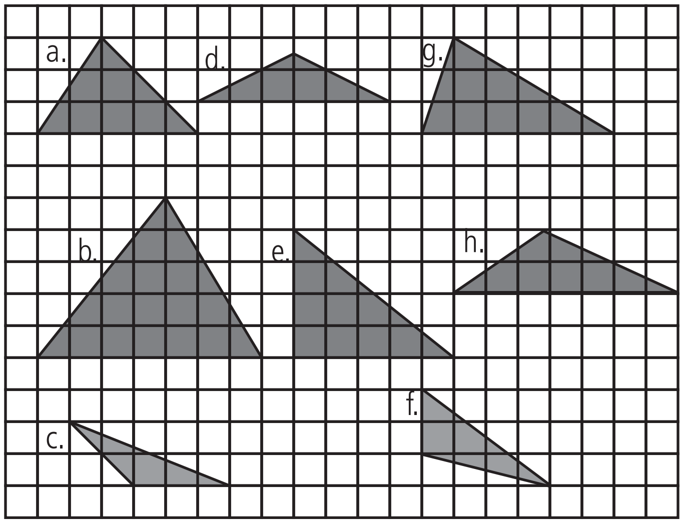
In addition to right triangles, there are other triangles that can be classified by their angles. Notice than an equilateral/equiangular triangle is a special kind of acute triangle with each angle equal to 60 degrees. A triangle with all three angles equal is an equilateral (equiangular) triangle. If all three angles of a triangle are acute, or less than 90°, the triangle is called an **acute** triangle. If one of the angles is larger than 90°, the triangle is called an **obtuse** triangle. Is it possible for a triangle to have two angles larger than 90°? Explain.



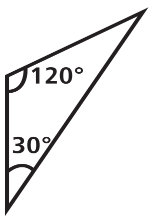
Two triangles are said to be congruent to each other if their corresponding sides and corresponding angles are **congruent** to each other. Experiment by drawing two different triangles on the same grid paper so that they each have the same lengths on all three sides. What would you say about their shapes? Measure their angles to verify that they also have the same angle measures. In some sense, they are exact copies of each other.

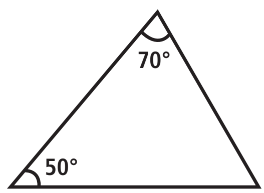
EXERCISES

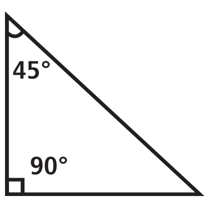
1. For each triangle below, classify the triangles by their angles (right, acute, obtuse) and then by their sides (equilateral, isosceles, or scalene). Explain your decision.

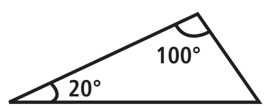


Find the measure of the angle missing in each of the triangles in Exercises 2–5.

2. 

4. 

3. 

5. 

6. A triangle has two angles whose measures are 35° and 65°. Find the measure of the third angle.

7. A right triangle has another angle whose measure is 45°. Find the measure of the third angle.

8. All of the angles in $\triangle ABC$ have the same measure. What is the measure of each of the angles?

9. A 120° angle is trisected, or divided into three equal angles. What is the measure of each of these angles? What would this look like?

10. A triangle has three angles with measures x, $2x$, and $3x$. Find the measure of each of these angles.

11. Use a protractor and ruler to draw an equilateral triangle with sides 5 inches long. What is the measure of each of the angles?

12. Consider each of the following sets of three numbers. Determine which of the sets contain numbers that can be the lengths of sides of a triangle. If a triangle cannot be formed using the lengths, explain your reason. If a triangle can be formed, explain your reason and identify the kind of triangle formed, such as scalene, isosceles, or equilateral.

    a. 3, 3, 3
    b. 1, 2, 3
    c. 3, 4, 5
    d. 5, 7, 5
    e. 1, 3, 1

Spiral Review:

13. Look at set R and set S below. List three additional numbers that would belong to both sets R and S.

    Set $R = \{1, 3, 5, 7, 9, 11...\}$

    Set $S = \{3, 6, 9, 12, 15, 18...\}$

14. An isosceles triangle has legs with length $2\frac{3}{5}$ inches and a base of $3\frac{1}{5}$. What is the perimeter of the triangle? Represent the perimeter as both a mixed fraction and an improper fraction.

15. **Ingenuity:**

    In triangle ABC, the measure of angle A is 6° more than the measure of angle B. If triangle ABC is an obtuse triangle, how many possible whole number measures are there for angle C?

16. **Investigation:**

A diagonal of a polygon is a line segment that connects one vertex of the polygon to another vertex, but is not a side of the polygon. We can divide a quadrilateral (a four-sided polygon) into two by drawing a single diagonal.

a. How many diagonals do we have to draw in order to divide a pentagon (a five-sided polygon) into triangles? How many triangles do we get?

b. How many diagonals do we have to draw in order to divide a hexagon (a six-sided polygon) into triangles? How many triangles do we get?

c. How many diagonals do we have to draw in order to divide an n-gon (an n-sided polygon) into triangles? How many triangles do we get?

SECTION 9.3
QUADRILATERALS AND OTHER POLYGONS

"Tri" in triangle gives us a clue about the number of sides in a polygon. Similarly, "Quad" in quadrilateral refers to its four sides. Other prefixes provide clues to the number of sides a polygon has. Here are some of the prefixes that are used with polygons:

Number of sides of a polygon	Prefix	Name of polygon
3	Tri	Triangle
4	Quad	Quadrilateral
5	Penta	Pentagon
6	Hexa	Hexagon
8	Octa	Octagon
10	Deca	Decagon
12	Dodeca	Dodecagon
n (in general)		n-gon

All **quadrilaterals** have the common characteristic that they are polygons with four sides. Just as with triangles, quadrilaterals can be classified by properties of sides and angles. We first make an important observation about lines and line segments.

Let's think carefully about what seems like a simple concept, the idea of "parallel" lines. The question is how you decide whether two lines are actually parallel. In fact, what does it mean to say that they are parallel in the first place?

One way to describe parallel lines is to say that two lines in a plane are **parallel** if they never intersect, even if they are extended forever in both directions.

As we examine different types of quadrilaterals, we consider the lengths of sides as well as whether opposite sides are parallel or not.

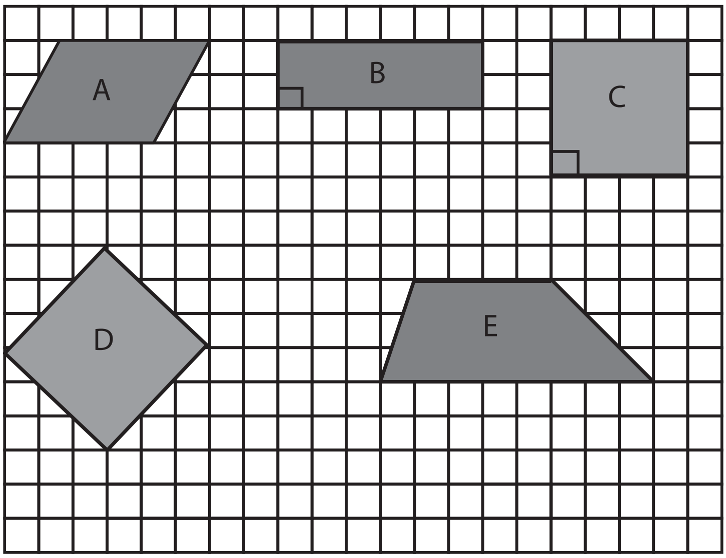
Common quadrilaterals include:

- **Parallelogram:** A quadrilateral with opposite sides parallel. The lengths of the opposite sides will be congruent.
- **Rectangle:** A parallelogram with a right angle.

- **Square:** A rectangle with all four sides of equal length.
- **Trapezoid:** A quadrilateral with exactly one pair of opposite sides parallel.
- **Rhombus:** A parallelogram with all four sides of equal length.

PROBLEM 1

Classify the following quadrilaterals with the appropriate term or terms that apply.



EXPLORATION 1

Draw three different parallelograms on a grid paper. Measure the angles in each parallelogram. Make two conjectures about the relationship among the angles that you think must be true for all parallelograms. Explain your reasoning.

In fact, opposite angles in a parallelogram are congruent to each other. Consecutive angles in a parallelogram are supplementary.

Rectangles and squares give us a clue that the sum of the angles is 360 degrees. It is not so clear when you examine general quadrilaterals. You could consider breaking a quadrilateral into two triangles. Can you see a way to use what you know about the sum of the angles in a triangle to make a conjecture about the sum of the angles in any quadrilateral?

Quadrilateral Angles Puzzle Activity

Draw a trapezoid and a parallelogram on grid paper. Label the four angles in each of your quadrilaterals. Use your protractor to carefully measure the four angles in each shape and record the measures. What do you notice in your findings? Make two conjectures about the relationships among the angles that you think must be true for all parallelograms.

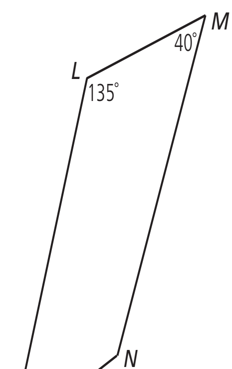
We summarize the observation you made about the angles in a quadrilateral: The sum of the angles in a quadrilateral is 360°.

The Polygon Riddles Activity will give you a chance to check your understanding of polygon classifications.

Sometimes one or more measures of angles are missing in a quadrilateral. How can knowing that the sum of the four angles is 360° help us to find the missing measures?

EXAMPLE 1

Without using a protractor, find the measure of $\angle N$. Show how you found the measure.



SOLUTION

$m(\angle K) + m(\angle L) + m(\angle M) + m(\angle N) = 360°$

$30° + 135° + 40° + m(\angle N) = 360°$

$205° + m(\angle N) = 360°$

$m(\angle N) = 155°$

EXAMPLE 2

Without using a protractor, find the measure of the missing angles in the parallelogram *ABCD* below.

SOLUTION

m ($\angle A$) = m($\angle C$) so m($\angle A$) = 65º.

m($\angle A$) + m($\angle B$) + m($\angle C$) + m($\angle D$) = 360º.

So m($\angle B$) + m($\angle D$) = 360º – 130º= 230º.

We also know m($\angle B$) = m($\angle D$) so, m($\angle B$) = 230 ÷ 2 = 115º = m($\angle D$).

We have m($\angle A$) = 65º, m($\angle B$) = 115º, m($\angle C$) = 65º, m($\angle D$) = 115º

The sides of polygons in general of varying lengths. If a polygon has all sides of equal length and the angles also of equal measure then we give it a special name. A polygon is said to be **regular** if the sides and the angles are all of equal measure.

Can you think of any polygon with 3 sides that are regular? With 4 sides?

EXERCISES

1. Draw the five shapes specified on graph paper. Be sure all the requirements are met.

   a. A parallelogram that is not a rectangle.

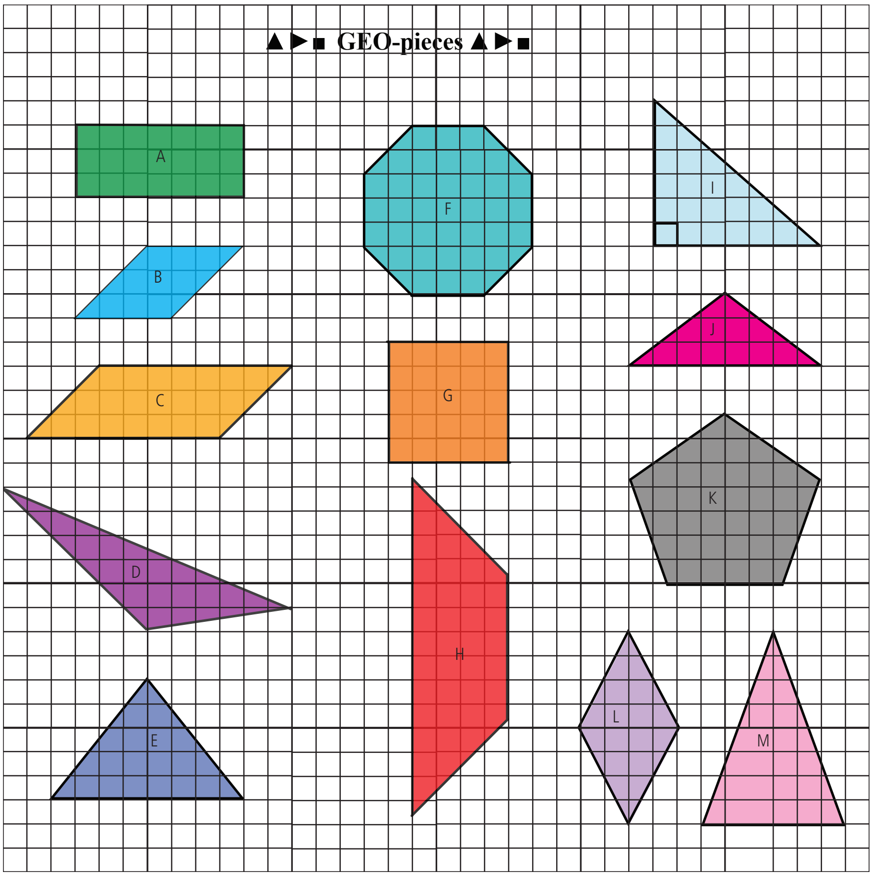
   b. A rectangle that is not regular.

   c. A trapezoid with a right angle.

   d. A parallelogram whose four sides are congruent that is not regular

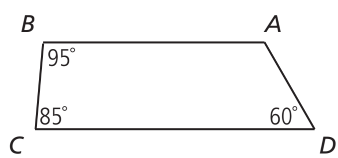
   e. A regular quadrilateral.

2. Classify the following shapes, writing all the names that apply to each figure.

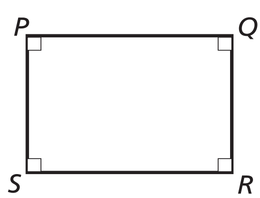


3. Find the measure of the specified part.

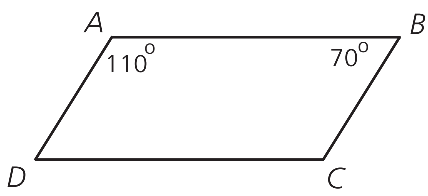
a. Angle *A* in the trapezoid *ABCD*



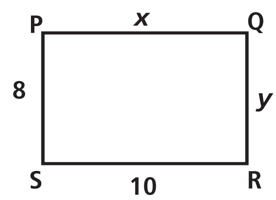
b. Sum of Angles *P* and *S* in the rectangle *PQRS*



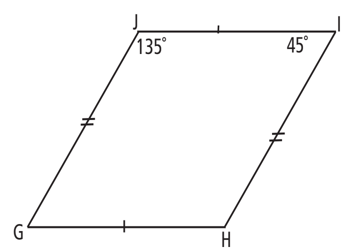
c. Angles ***C*** and ***D*** in the parallelogram ***ABCD***



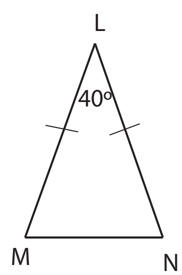
d. Lengths of sides ***x*** and ***y*** in the rectangle ***PQRS***



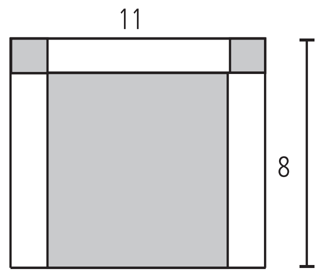
e. Angle ***G*** and Angle ***H*** in rhombus ***GHIJ***



f. $\angle$***M*** and $\angle$***N*** in isosceles triangle ***LMN***.



4. The shaded regions inside the 11 x 8 rectangle below represent squares. Find the side lengths of each of the squares.

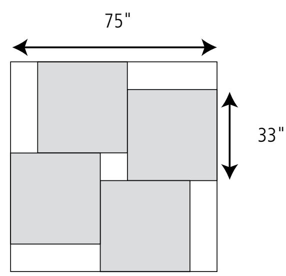


5. Determine the sum of the interior angles of a hexagon. Explain how you arrived at this measure. If the hexagon is regular, what is the measure of each interior angle?

Spiral Review:

6. The student council raised $672 by washing cars on Saturday. If they charged $8 to wash each car, how many cars did they wash?

7. In the figure below, four squares, each having sides of length 33 inches, are nested inside a larger square with sides of length 75 inches. What is the length of the sides of the small square in the middle?



8. **Ingenuity:**

The table below shows the liquid capacity of 4 sizes of floor cleaners.

Size	Capacity
Mini	45 fluid ounces
Regular	100 fluid ounces
Large	130 fluid ounces
Super	160 fluid ounces

a. Which containers hold more than a quart?

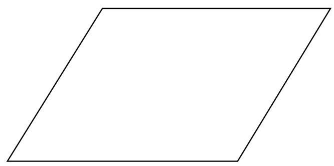
b. Which containers hold more than a gallon?

c. How many cups does the super size contain?

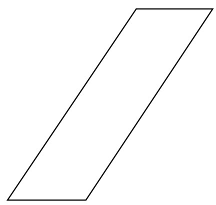
9. **Investigation:**

For each of the following figures, determine whether it is possible to cut a figure into several pieces and rearrange the pieces to form a rectangle.

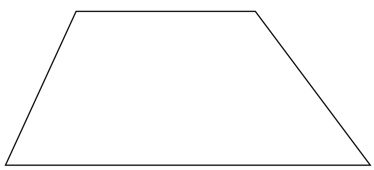
a. A parallelogram:



b. Another parallelogram:



c. A trapezoid:

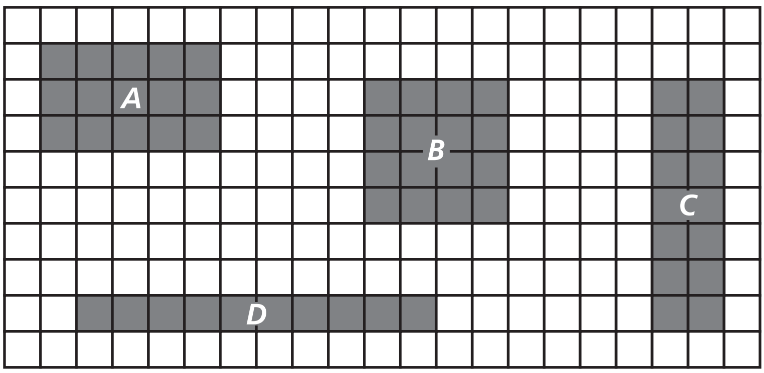


SECTION 9.4
PERIMETER AND AREA

How do you find the area of a 3 cm × 4 cm rectangle? How is this different from the distance around a 3 cm × 4 cm rectangle?

EXPLORATION 1

Of rectangles *A*, *B*, *C*, and *D*, which is the biggest? Explain your answer.



Discussion

There are several ways to think about what "biggest" means. One way to measure "biggest" is to find the area by counting the number of unit squares that are needed to cover each figure.

1. What are the areas of rectangles *A*, *B*, *C*, and *D*?
2. Which one has the largest area?
3. Does this agree with the rectangle you chose?

Another way to measure the size of a rectangle is to add the lengths of all the sides. This sum is called the **perimeter**. Its name comes from the Greek words *peri*, meaning "around," and *metron*, meaning "measure."

4. What are the perimeters of the 4 rectangles?
5. Which one has the largest perimeter?
6. Does this agree with the rectangle you chose?

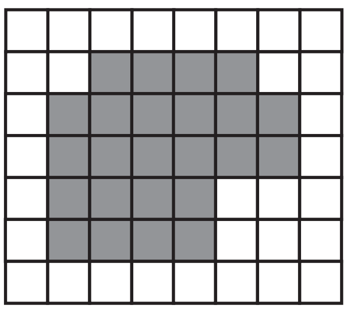
What is the definition of perimeter? What is the perimeter of a 3×4 rectangle? In general, if a rectangle has length L and width W, what is the perimeter of the rectangle? You saw that the perimeter of a rectangle is the sum of the lengths of the four sides. You can write this as perimeter of the rectangle, $P = L + L + W + W = 2L + 2W = 2(L + W)$.

Notice that squares are special rectangles whose sides are all the same length. If we call the side length as s, then the perimeter of the square, $P = s + s + s + s = 4s$.

Just as you found the perimeter of a rectangle by adding the lengths of the four sides, you can find the perimeter of any polygon.

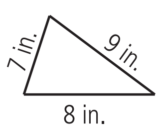
DEFINITION 9.8: PERIMETER
The **perimeter** of a polygon is the sum of the lengths of all of its sides.

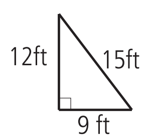
If the units of the length and width are inches, then the perimeter is measured in inches. If the unit of measure is not given, then the perimeter is measured in units. Find the perimeter of the shaded figure.



PROBLEM 1

Find the perimeter for each triangle.

a. 

b. 

DEFINITION 9.9: AREA
Area is the measure that it takes to cover the space inside the shape. When you formed shapes using tiles, the number of tiles used was the area of the shape.

Area is measured in square units. It can be labeled as square units or units^2 or u^2. What is the area of the shaded region above?

Draw the 2×6 rectangle that you formed with tiles in the Launch activity. What is its length? What is its width?

Now remember that 2 and 6 are dimensions that multiplied together give the area of this rectangle. There is a formula for this that says: Area = length · width or $A = L \cdot W$. Can you use this formula to find the area of a square? Draw a 6 x 6 square. L and W can be any positive rational number, not only positive integers. For example, a rectangle with length $L = 2.5$ m and $W = 1.2$ m would have area = (2.5)(1.2) = 3 square meters.

FORMULA 9.1 AREA OF A RECTANGLE
$A = L \cdot W$ where L is the length and W is the width of the rectangle.

If you use the formula $A = L \cdot W$, you will see that $6 \times 6 = 36$ square units. Because the square is a special type of rectangle, it has its own formula.

Area $= s \cdot s = s^2$

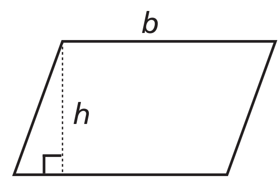
FORMULA 9.2 AREA OF A SQUARE
$A = s^2$ where s is the length of the side of the square.

EXPLORATION 2

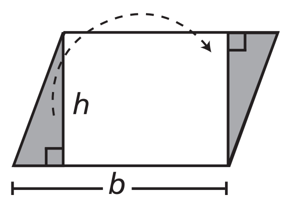
Draw four parallelograms using grid paper. For this exploration, make sure the longest side is on one of the grid lines.

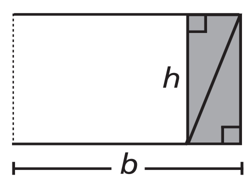
a. Measure the length of each of the sides and the measure of each angle. What do you observe?

b. Find the area of one of the parallelograms by cutting the parallelogram apart as illustrated below, and reassembling it to make a rectangle.



Label one of the horizontal parallel sides of the parallelogram the **base**, with length b. To find the height, draw a line segment between the two bases perpendicular to each base. The **height**, h, is the length of the perpendicular distance between the two bases. Notice that the height in parallelograms, unless they are rectangles, is not the same as the length of either of the two non-horizontal sides.





When reassembled, the parallelogram creates a rectangle with length, or base, b, and width, or height, h. That means the formula for the area A of a parallelogram is:

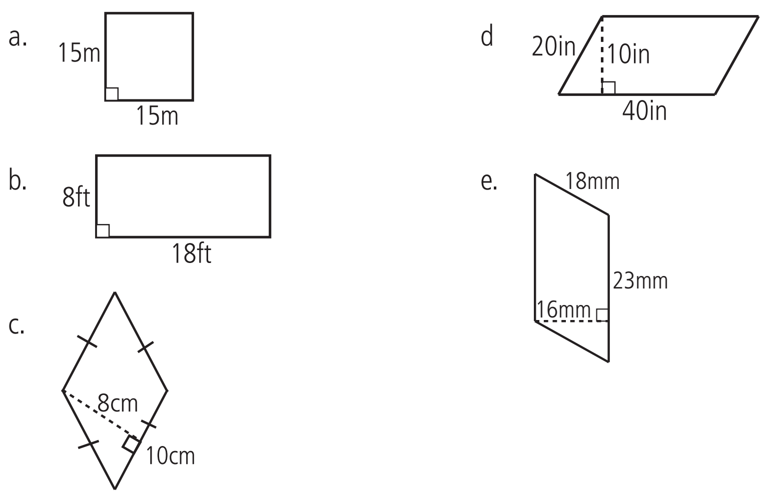
FORMULA 9.3 AREA OF A PARALLELOGRAM
$A = b \cdot h$ or $A = bh$.

The formula can be used for parallelograms with any positive rational number. For example, a parallelogram with base = 3.6 and height = 2.4 would have area = (3.6)(2.4) = 8.64 square units.

Rectangles are special parallelograms with four right angles. In this case, the height of the parallelogram can be thought of as the width of the rectangle and the base of the parallelogram the length. The area formula for a rectangle as we saw earlier can be written as $A = L \cdot W$.

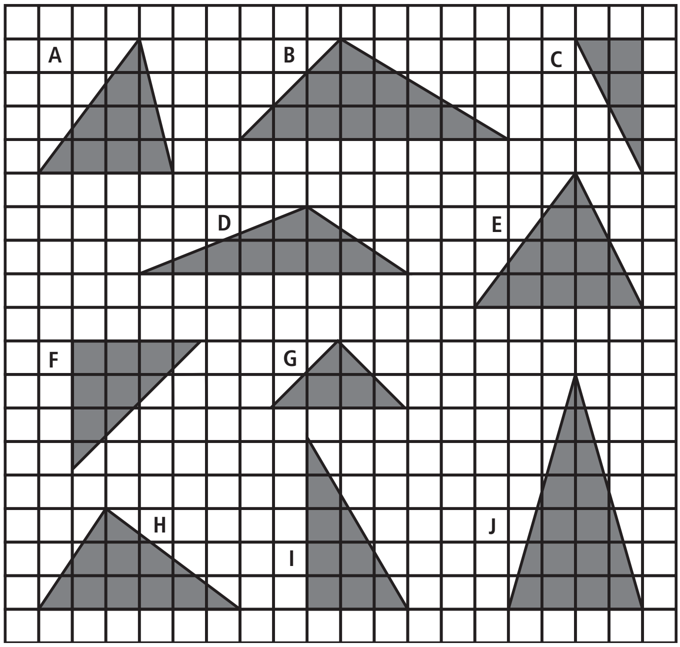
PROBLEM 2

Find the area and perimeter of each figure. Decompose and rearrange the parts of each parallelogram to create a rectangle and then use the rectangle formula to find its area.



EXPLORATION 3

Using the grid below, estimate the area of each triangle in square units. The bottom is usually called the **base** of the triangle, though any side can be considered a base of a triangle with its corresponding altitude. Sometimes you may have to rotate the triangle to find its base.

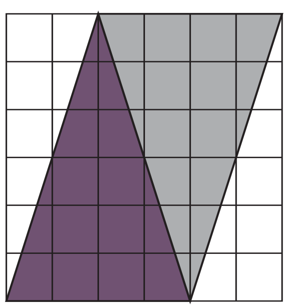


a. How did you compute the areas of each triangle?

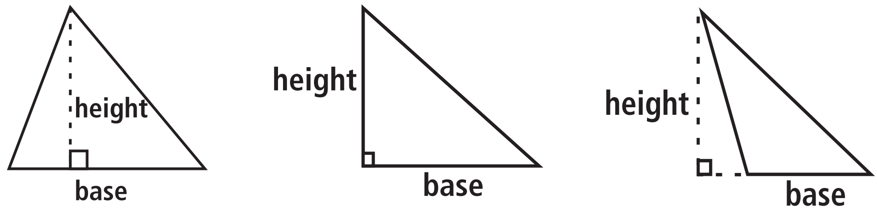
b. What patterns did you notice? Explain.

c. When you found the sum of the angles in a triangle in Section 9.2, you pasted two triangles together to form a four-sided figure. Using the triangles above, make a copy of each triangle and paste it together with the original triangle. What shape do you get? Use this process to find a rule for the area of these triangles.

What is the area of a triangle and is there a formula to compute this area? You have seen that taking any triangle, copying it exactly and putting the two triangles together creates a parallelogram.



You can use the formula for the area of the parallelogram and take one-half of it to compute the area of the triangle. Be careful in identifying the base and the height of the triangle. The base must be a side of the triangle, and the height, or **altitude**, must be perpendicular to the base or an extension of the base, and be drawn from the vertex opposite the base.



Notice that a height of a triangle may be a side of a triangle as in a right triangle. In general, a height of a triangle is not a side of a triangle but an additional line segment that must be drawn in.

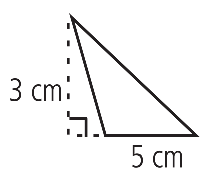
With those restrictions, the formula for the area A of a triangle with height h and base b is:

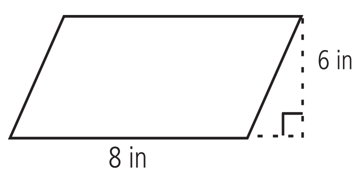
FORMULA 9.4: AREA OF A TRIANGLE
$A = \frac{1}{2} b \cdot h$ or $A = \frac{1}{2} bh$ or $A = \frac{bh}{2}$.

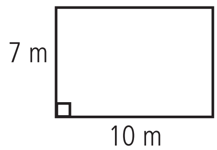
In the three triangles given above, a base and a height are specified. Can you see another possible base and height pair? Remember there are three sides to a triangle so there are three possible choices for the base. If you use the area formula, do you suppose the area of the triangle would be the same for any base and height combination?

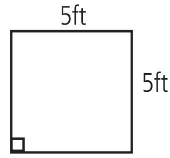
EXAMPLE 1

Find the area of the following polygons.

a. 

b. 

c. 

d. 

SOLUTION

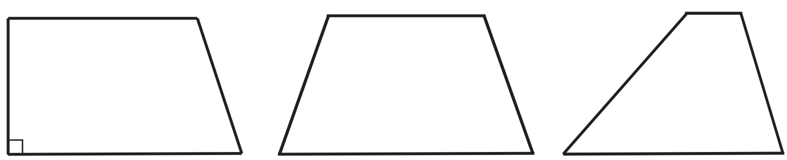
a. $A = \frac{bh}{2} = \frac{5 \cdot 3}{2} = \frac{15}{2}$ sq. cm.

b. $A = bh = 6 \cdot 8 = 48$ sq. in.

c. $A = bh = LW = 10 \cdot 7 = 70$ sq. m.

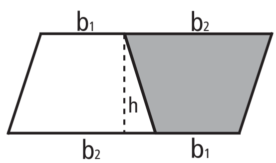
d. $A = LW = s \cdot s = s^2 = 5^2 = 25$ sq. ft.

A quadrilateral with two pairs of opposite sides parallel is called a parallelogram. If a quadrilateral has exactly one pair of opposite sides parallel, this 2-dimensional figure is called a **trapezoid**.



Draw a trapezoid on a grid paper. Draw an upside-down congruent trapezoid next to the first. What is the resulting shape?

The two trapezoids should form a large parallelogram similar to the picture below.



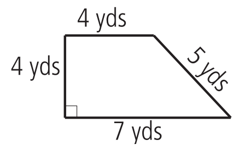
Using what you know about parallelograms, what is the area of this parallelogram consisting of two trapezoids? Because the parallelogram consists of the same trapezoid twice, what should the area of one trapezoid equal?

FORMULA 9.5: AREA OF A TRAPEZOID
$A = \frac{1}{2} \cdot (b_1 + b_2) \cdot h = \frac{(b_1 + b_2)h}{2}$

The formula can be used for trapezoids with any positive rational numbers. For example a trapezoid with $base_1 = 5.5$, $base_2 = 7.3$, and height = 3.1 would have area = $\frac{1}{2}(5.5 + 7.3)3.1 = 19.84$ square units.

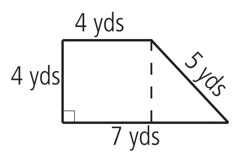
EXAMPLE 2

Find the area and perimeter of the following trapezoid. For the area, consider decomposing the figure into two recognizable shapes and finding these two smaller areas to attain the total area of the trapezoid:



SOLUTION

The trapezoid can be thought of as a composite figure that is made up of other figures put together. In our case, the trapezoid is a composite of a square and a triangle as you can see below:



The area of the square is $4^2 = 16$ yd^2.

The area of the triangle is $\frac{1}{2}(4 \cdot 3) = 6$ yd^2.

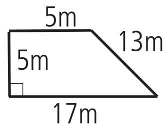
The area of the trapezoid is $16 + 6 = 22$ yd^2.

If we use the trapezoid formula we also get the same value for the area:

$$A = \frac{1}{2} \bullet (\boldsymbol{b}_1 + \boldsymbol{b}_2) \bullet \boldsymbol{h} = \frac{1}{2} \cdot (4 + 7) \cdot 4 = 22\text{yd}^2$$

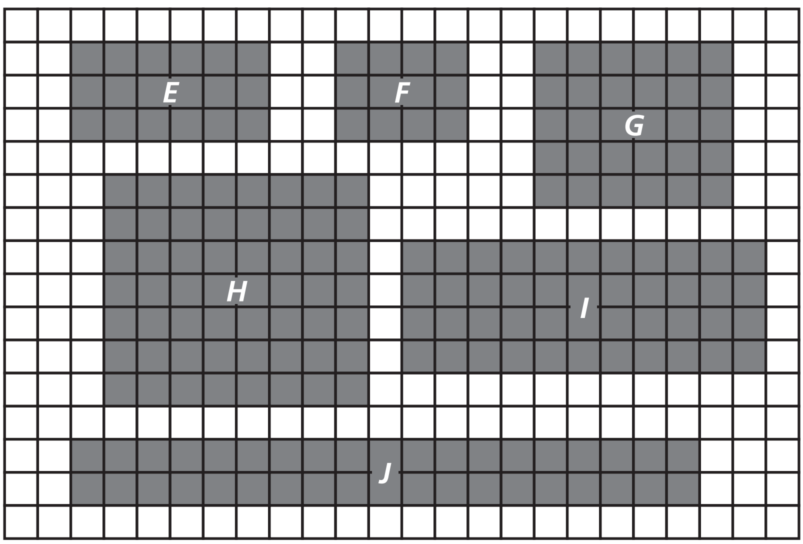
PROBLEM 3

Find the area and perimeter of the following trapezoid. For the area, try decomposing the figure into two recognizable shapes, then finding the two smaller areas to find the total area of the trapezoid. Then use the formula for the area of a trapezoid to compute the area and compare your results from the two methods.



EXERCISES

1. Calculate the area and perimeter of each rectangle on the following grid. Make a table like the one that follows, showing the length, width, area, and perimeter of each rectangle. Assume the length is the horizontal distance.

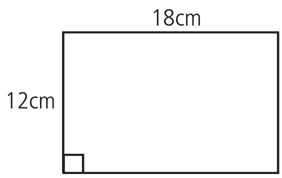


Rectangle	Length	Width	Area	Perimeter
E				
F				
G				
H				
I				
J				

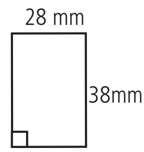
What patterns do you notice in the values in the table? Do you see any relationships between the four categories that you can state as a rule?

2. Find the area and perimeter of the following rectangles and squares. Refer to the area formulas for rectangles and squares.

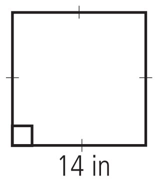
a.



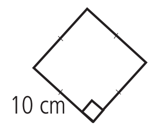
c.



b.

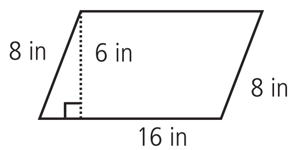


d.

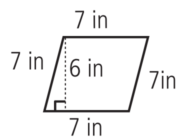


3. Find the area and perimeter of the following parallelograms.

a.

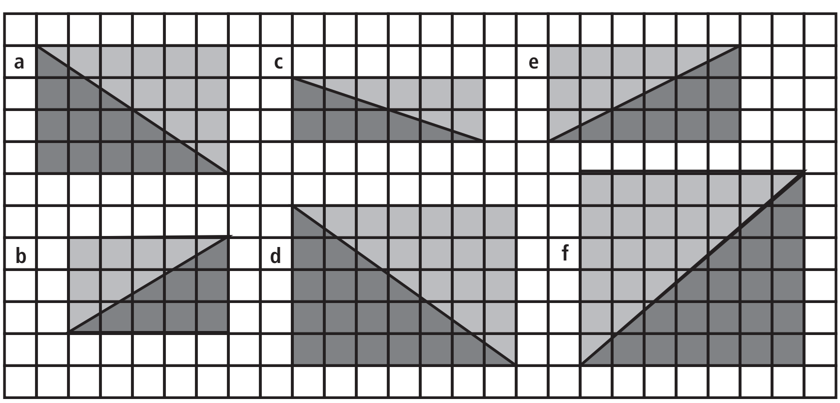


c.

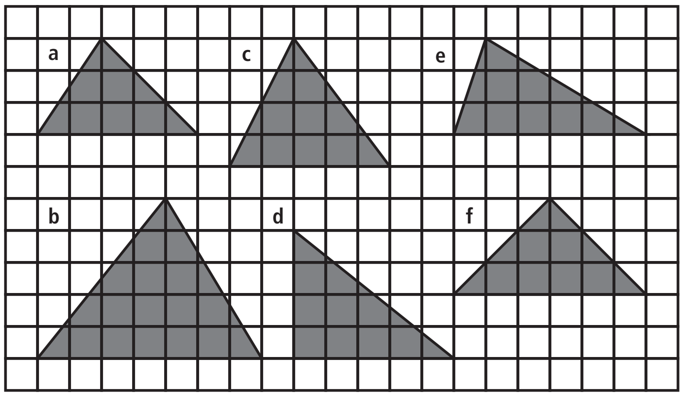


b.

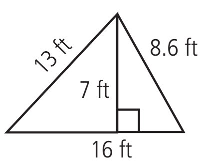
4. Calculate the area of the darkly shaded triangles below. How does the area of each rectangle relate to the area of the triangle inside it?

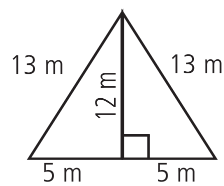


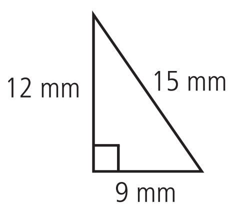
5. Calculate the areas of the triangles below using the area formula for a triangle.



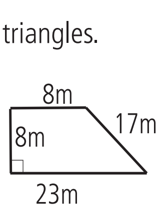
6. Find the area and perimeter of the following triangles.

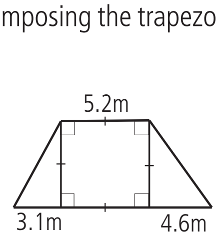
a.


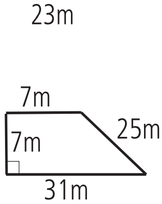
c.


b.


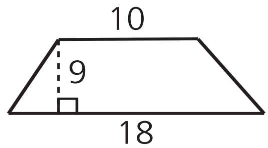
7. Find the area of the following trapezoids by decomposing the trapezoid into squares and triangles.

a.


c.


b.


8. Find the area of the following trapezoid by arranging a copy of the given trapezoid with the original to form a parallelogram.



9. The school district is going to buy a cover for the artificial turf on the district football field. What must they know about the field so they can order the correct size to cover the entire field?

For problems 10 and 11, draw a picture to match the information provided. Solve the problems and write your answer in a complete sentence.

10. Mrs. Moreno wants to put a border around the rectangular bulletin board in her classroom. The board is twice as long as it is wide. If the width of the board is 28 inches, how much border will she need?

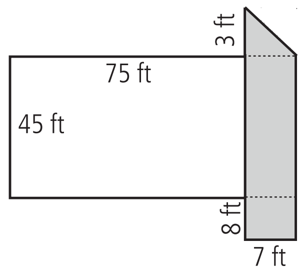
11. A regular hexagon has sides of length 4 inches. Determine the perimeter of the hexagon.

12. Using another sheet of grid paper, make and label as many different rectangles of area 36 square units as possible. Make a chart to organize the information about these rectangles. How is the perimeter of a rectangle related to the area of the rectangle?

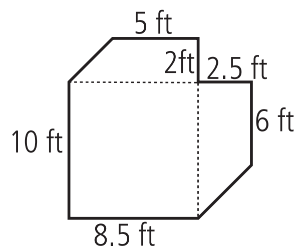
13. Draw as many different rectangles as you can that have perimeter 24 units.

14. A rectangular house has a square porch on the rear of the house as shown. Find the area of the porch, the house without the porch, and the house and the porch together.

15. A rectangular house has a porch on the rear of the house as shown. What is the area of the house and porch combined? Note, this picture is not drawn to scale.



16. A room has the following floor plan and dimensions.



a. Find the area of the room.

b. If the approximate perimeter is 43 feet and the walls in the room are 8 feet high, what is the total area of the walls?

c. It takes one gallon of paint to cover 200 square feet of wall. How many gallons will it take to paint the room?

Spiral Review:

17. List the fractions in order from least to greatest, representing the fractions as percents $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{3}{4}$

18. Which two expressions have the same value?

a. $3(2)^2 + 2(7 - 4)$

b. $2(6 - 2) + 4^2$

c. $5(4 - 1) + 3 + 6 \div 2$

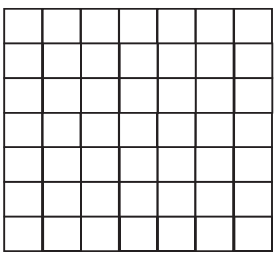
d. $6 + 2 \cdot 3^2$

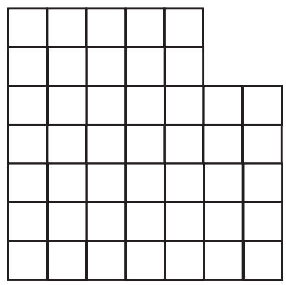
19. **Ingenuity:**

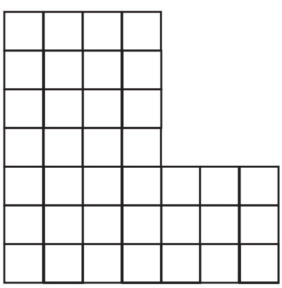
A three-foot-wide sidewalk (shaded part) has been built around a rectangular field, as shown in the figure below. (Note: the figure is not drawn to scale.) If the total area of the sidewalk is 1743 square feet, what is the outside perimeter of the sidewalk?

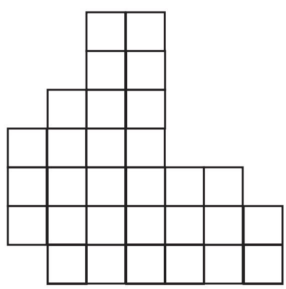
20. **Investigation:**

Find the perimeter of each of the following figures:

a. 

c. 

b. 

d. 

e. Create a figure that has perimeter 28 and area 19. Is there only one possible figure?

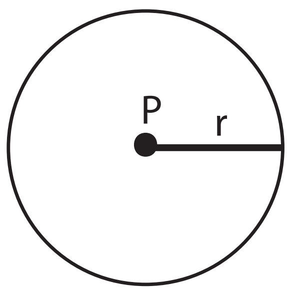
SECTION 9.5
CIRCLES

Everyone has seen circles of various sizes, but what is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word "circle."

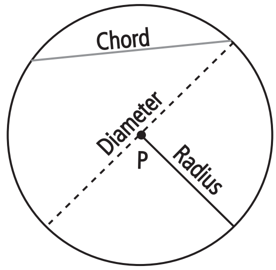
EXPLORATION 1

How do you draw a circle? Once you have drawn a circle, write directions that someone could use to draw a circle. Then state your definition for a circle.

In general, one way to draw a circle is by marking a point ***P***, called the **center** of the circle. Then take a length of string, ***r*** units long, place one end of the string at point ***P*** and attach a pencil to the other end. Stretch the string to its full length and draw the circle with the pencil. Each point on the circle is r units from ***P***. A circle is often named by its center. In this case, we have circle ***P***.



Circle ***P*** is made up of all points that are distance ***r*** from ***P***. The fixed distance ***r*** from the center ***P*** to a point on the circle is called the **radius** of the circle. All line segments that can be drawn from one point of the circle to another are called **chords**. A circle can have many chords. A line segment connecting two points on the circle and passing through the center ***P*** is called a **diameter**. The diameter is a special chord because it passes through the circle's center. The length of the diameter is equal to the length of 2 radii. Radii is the plural of radius. Notice that the diameter cuts the circle in half, forming two **semicircles**. Also notice that it is the longest line segment that can be drawn from one point on the circle to another. Are all diameters considered chords? Are all chords considered diameters?



The distance around the circle is called the **circumference** and is like the perimeter of a polygon.

Use the circles on the Circle Handout. Using a piece of string, carefully measure the radius and circumference. Place the circle on grid paper, measure the radius, diameter and circumference, and estimate the area. Use the information to complete the table below:

Circle	Radius	Diameter	Circumference
A			
B			
C			
D			
E			

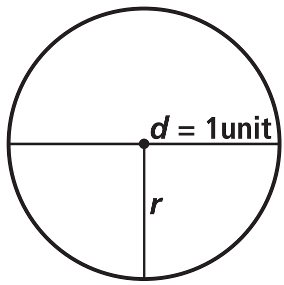
Do you notice a relationship between the radius and the diameter? Using the variable d to denote the length of the diameter, express the diameter in terms of the radius r.

What is the relationship between the circumference of a circle, C, and its diameter, d? Add a column to your table and compute the ratio of the circumference to its diameter for the five circles. What do you notice about the ratios? The ratio you computed approximates the exact ratio of the circle's circumference to its diameter, the number pi, written as the Greek letter π. This ratio, $\frac{C}{d} = \pi$, of the circumference to its diameter is the same regardless of the size of the circle.

FORMULA 9.6: CIRCUMFERENCE OF A CIRCLE
The formula for the circumference of a circle with radius r and diameter d is $C = 2\pi r$ or $C = \pi d$.

In either version of the circumference formula, π can be thought of as a constant rate that is multiplied by the diameter of the circle.

Look at a circle with diameter 1 unit.



Remember, a unit can be any length you choose. Take out a string or tape ruler and measure the circumference. The circumference has a length of π units. The number π is approximately equal to the fraction $\frac{22}{7}$ or the decimal 3.14. These two approximations are not exactly equal to π. However, the two approximate values are very close to the actual value of π, which begins 3.1415926….

What happens when the circle is scaled by a factor of 2, making the radius twice as large? What happens to the diameter? What happens to the circumference? What pattern do you notice when you measure the scaled circumference and compare it to the original circumference? Just as with a square, when scaled by a factor of 2, the perimeter, or circumference, doubles. The ratio $\frac{C}{d}$ of the circumference to the diameter remains the same, π.

In summary, call C the circumference of a circle, d the diameter, and r the radius. Then,

$$\begin{aligned} d &= 2 \cdot r \\ &= 2r \end{aligned} \qquad \text{and} \qquad \begin{aligned} C &= \pi \cdot d \\ &= \pi d \end{aligned} \qquad \text{or} \qquad \begin{aligned} C &= \pi \cdot 2 \cdot r \\ &= 2 \cdot \pi \cdot r \\ &= 2\pi r \end{aligned}$$

EXAMPLE 1

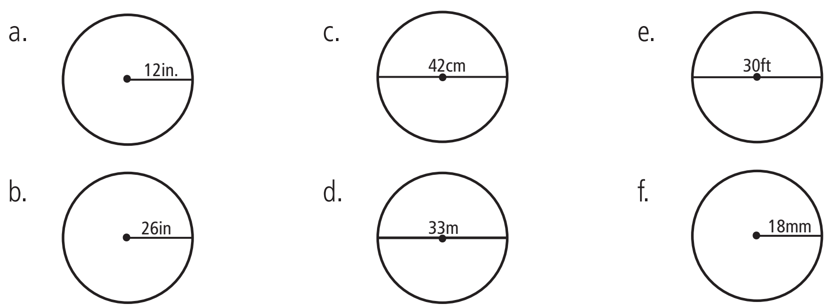
The diameter of a circle is 8 ft. Find the circumference. Use $\frac{22}{7}$ as an approximation for π. Round your answer to the nearest foot.

SOLUTION

Circumference of a circle is $C = \pi d$ and $d = 8$, so $C = 8\pi$. Using $\frac{22}{7}$ for π, we have $C = \frac{22}{7} \cdot 8 = \frac{176}{7} = 25\frac{1}{7}$ ft. Rounding to the nearest foot, we have $C = 25$ ft.

PROBLEM 1

Find the circumference of the following circles with the given radius or diameter. First, write your answer in terms of π. Then find the circumference using $\frac{22}{7}$ as an approximation for π.



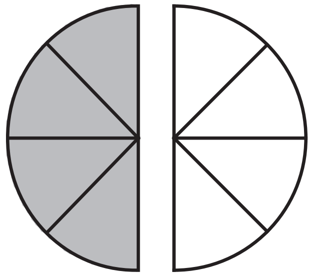
PROBLEM 2

The radius of a circle is 6 cm. Find the circumference of this circle.

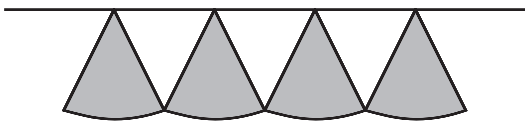
While circumference is the special term for the perimeter of a circle, its area has no special name other than area. The following exploration gives us a way to formulate the area of a circle.

EXPLORATION 3

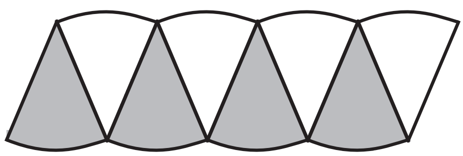
What is the area A of a circle whose radius is 1? Draw a circle with radius 1 and circumference 2π. Cut this circle in half. Then cut each half into many small pie slices of equal size:



Take the slices from one half of the circle and lay the points of the slices along a line:

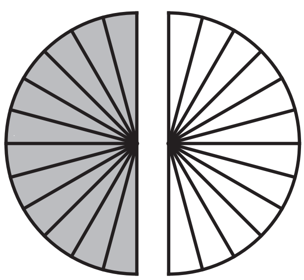


Fill in the spaces with the other half of the circle.

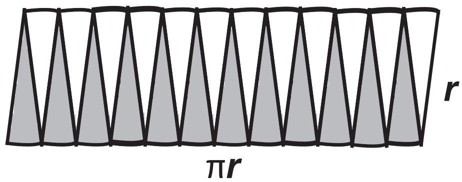


The shape looks a little like a parallelogram. The more slices, the closer the shape is to a rectangle. If this cutting process continued infinitely, the area of the circle with radius 1 would approximate the area of the rectangle with length π and width 1.

What happens to the area of the circle when its radius is a number r? One way to visualize this is to create slices in the circle with radius r, like the previous process when the radius was equal to 1.



Cut the circle into two equal semicircles as you did in the unit circle and fit one semicircle into the other semicircle.



What is the length of this rectangular shape? What is its width? What is the area of the rectangle?

In this rectangle the length is $\pi \cdot \boldsymbol{r}$, which is half the circumference $2 \cdot \pi \cdot \boldsymbol{r}$ and the width is $\boldsymbol{r}$. The area of the rectangle is length times width, so we have $\pi\, \boldsymbol{r} \cdot \boldsymbol{r}$ or $\pi\, \boldsymbol{r}^2$. Any area is measured in square units, so if r is measured in inches, then $\pi\, \boldsymbol{r}^2$ is measured in square inches. The formula for the area of a circle A with radius r can be summarized as:

FORMULA 9.7: AREA OF A CIRCLE
The area of a circle with radius $\boldsymbol{r}$ is $\boldsymbol{A} = \pi \boldsymbol{r}^2$ square units.

PROBLEM 3

Write an expression for the area of a circle with diameter 6 cm. First, give the answer in terms of π and then use $\frac{22}{7}$ as an approximate value for π and determine the area to the nearest whole number.

EXAMPLE 2

A circle has radius 4 inches.

a. Find the exact circumference of the circle.

b. Approximate the circumference using $\frac{22}{7}$ for π to the nearest tenth of an inch.

c. Find the exact area of the circle.

d. Approximate the area using 3.14 for π to the nearest hundredth of an inch.

SOLUTION

a. Apply the above formulas. The circumference is $C = 2\pi \cdot 4$ inches $= 8\pi$ inches.

b. If we approximate π by $\frac{22}{7}$, then the circumference $= 2 \cdot \frac{22}{7} \cdot 4 = \frac{176}{7} = 25\frac{1}{7}$, which is approximately 25.1 square inches.

c. Apply the area formula for a circle, $A = \pi r^2$. $A = \pi \cdot 4^2 = 16\,\pi$ in^2.

d. If we approximate π by 3.14, then the area $= 3.14 \cdot 4^2 = 3.14 \cdot 16 = 50.24$ square inches square inches.

In mathematics, a product that includes a **constant**, or number, times a variable is written with the constant first. For example, in the product of a constant 2 and a variable x, 2 is called the **coefficient** of the product $2x$. Even though π is a constant, not a variable, the product of π and a constant like 16 is usually written 16π.

PROBLEM 4

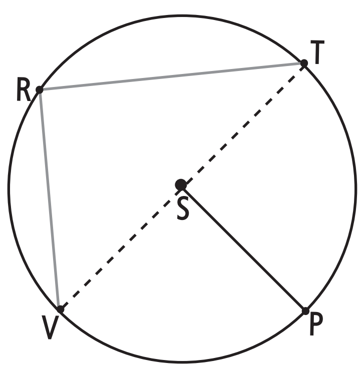
A circle has circumference 12 inches. Use 3 for π to approximate:

a. The diameter of this circle.

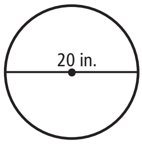
b. The radius of this circle.

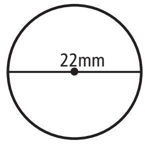
EXERCISES

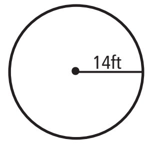
1. Use the diagram below to label the parts of the circle. Remember to use the correct labeling.

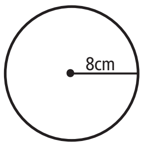


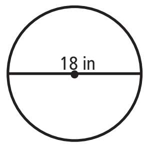
2. Find the circumference of each circle below. Use the circumference formula that corresponds to the circle part shown. Find the exact circumference and then approximate using $\pi \approx 3.14$. Remember to include your units.

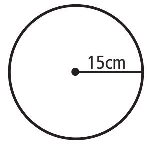
a. 

c. 

e. 

b. 

d. 

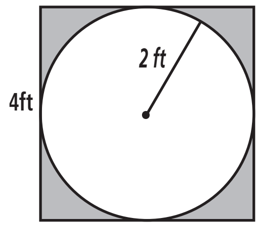
f. 

3. Use the circles from Exercise 2 to find the area of each circle. Find the exact area and then approximate using $\pi \approx 3.14$. (Think carefully about the circle parts given and the part needed for the formula.) Remember to include your units.

Use the formulas for area and circumference of a circle to answer the following questions. For approximations, use $\pi \approx \frac{22}{7}$. Label your answers appropriately.

4. The diameter of a circle is 30 mm.
   a. Write an equation you would need to find the circle's circumference.
   b. What is the circumference of the circle?
5. The spokes of a bike tire are 21 inches. What is the circumference of the bike tire?
6. A medium pizza has a diameter of 10 inches. What is the area of the pizza?
7. The large circular clock at the top of the church tower has radius of 4 feet. What formula would you need to find the area of the clock?
8. Local artists were asked to create a circular mural on the wall of the new library. The circle has diameter of 12 feet.
   a. The arc of the circle will have a rubber edging. How much rubber edging will they need?
   b. The circle will be covered with crystallized paint. How much of the wall will they need to paint with this special paint?

9. A circle with radius 2 ft lies inside a square with each side 4 ft long.

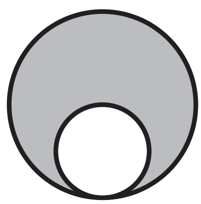


a. What is the area of the circle?

b. What is the area of the square?

c. Find the shaded area inside the box but outside the circle.

10. A circle with radius 3 in. is contained in a larger circle with radius 6 in. They touch at the bottom.



What is the shaded area outside the smaller circle and inside the larger circle?

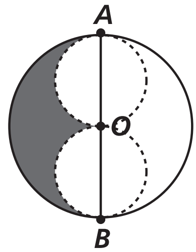
Spiral Review:

11. Casey is C years old. Melissa's age, M, is two times Casey's age. Write an expression representing Melissa's age in terms of Casey's age. If Casey is 10 years old, how old is Melissa?

12. Susan's sugar cookie recipe requires 1 egg for 36 cookies. How many eggs does Susan need for 6 dozen cookies?

13. **Ingenuity:**

In the following figure, $\overline{AB}$ is the diameter of the larger circle and O is the center of the larger circle. $\overline{AO}$ and $\overline{OB}$ are diameters of the two smaller circles. The length of the diameter $\overline{AB}$ of the larger circle is 6 cm. What is the shaded area?



14. **Investigation:**

    The earth's radius is approximately 3960 miles. For the purposes of this problem, we will assume that the earth is a perfect sphere without any irregularities. Suppose we make a belt that is just large enough to encircle the earth's equator.

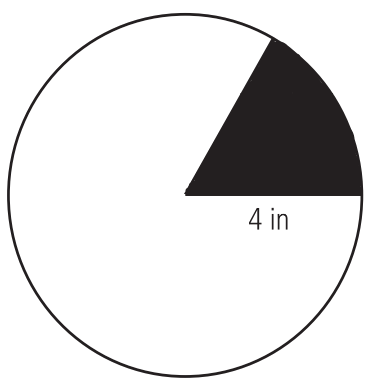
    a. What is the approximate length of this belt?

    b. Guess the answer to the following question: suppose we now make a belt that is 6 feet longer than the original belt. We then encircle the equator with this belt, and hold it over the original belt so that the amount of room under the belt is constant all the way around the equator. How far above the ground will the new belt be?

    c. Once you have guessed the answer to this question, use your knowledge of geometry to get a definitive answer.

15. **Challenge:**

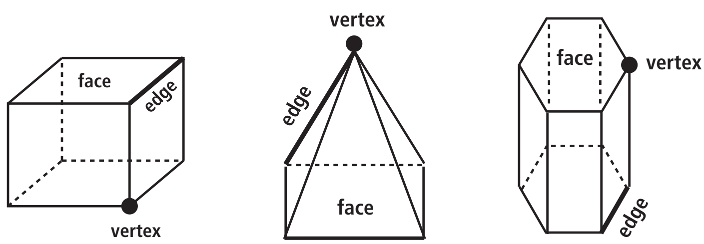
    A **sector** of a circle is the part of the interior of the circle between two radii, like a slice of pie. A circle has radius 4 inches, and two radii make a sector with a 60° angle. Find the exact area of the sector these radii enclose.



SECTION 9.6
THREE-DIMENSIONAL SHAPES

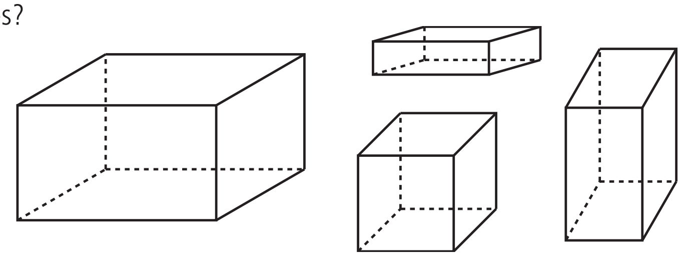
In the previous sections, you studied shapes in two dimensions: triangles, squares, rectangles, parallelograms, trapezoids, and circles. In this section, you will learn about three-dimensional shapes. Some of these shapes appear as familiar objects like beach balls, blocks, paper towel rolls, or cardboard boxes. In this section, you will learn some mathematical terminology and ways to measure volume. We will start with the easiest three-dimensional shapes.

A basic kind of three-dimensional figure is called a polyhedron. This word comes from the Greek words *poly*, meaning "many," and *hedra* meaning "faces." So a **polyhedron** is a three-dimensional figure with many faces. The plural form of polyhedron is polyhedra. Each **face** of a polyhedron is a polygon. The vertices of the polygons are the **vertices** of the polyhedron. The **edges** are the borders of the faces, which are also the line segments that join the vertices.



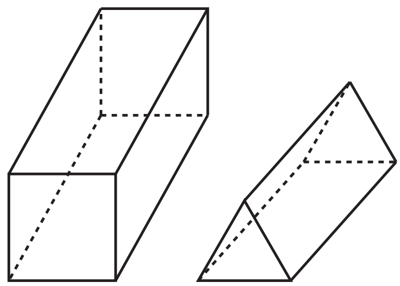
A box shape is an example of the most common type of polyhedron called a **prism**. In a prism, two of the faces, called **bases**, are parallel and congruent. Prisms are named by their bases. In the case of a box, the polyhedron is a **rectangular prism**, because the bases are rectangles. The faces that connect the two bases are parallelograms, and in this case rectangles. They are called **lateral faces**. In fact, although the faces of prisms are not always rectangles, in rectangular prisms all the faces are rectangles. How many total faces are there in rectangular prisms? How many are bases? How many are lateral faces?

9.6

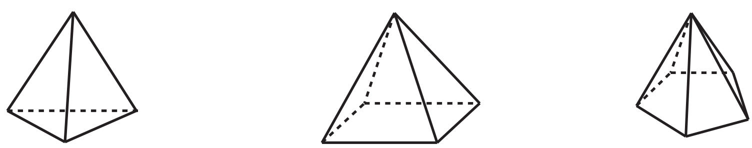


PROBLEM 1

Pictured below are three different prisms. What shape are the base faces? What shape do the non-base or lateral faces have? Give each prism an appropriate name.



Another type of polyhedron is called a pyramid. A **pyramid** consists of a polygon for a base and lateral faces that are triangles that meet at a point called the **apex**.



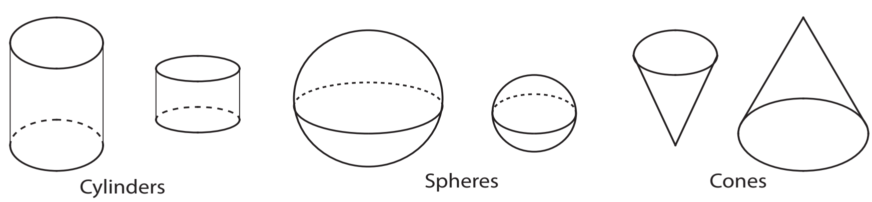
The pyramids are named by their base. Identify the names of the pyramids above.

PROBLEM 2

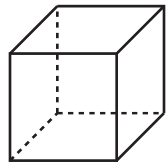
You found that rectangular prisms have 6 faces. How many edges and how many vertices does a rectangular prism have? Consider a rectangular pyramid. How many faces, edges, and vertices does it have?

Other common three-dimensional shapes include cones, cylinders, and spheres.

Notice that **cones** are related to pyramids but with a circular base. A **cylinder** in a similar way is related to a prism with circular bases. A **sphere** is a three-dimensional version of a circle, a figure formed by all points a fixed distance from a fixed point, called a center. Examples of cones, cylinders, and spheres are below:

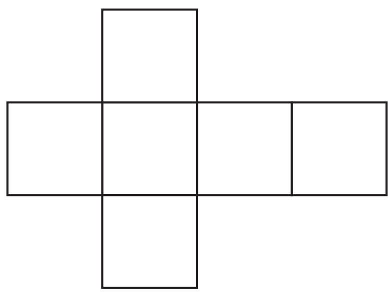


The simplest three-dimensional shape to measure is a **cube**, two parallel congruent square bases connected by four perpendicular congruent squares.



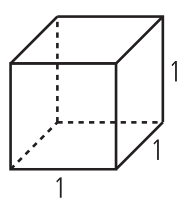
A cube is a regular rectangular prism. A cube is **regular** because each of its faces has equal sides and angles. In other words, all the faces are congruent to each other. All the cube's faces are squares.

A cube one unit long, one unit wide and one unit high has a volume of one cubic unit. Recall that the area of a two-dimensional figure is measured by the number of unit squares needed to cover it.



Just as you can cut a cardboard box open by cutting along some of the edges, we can cut a cube along some of the edges and flatten the cube to create a **net** similar to the figure above. How many square units is the net? Can you think of other ways in which the cube can be cut to create a net? You will study more about nets when you examine surface area.

The **volume** of a three-dimensional shape is measured by the number of unit cubes needed to fill it. For example, if each side of a cube is 1 foot long, the volume of the cube is 1 cubic foot, written 1 cu. ft. or 1 ft^3.



EXPLORATION 1

How many centimeter cubes (also called cubic centimeters) are there in a cube that is 2 cm long on each side? How many cubic cm are there in a cube that is 3 cm long on each side?

Use centimeter cubes to cover a rectangular area that is of length 2 cm and width 2 cm. Discuss the number of cubes used. Record your data as follows:

Area of first layer = 2 cm × 2 cm = 4 sq. cm.

On top of the first layer, which is called the base, add an identical layer of centimeter cubes. How many total cubes are used? To find the answer you can take the answer for the number of cubes used in the base and multiply it by the number of layers. We now have 2 cm × 2 cm × 2 cm = 8 cubic centimeters. Common abbreviations for cubic centimeters are cu cm and cm^3.

Now investigate using a cube with sides of 3 cm.

FORMULA 9.8: VOLUME OF A CUBE
The volume of a cube with each side of length s inches is s^3 cubic inches. $V = s^3$.

EXPLORATION 2

How many cubic inches are there in one cubic foot?

In order to think about this problem, let's begin by reviewing how to change units in computing areas. A square that is one foot long on each side has area one square foot. Thinking in terms of smaller units, each side of the square foot is 12 inches long. Using this ratio of 1 foot to 12 inches, we have:

$$\begin{aligned} 1 \text{ square foot} &= (1 \text{ foot})(1 \text{ foot}) \\ &= (12 \text{ inches})(12 \text{ inches}) \\ &= 144 \text{ square inches.} \end{aligned}$$

Using the same pattern,

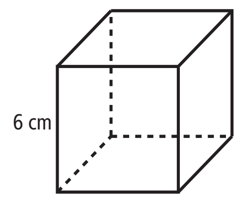
$$\begin{aligned} 1 \text{ cubic foot} &= (1 \text{ foot})(1 \text{ foot})(1 \text{ foot}) \\ &= (12 \text{ inches})(12 \text{ inches})(12 \text{ inches}) \\ &= 1728 \text{ cubic inches.} \end{aligned}$$

Another way to compute the volume of this cube is to use the formula you just learned. Since one foot = 12 inches, each side of the cube is 12 inches long. The volume of your cube then is $12^3 = (12)(12)(12) = 1728$ cubic inches.

In three-dimensions, conversions to smaller units sometimes make volumes seem much larger even though the shape and size have not changed at all!

PROBLEM 3

Find the volume of the cube with sides of length 6 cm.

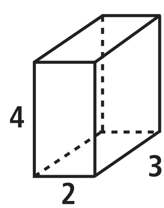


EXAMPLE 1

Draw a rectangular prism with edges that are 2, 3, and 4 units long. Find its volume.

SOLUTION

The first step is to draw the two base rectangles. For example, make the bases 2×3 rectangles. Place these rectangles 4 units apart to make the height of the box.



How many unit cubes does it take to fill the box? The 2×3 base rectangles have area 6 square units so there are 6 cubes in the first layer. Because the rectangular prism is

4 units in height, we have a second layer of 6 cubic units, a third layer of 6 cubic units, and finally a fourth layer of 6 cubic units.

In other words, to find the volume, multiply the area of the base, $2 \cdot 3 = 6$, by the height 4 to get $6 \cdot 4 = 24$ cubic units.

In general the volume of a prism is equal to the area of the base times the height. This formula is often written as $V = B \cdot h = Bh$. The variable B is the area of the base. This general formula is true for any prism, regardless of the shape of the base, whether it is a rectangle, a triangle, a hexagon, or any polygon.

FORMULA 9.9: VOLUME OF A PRISM
The volume formula for a prism can be written by volume = area of base · height $V = Bh$ with B = area of the base and h = height of the prism. In particular, for a rectangular prism, $B = lw$ so $V = lwh$ where l is length and w is width

All the sides of a cube are equal, that is $l = w = h$. If we call the side of a cube s, then the volume formula for a cube based on the volume formula for a rectangular prism becomes $V = Bh = lwh = s \cdot s \cdot s = s^3$. Volume of a cube is given by $V = s^3$.

Remember, the volume in cubic inches of three-dimensional shapes is the number of one-inch cubes it takes to fill the shape exactly. Because some shapes cannot be easily filled with one-inch cubes, the volume might be a fraction or a decimal part of a unit cube. As in the case of prisms, you can examine volumes and arrive at formulas that will make the computation much easier than counting blocks every time.

EXAMPLE 2

Determine the volume of a cube that is 2 units long, 2 units wide, and 2 units high. Answer in cubic units. Then double each of the cube's dimensions. Predict the volume of this new cube. Verify your prediction by calculating the volume of the larger cube.

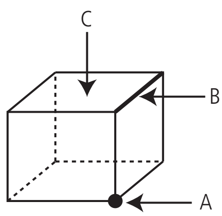
Make a prediction about what happens to the volume of any cube when its dimensions are doubled. What is the relationship of the volume of the original cube to the volume of the enlarged cube?

SOLUTION

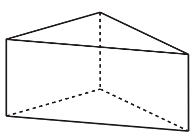
The original cube's volume is 8 cubic units, the product of the area of the base ***B***, which is $2 \cdot 2 = 4$ square units, and the height, which is 2 units. When each of the cube's dimensions is doubled, the volume of the resulting larger cube is (4 units)(4 units)(4 units) = 64 cubic units. The original volume was 8 cubic units. The new volume is 64 cubic units which is 8 times the original volume.

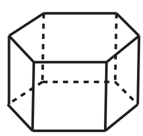
EXERCISES

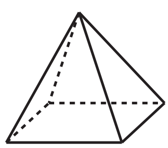
1. Identify the parts of the polyhedron shown below.

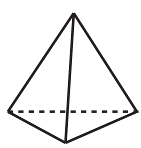


2. Name the following polyhedra. Identify the number of faces, edges, and vertices in each.

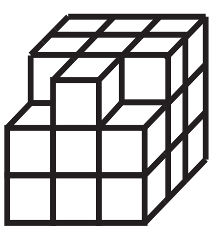
a. 

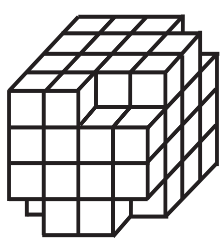
c. 

b. 

d. 

3. Find the volume of the shapes below.

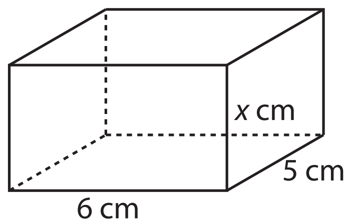
a. 

b. 

4. What is the volume of a rectangular prism with edges 3 ft, 4 ft, and 5 ft?

5. What is the volume of a cube with side lengths of 6 inches?

6. A rectangular prism has volume 210 cm^3 and edges with lengths 5 cm, 6 cm, and x cm. What is the value of x? Refer to the picture below.



7. Consider the following rectangular prisms.

   Prism A with dimensions: 15 inches × 8 inches ×1 inch.

   Prism B with dimensions: 10 inches × 10 inches × 3 inches.

   Prism C with dimensions: 7 inches × 7 inches × 7 inches.

   Which is the biggest rectangular prism? Explain your reasoning.

8. A swimming pool has dimensions 35 yards × 10 yards × 2 yards. Determine the volume of the pool.

9. The ice cream will be stored in a cube-shaped container with sides of length 11 inches. What is the volume of the container?

10. If a cube has a volume of 125 cubic inches, what is the length of the cube's sides?

11. The table below shows the length and area of several rectangles. All these rectangles have a width of 5 cm. Which of the following equations best represents the relationship between the length, L, and area A of these rectangles?

Rectangles

Length (L)	Area (A) cm2
4	20
6	30
9	45
11	55

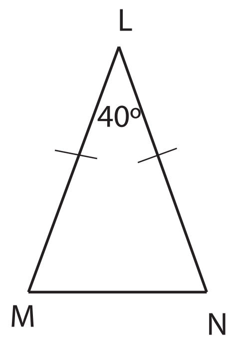
a. $A = \frac{L}{5}$

b. $A = L^2$

c. $A = 5L$

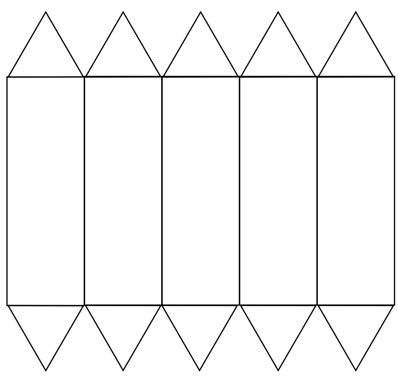
d. $A = L + 5$

12. Triangle LMN is an isosceles triangle. If the measure of angle L is 40º, what is the measure of angle N?



13. **Ingenuity:**

Suppose we fold the faces in the diagram below and glue the faces along their edges to form a polyhedron.



a. How many vertices, how many edges, and how many faces will the resulting polyhedron have?

b. The figure above consists of five pieces, each of which is a rectangle with triangles on both ends. Suppose that the figure instead had six such pieces. What would the figure look like if we glued the faces together to form a polyhedron?

14. **Investigation:**

A certain $3 \times 3 \times 3$ cube is made of 27 unit cubes. The surface of the $3 \times 3 \times 3$ cube is then painted red.

a. How many of the 27 unit cubes have no faces painted red?

b. How many of the 27 unit cubes have exactly one face painted red?

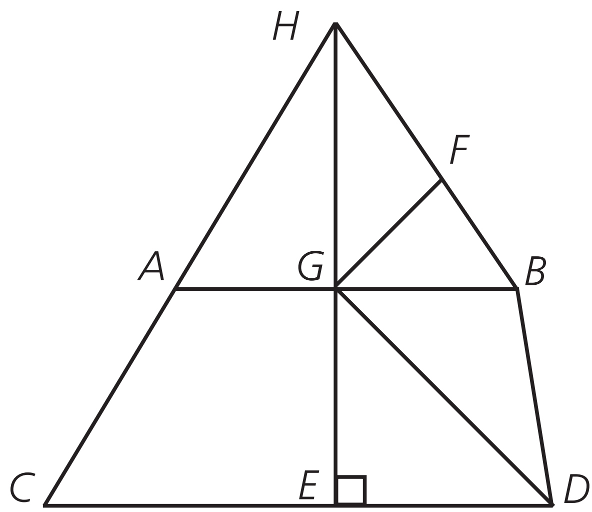
c. How many of the 27 unit cubes have exactly two faces painted red?

d. How many of the 27 unit cubes have exactly three faces painted red?

e. How would the answers to parts a through d change if we replaced the $3 \times 3 \times 3$ cube with a $4 \times 4 \times 4$ cube made using 64 unit cubes?

SECTION 9.R
CHAPTER REVIEW

1. Answer the following question using the figure below, assuming $\overline{AB}$ and $\overline{CD}$ are parallel.



   a. Name 2 right angles

   b. Name 2 obtuse angles

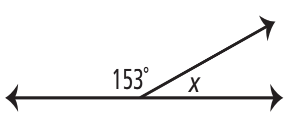
   c. Name 2 acute angles

   d. Name 1 pair of Complementary angles

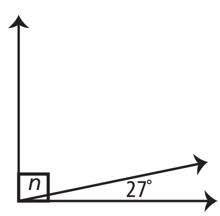
   e. Name 1 pair of Supplementary angles

2. Find the measures of the indicated angles:

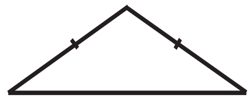
   a.

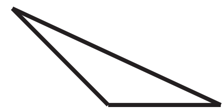


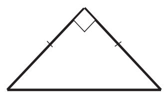
   b.

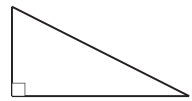


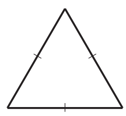
3. Classify each of the following triangles by its sides and angles:

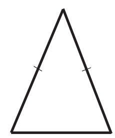
a. 

d. 

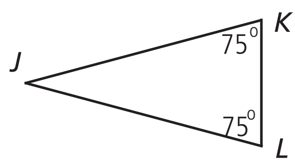
b. 

e. 

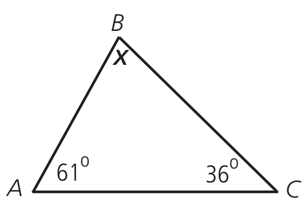
c. 

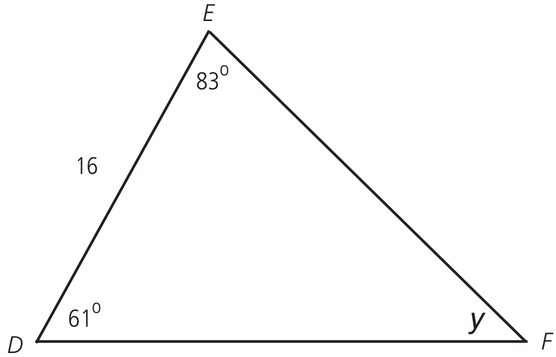
f. 

4. In the figure below, fill in the missing angle measure using the information you are given.



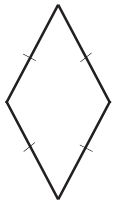
5. Find the measure of the missing angles ***x*** and ***y*** in the triangles below.

a. 

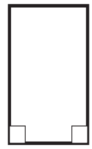
b. 

6. Classify each quadrilateral. Give all names that apply.

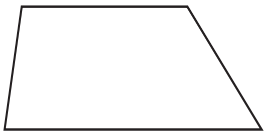
a.



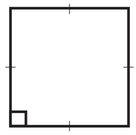
d



b.

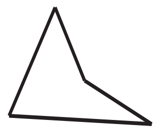


e.



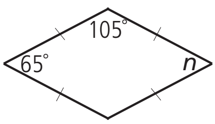
c.

f.

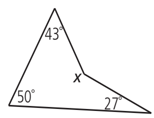


7. Find the missing angle or angles in each of the following quadrilaterals:

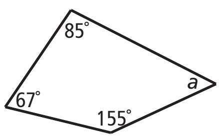
a.



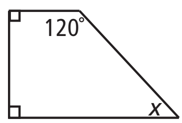
c.



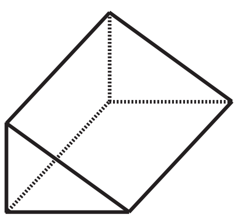
b.



d.

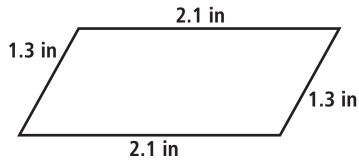


8. We have a right triangular prism as pictured. How many faces, edges, and vertices does it have?

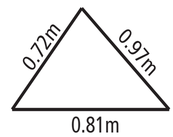


9. Find the perimeter of each figure. Remember to label your answer with the appropriate unit of measure.

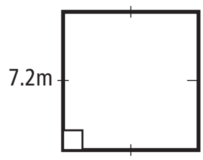
a.



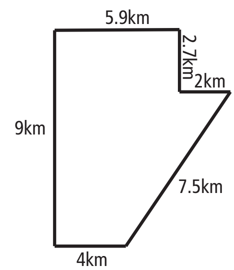
d.



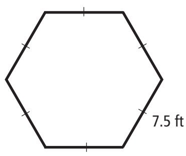
b.



e.

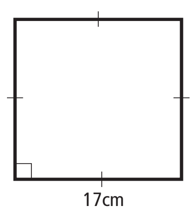


c.

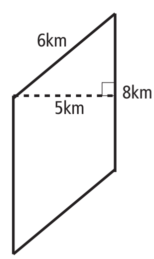


10. Find the area for each quadrilateral. Label the unit of measure appropriately.

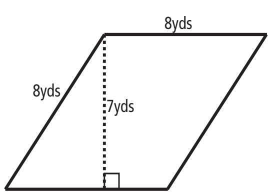
a.



c.



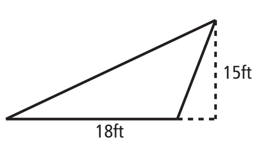
b.



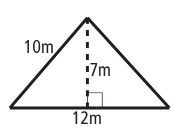
d.

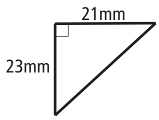
11. Find the area for each triangle. Label the units of measure appropriately.

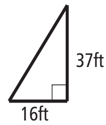
a.



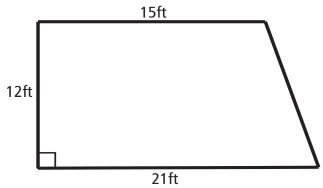
b.

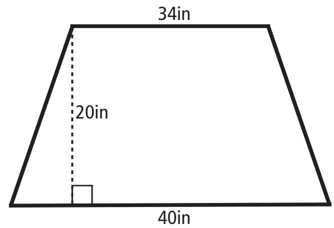


c. 

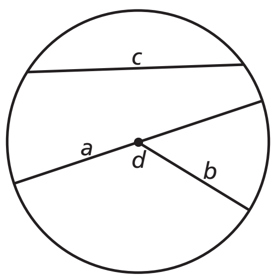
d. 

12. Find the area of each trapezoid. Label the units of measure appropriately.

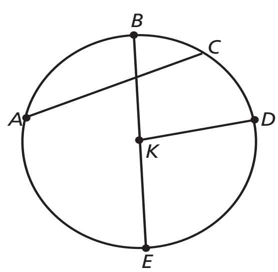
a. 

b. 

13. Name the parts of the circle in the diagram below (For example, the center of the circle is at *d*):



14. Identify the following from the diagram below (The center of the circle is at *K*):



a. Name the diameter

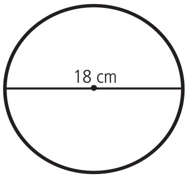
b. Name all the radii.

c. Name all the chords.

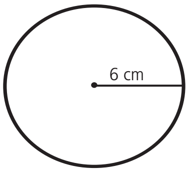
d. Name the circle.

15. Find the area and circumference for each circle. Use $\pi = \frac{22}{7}$.

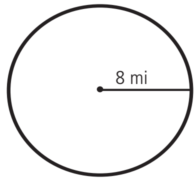
a.



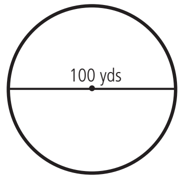
b.



c.

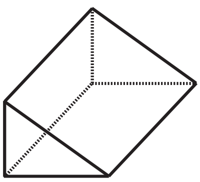


d.

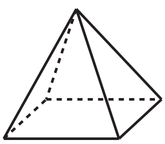


16. Name each 3-D shape. Tell the number of faces, edges, and vertices in each.

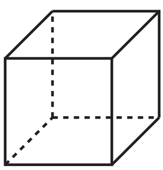
a.



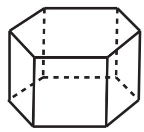
b.



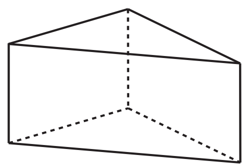
c.



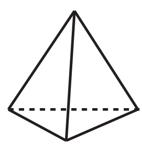
d.



e.

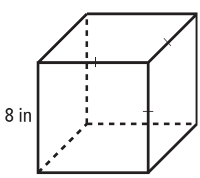


f.

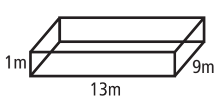


17. Find the volume of each figure. Label the unit of measure appropriately. Assume figures a and d are cubes.

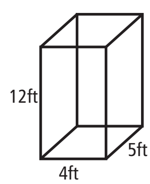
a.



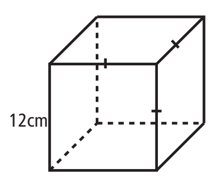
b.



c.



d.



DATA ANALYSIS 10

SECTION 10.1
MEASURES OF CENTRAL TENDENCY

Sets are useful for grouping interesting and related numbers. One such set is the heights of all of the people in your class. A set like this, in most cases, includes various heights. In order to use these sets, we need to analyze the numbers, or data, in context. The first step in data analysis, the process of making sense of a set, is collecting data. In **data analysis,** the idea of a data set is slightly different from that of a set. Unlike regular sets, data sets can have repetition of elements, and the order or arrangement matters.

Ms. Dibrell's math class is interested in data sets. Ms. Dibrell has each student record their name, height in inches, and age in months in order to create a collection of numbers that represents her class.

Name	Height (in)	Age (months)
Sophia	52	113
Rhonda	51	112
Edna	57	112
Danette	61	115
Hesam	55	117
Eloi	62	110
Vanessa	58	113
Michelle	60	108
Mari	58	125
Calvin	56	129
Moises	57	124
Amanda	57	120
Hannah	55	131
Tricia	55	129

Name	Height (in)	Age (months)
Kristen	57	130
Max	52	135
Jim	50	142
Karen	57	136
Diane	49	138
Tankai	58	138
Oscar	51	137
Jenny	60	138
Bence	59	142
Pat	53	134
Teri	59	135
Sally	57	139
Will	57	140

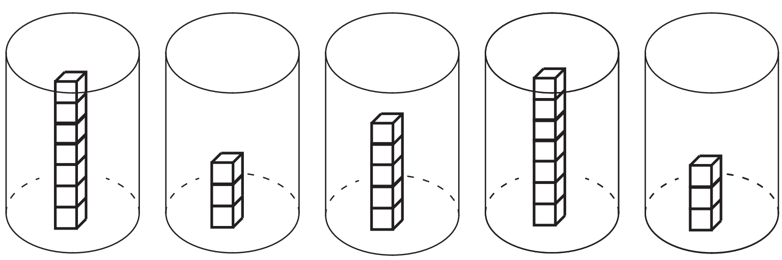
Ms. Dibrell points out that the entire collection of numbers is called the **data** and each individual piece of information is called a **data point**. Data is plural for datum.

A major goal of data analysis is to find a simple measure of the data, called a **measure of central tendency**, that summarizes or represents, in a general way, the majority of the data. There are three common measures of central tendency: the mean, median, and mode. The mean, median, and mode are different ways to identify the location or center of the data. We are also interested in how spread out our data is. The **range**, the difference between the largest and smallest values of the data, provides a simple measure of how much the data varies.

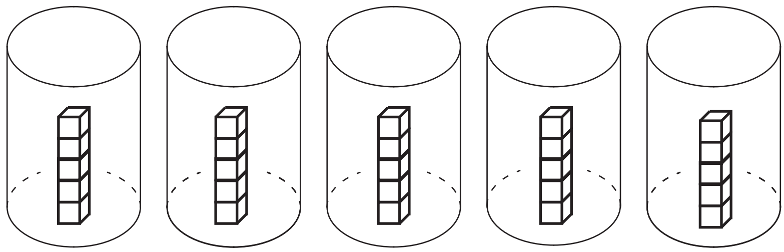
EXPLORATION 1

Measure the height of each person in your class in inches and record their name, age in months, and height in inches in a table like the one from Ms. Dibrell's class. Your own class will have different results. Try to find ways to summarize the information in the table so that you can share your results with a friend without showing her the whole table. Would your strategy still work if there were 100 people in the survey? 1000 people?

The **mean**, also called the **arithmetic mean** or **average**, is the sum of all the data values divided by the number of data points. For a visual example, suppose we have five containers, each containing a certain number of blocks:



These data can be grouped into a data set: {7, 3, 5, 7, 3}. We will notice the importance of the order of arrangement and the repetition. There are 25 blocks total. The *mean* number of blocks in a container is the number of blocks each container has if these 25 blocks are distributed evenly among the 5 containers: $\frac{25}{5} = 5$.



EXAMPLE 1

Find the mean of the following data set.

{91, 100, 83, 76, 37, 98}

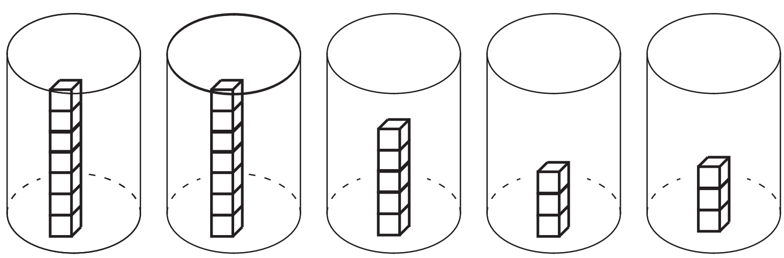
SOLUTION

Add the data points. Divide the sum by the number of data points.

Sum = 91 + 100 + 83 + 76 + 37 + 98 = 485.

Mean = 485 ÷ 6 = 80.83

The **median** is the value of the middle data point when the values are arranged in numerical order. If the data set has an even number of data points, the median is the average of the two middle values. To find the median value for the container example, order the data, with the largest number of blocks first and the smallest number last:



The ***median*** is the number of blocks in the middle, or third container ***with respect to the sorted ordering***. The median is a helpful measure of central tendency because half of the values are less than or equal to the median and the other half of the values are greater than or equal to it.

EXAMPLE 2

Find the median and range of the following data set.

{91, 100, 83, 76, 37, 98}

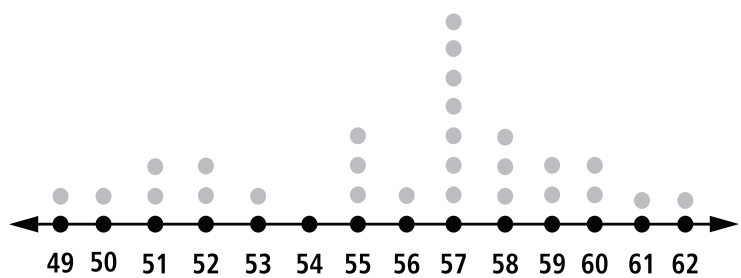
SOLUTION

Arrange the data points in decreasing order

100, 98, 91, 83, 76, 37

Because there are an even number of data points, take the middle two points, 91 and 83 and find their mean. $\frac{91+83}{2} = \frac{174}{2} = 87$. The median for the data set is 87.

Frequency is the number of times a data point appears in a data set. For example, if there are 4 people in the class who are 56 inches tall, then the frequency of the height 56 inches in the class is 4. The **mode** is the value or element that occurs the most often or with the highest frequency in the data set. One way to illustrate the frequency of a relatively small data set is to use a **dot** or **line plot**. The dot plot below uses the data set from Exploration 1.



In Exploration 1 the mode in our data is 57 because it appears 7 times, or the most times in the data. What is the range for this data set? The dot plot is helpful in seeing the largest and smallest values. The range for Exploration 1 is the difference between the largest value of 62 and smallest value of 49 in the data set. The range is 62 – 49 = 13 for this data set.

A set of data can have more than one mode. The modes for the containers of blocks discussion earlier are 3 and 7. Sometimes a set of data has no repeated data points. What would the mode be in such a data set? If you said no mode, you were correct. It would be incorrect to say that the mode was zero.

Another way to represent a numerical data set is to use a **stem and leaf plot**. The leaf is usually the last digit of the numbers in the data and the stem is the rest of the number to its left, arranged in a vertical numerical order. Let's look at an example. This type of display lists all data points in a condensed form.

EXAMPLE 3

Create a stem and leaf plot for the heights of students from the table in Exploration 1.

SOLUTION

Stem	Leaf
4	9
5	0112235556777777788899
6	0012

As you can see, this graph shows how most of the data is in the 50's and indicates a shape that peaks in the 50's and well distributed among the other values.

The stem and leaf plot can show how the distribution of large number of data is shaped. Some data can be very clustered around certain numbers as in Example 3 where most of the numbers were in the 50's. Other sets of data can be spread out, or even skewed towards one end or another. It can even show that there may be a few data points that are very different from the others.

One way to see the contrast in shape and distribution is to examine two sets of data as in the following exploration.

EXPLORATION 2

Data on daily temperatures in two cities are given:

Daily average temperature in a Texas city in April.

61	59	65	68	82	72	77
76	73	65	65	62	70	67
57	50	62	61	70	69	64
80	77	82	75	79	71	79
87	80					

Daily average temperature in a New York city in April.

41	57	59	40	34	33	40
51	54	64	65	45	47	63
63	54	59	57	42	45	46
70	62	48	48	43	51	59
70	89					

1. Create a stem and leaf plot for each of the two cities.
2. Find the median of each of the data sets.
3. Find the mean of each of the data sets.
4. Describe some features of the individual stem and leaf plots that you notice. Write down differences and similarities between the two data sets.
5. Did you notice the lowest and the highest temperatures for each of the cities?
6. What is the range of each of the data sets?

We close this section with the **box plot**, sometimes called the **box and whisker plot**, which is another way of organizing data.

The first step is to order the data in increasing order. As an example, we use the data from Exploration 2 for the two cities.

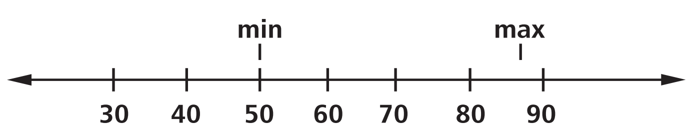
Texas: 50, 57, 59, 61, 61, 62, 62, 64, 65, 65, 65, 67, 68, 69, 70, 70, 71, 72, 73, 75, 76, 77, 77, 79, 79, 80, 80, 82, 82, 87

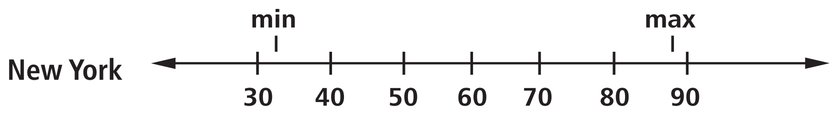
New York: 33, 34, 40, 40, 41, 42, 43, 45, 45, 46, 47, 48, 48, 51, 51, 54, 54, 57, 57, 59, 59, 59, 62, 63, 63, 64, 65, 70, 70, 89

The instructions for constructing the box plot are as follows:

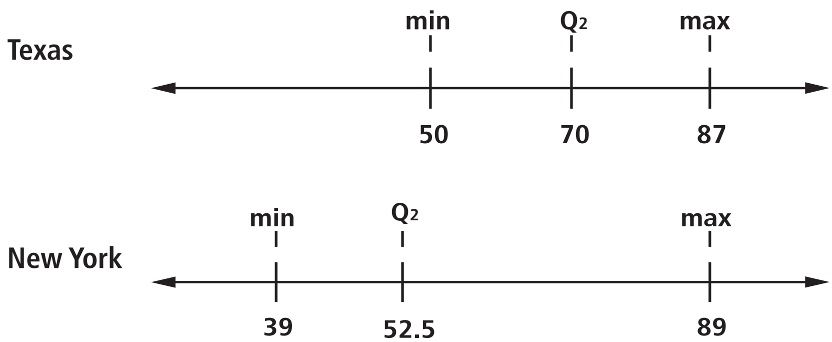
Step 1: Place the largest and smallest values on the respective number lines and put notches above those numbers as shown below.

Texas

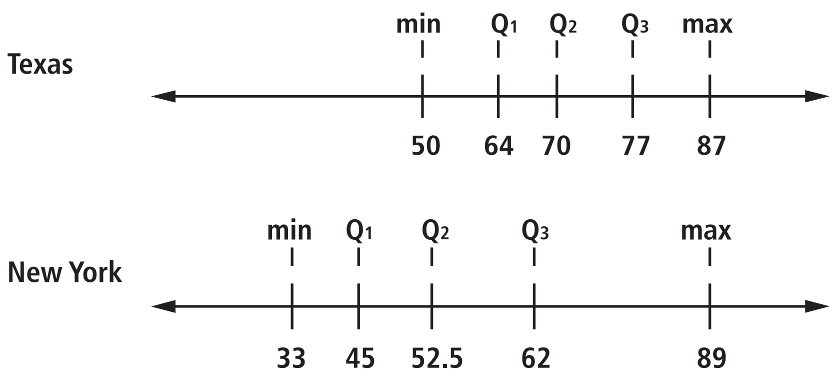




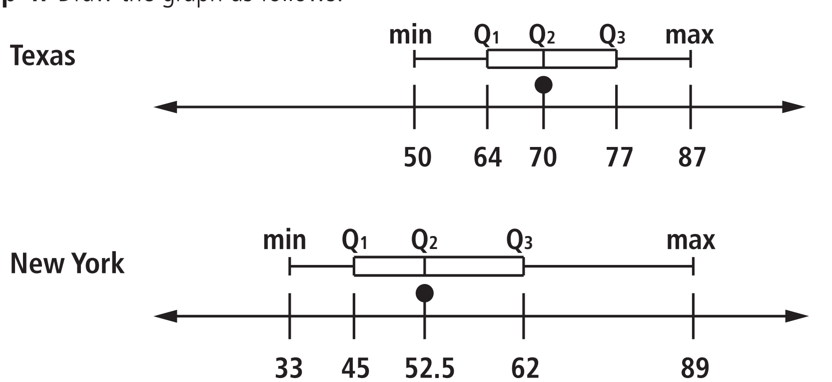
Step 2: Locate the median (Q_2) for the data set.



Step 3: Locate the median for the lower half, called the lower or first quartile (Q_1), and the median for the upper half, called the upper or 3rd quartile (Q_3).

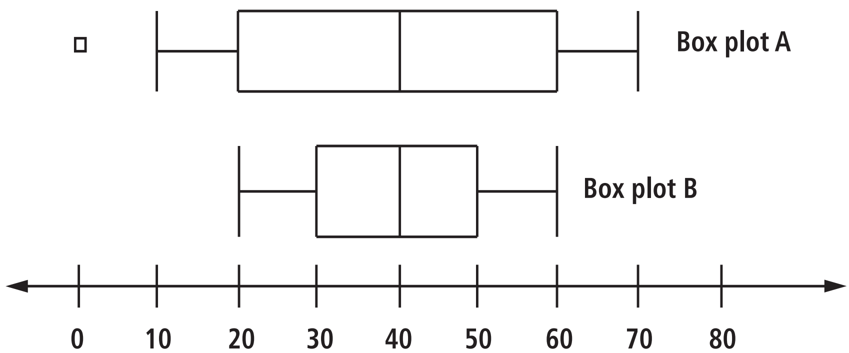


Step 4: Draw the graph as follows:



Along with the range and medians that are summarized in the box plot, another number referred to as the Interquartile Range (IQR) tells us how the distribution of the 50% is concentrated. The IQR = upper quartile – lower quartile.

Two boxplots below represent two datasets.



A summary for each boxplot is given below:

Box plot A

Minimum value is 10

Maximum value is 70

The range is 70 – 10 = 60

The median is 40

First quartile is 20

Third quartile is 60

IQR is 60 – 20 = 40

Outlier(s) at 0

(very different from rest of data)

Box plot B

Minimum value is 20

Maximum value is 60

The range is 60 – 20 = 40

The median is 40

First quartile is 30

Third quartile is 50

IQR is 50 – 30 = 20

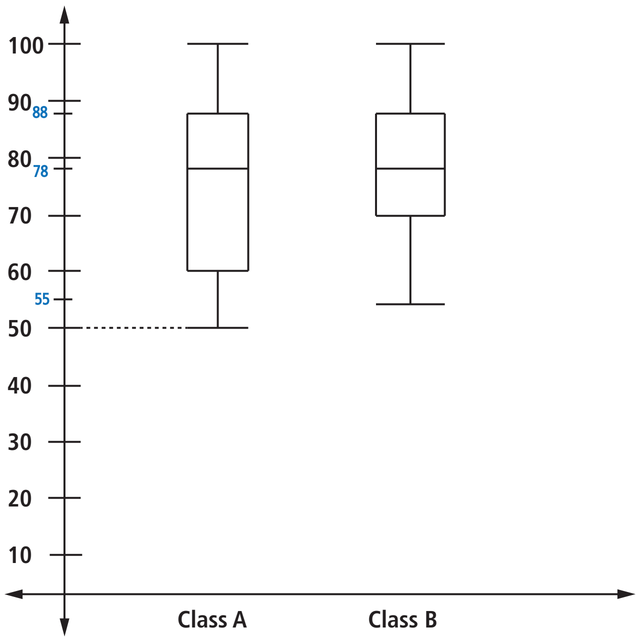
No outliers

Notice that the median for both data sets is 40. The data set for A has a greater spread than B because the range of A is 60, while the range of B is 40. The IQR that shows the range of the 2nd and 3rd quartiles is 40 for data set A, while B has a smaller IQR of 20. This suggests that more data points are concentrated near the center for B. Another difference between the two data sets is that A has an outlier while B does not.

In our example above, the box plots were situated horizontally. The box plots can also be situated vertically as in the problem to follow.

PROBLEM 1

Consider the two box plots below for the test grades from two 6th grade classes. Describe the center, spread, and shape of the data distribution using the ideas of range, median, upper and lower quartiles, and the interquartile range.



PROBLEM 2

Create a box plot for the data set of heights used in Exploration 1. Use the box plot to summarize your observations.

EXAMPLE 4

Find the mean, median, mode, and range of the following data set. Create a dot plot with the given data.

$$\{2, 8, 4, 8, 8, 6, 5, 7, 9, 3, 7, 5\}$$

SOLUTION

The mean is found by adding the values together and dividing by the number of values.

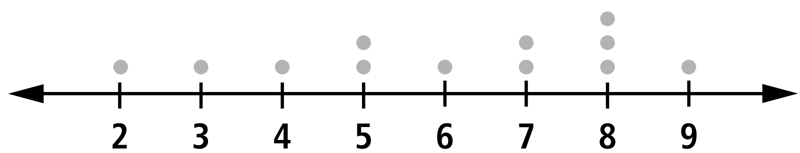
The sum of the values is $2 + 8 + 4 + 8 + 8 + 6 + 5 + 7 + 9 + 3 + 7 + 5 = 72$. The number of values in the set is 12. The mean is $\frac{72}{12} = 6$.

Putting the data set into order from smallest to greatest value results in {2, 3, 4, 5, 5, 6, 7, 7, 8, 8, 8, 9}. Because there are an even number of values in the set, the median is the average of the two middle values.

The median is the average of 6 and 7, $\frac{(6+7)}{2} = \frac{13}{2} = 6.5$.

The most commonly occurring value in the data set is 8, so 8 is the mode.

The range is the difference between the highest and lowest value. The range is $9 - 2 = 7$.



PROBLEM 3

In the following data set, what is the mean? the median? the mode? the range? Include a dot plot of the given data.

{4, 9, 9, 12, 5, 9, 2, 5, 6, 7, 28, 29, 5}

The mean depends on all the numbers in the data, but the median only depends on the value of the data point in the middle position. That does not, however, suggest that the mean is a better measure of central tendency than the median.

PROBLEM 4

Find the mean and median of the following six weeks test grades:

{95, 30, 98, 93, 100}.

Compare the value of each as a measure of the data.

PROBLEM 5

Find the mean, median, mode, and range of your class height data from Exploration 1.

PROBLEM 6

You are given a data set represented by the following stem and leaf plot:

Stem	Leaf
10	0
9	6442
8	98777553
7	98841
6	73

Use the information to determine the following, if possible. Round any value to the nearest one:

1. The mean of the data set.
2. The median of the data set.
3. The mode of the data set.
4. The range of the data set.
5. A dot plot of the data set.

EXPLORATION 3

Using the data from Exploration 1, compute the mean and the median of the heights of the class. Then, imagine that a giant who is 400 inches tall joins the class. Compute the new mean and find the new median. How has each changed?

If the data is **skewed**, or uneven, a median value is a more accurate picture of the representative value than the mean is. Exploration 3 had a very tall giant join the class. The mean was affected by this **outlier**, a term used to refer to a value that is drastically different from most of the data values. The median, however, was not affected. The mean is usually more influenced by extreme values than the median.

A distribution that has a behavior in the lower quartile similar to the upper quartile would be symmetric about the median. The distribution would be shaped symmetric to the median. We will discuss this some more in the next section.

Let us review the ways in which we summarized data in this section.

If we have a set of n values, then we can find the following measures:

- Find the mean by adding the values and dividing by n.
- Find the median by ordering the values and finding the value that is in the middle, if n is odd, or by taking the average of the middle values, if n is even.
- The mode is the most frequent value that occurs. There could be two or more such values. There could also be no mode for a data set.
- The range is the difference between the largest and the smallest values in the set.
- The interquartile range (IQR) is the difference between the median of the upper half (the third or upper quartile) and the median of the lower half (the first or lower quartile).
- Q_2 is found above as the median. Q_1 and Q_3 can also be determined by finding the median of the lower half for Q_1 and median of the upper half for Q_3.

EXERCISES

1. Find the mean, median, mode, and range of the following data sets:

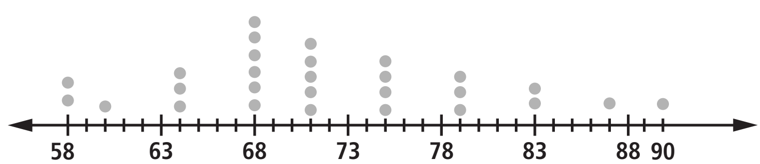
   a. {8, 16, 0, 8, 6, 2, 15, 2, 12, 8, 16, 5}

   b. {71, 66, 74, 64, 66, 73, 71, 60, 71, 65, 63, 74}

2. Take the data from problem 1a and create a dot plot.

3. Take the data from problem 1b and create a stem and leaf plot.

4. Summarize the information given by the dot plot below representing the pulse rate, in beats per minute, for a group of 28 students.



   Determine:

   a. the range of the data set

b. the mode of the data set

c. an estimated normal pulse for this class

5. Which measure of central tendency is most helpful in representing the following situations? Choose from mean, median, mode, and range.

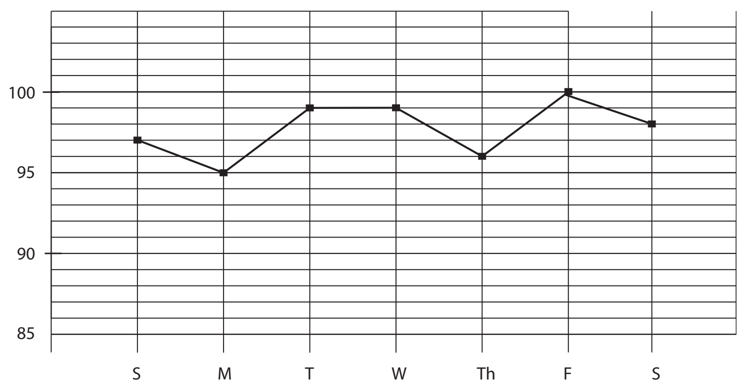
   a. Determining your report card grade in a subject.

   b. The number of bowling pins knocked out the most often during a bowling game.

   c. The fluctuation or change between the high and low temperature of a city in one day.

   d. The midpoint age among a class of college students.

6. Use the following data to answer the questions that follow:



   a. What is the median for the week's temperature?

   b. What is the mode of the week's temperature?

   c. Identify the range of the week's temperature.

7. The total height of all the students of a class of 15 is 945 inches. What is the mean height of the class?

8. The January mean daily temperatures for Castolon, TX, and Galveston, TX, are approximately the same. However, their ranges are quite different. The temperature data, in degrees Fahrenheit, from the National Oceanic and Atmospheric Administration (NOAA) are:

City	Maximum	Minimum	Mean	Range
Galveston	61.9	49.7	55.8	12.2
Castolon	67.7	33.6	50.7	34.1

Even though Galveston and Castolon have about the same daily mean temperature for January, would you consider packing different clothes for the two places? Which measure of central tendency influenced your decision? Why?

9. Below are estimated national median heights in inches for 9- through 14-year-olds in 2000, according to the National Center for Health Statistics (NCHS). Based on this data, what is your estimate for the median height for 15-year-olds? Do you think the median heights for 24-year-olds and 25-year-olds are that much different? Explain.

Age Group	Height (in)
9-year-olds	52.5
10-year-olds	54.5
11-year-olds	56.5
12-year-olds	59.0
13-year-olds	62.0
14-year-olds	65.0

10. Use the data in the stem-and-leaf to draw a box plot. Summarize your observations (for example, range, median, etc.).

Stem	Leaf
3	9
4	12334444556789
5	01

11. Sam and Matt went fishing in the Gulf of Mexico. Over the week each caught a total of 13 fish. They recorded the length, in inches, of each fish that they caught.

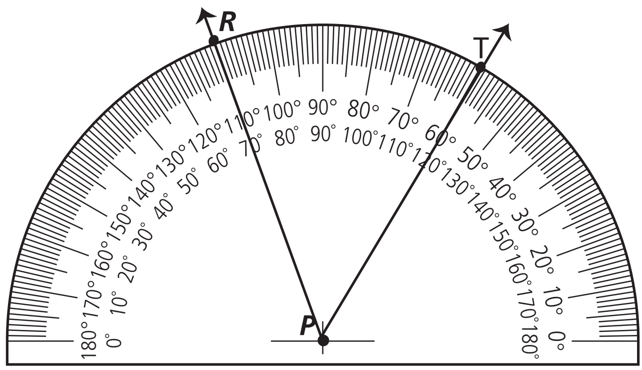
Sam: 13, 12, 9, 5, 8, 14, 20, 16, 14, 9, 6, 12, 12

Matt: 11, 12, 10, 13, 15, 11, 12, 8, 13, 12, 9, 9, 14

Draw a box plot for the two data sets. Summarize and compare their medians, ranges, interquartile ranges, and their upper and lower quartiles.

Spiral Review:

12. What is the measure of $\angle RPT$?



13. Which statement is not true?

a. $\frac{4}{5} > 70\%$ b. $^{-}1\frac{3}{5} < -\frac{4}{5}$ c. $40\% > \frac{2}{5}$ d. $1\frac{5}{6} > 1\frac{2}{3}$

14. **Ingenuity:**

Ling's grades in her math class are 85, 94, 96, 87, 98, and 89. If Ling has one more assignment to do, and she wants to make sure that she has an average of at least 90 in the class, what grade does she need to make on the assignment?

15. **Investigation:**

For each of the following sets of numbers, determine whether the median is greater than, less than, or equal to the mean. Can you find a way to do this without actually calculating the median and mean of the numbers?

a 1, 1, 1, 1, 2, 2, 3

b. 1, 8, 9, 10

c. 13, 13, 13, 13, 13

d. 42, 43, 44, 45, 46, 47, 48

e. 5, 5, 5, 5, 5, 5, 11, 11

f. 42, 57, 75, 80, 84, 89, 91

16. **Challenge:**

A class of 11 students was given the following extra credit question on a test: "Pick a positive integer between 1 and 10, inclusive: ______"

The least integer greater than or equal to the nonnegative difference between the mean and the median of the answers given will be added to everyone's score. What was the maximum bonus that the class could have earned?

SECTION 10.2
GRAPHING DATA

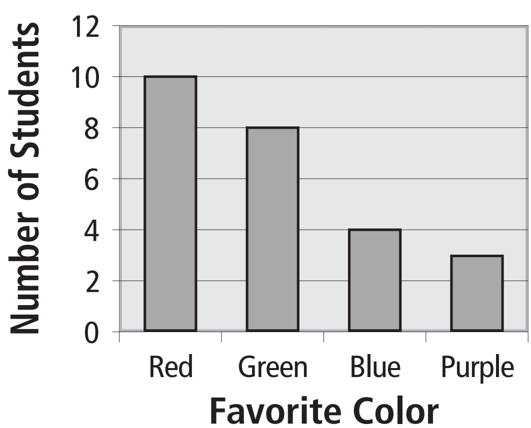
When collecting data, it is often useful to draw a picture or graph to represent the data that has been collected. A graph of the data gives a quick, easy way to see what the data represents.

EXAMPLE 1

Ms. Tate's class has 25 students. Each student was asked which color they like best. The survey shows that 10 students prefer red, 8 students prefer green, 4 students prefer blue, and 3 students prefer purple. What are the best ways to represent this information?

SOLUTION

One way to display the data is to make a special kind of graph called a **bar graph**. A bar graph is generally used when the categories are not numerical, as in this case when we examine kinds of color. To construct a bar graph, draw an ***x***- and ***y***-axis, subdivide the horizontal or ***x***-axis into four equally-spaced intervals and label the intervals with the categories Red, Green, Blue, and Purple. Then label the vertical or ***y***-axis with points from 0 to 12. For each color, draw a vertical bar equally separated from the other bars. The height of each bar represents the number of people who liked a particular color best. The bar graph should look like this:



Explain why the vertical axis has a number scale from 0 to 12. Why is there no number scale on the horizontal axis? Explain whether the order of the colors is important.

PROBLEM 1

The cafeteria at Summit Ridge Middle School took a survey to determine what were student's favorite cafeteria foods. The results of the survey were as follows: 120 liked pizza, 75 liked hamburgers, 60 liked hot dogs, and 45 liked tacos. Construct a bar graph to represent the survey results.

A double bar graph is another useful way of representing certain kinds of data. Notice in the previous example the data gathered was for one category: color. If there is more than one category, such as boys and girls, then a double bar graph might be called for.

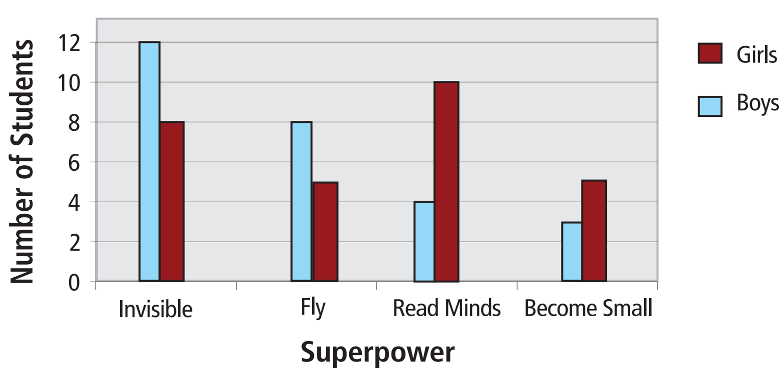
EXAMPLE 2

A survey asked boys and girls in Ms. Lowrance's class what one super power they would choose if they were superheros. There were four choices of super powers and the results of the survey is as follows:

Be invisible was chosen by 12 boys and 8 girls
Be able to fly was chosen by 8 boys and 5 girls
Be able to read people's minds was chosen by 4 boys and 10 girls
Become small was chosen by 3 boys and 4 girls.

SOLUTION

Just as with bar graphs, we let the vertical axis be frequency, in this case from 0 to 12, and the horizontal axis the four categories of super powers. But instead of just one bar, we put the boys and girls next to each other and are able to compare the categories as well as compare the boys and girls.



PROBLEM 2

A survey was conducted in Ms. Soto's class to determine favorite sports among boys and girls. The results were as follows:

Football	18 boys	3 girls
Baseball	10 boys	8 girls
Volleyball	3 boys	12 girls
Bowling	8 boys	10 girls

Construct a double bar graph for the data.

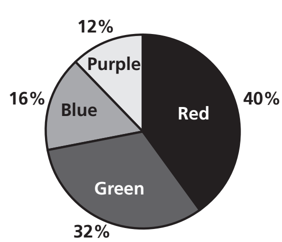
PROBLEM 3

Another way to represent the data from Example 1 is by using percents. Because there are 25 students in all, $\frac{10}{25}$ of the class likes the color red. Convert $\frac{10}{25}$ to the decimal 0.40 and then to the percent 40%. Similarly, $\frac{8}{25} = 0.32 = 32\%$ of the class likes green. Calculate the percents of the remaining colors from Example 1. Use these percents as data on the vertical axis to build another bar graph. Label the axes and draw the graph.

Although the shape of the bar graph is the same, the percentage graph gives a picture of the relative quantity of the class' preference for each color, not the number of students directly. This bar graph shows the relative number or percentage of students immediately.

Another way to represent the percentage data is to use a **circle graph** or a **pie graph**. Use your protractor or compass to draw a circular outline for the circle graph. You have already computed the percentage of each color. The proportion of the circle graph with a given color corresponds to the percentage of students who prefer that color. The larger the sector of the circle graph, the greater the percentage of people who liked the color.

The completed pie graph below clearly represents which color students like best and makes the result of the survey visually obvious.

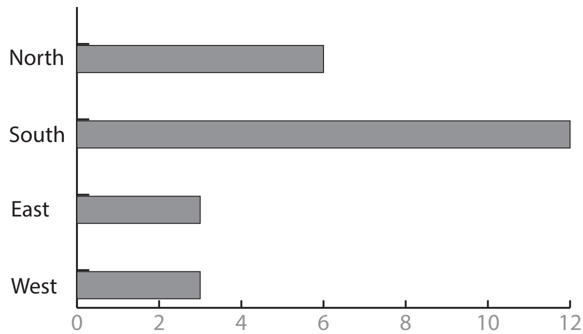


PROBLEM 4

Mr. Ruiz asked his 12 students which color they liked best as well. In his class, he found that 6 students prefer green, 4 students prefer red, 1 student prefers blue, and 1 student prefers purple. Make a bar graph to compare the data from Mr. Ruiz's class to the data from Ms. Tate's class in Example 1. In which class is green more popular? Explain.

PROBLEM 5

The bar graph represents the number of students in Mr. Mungia's class who live in various parts of town.



Use the information to determine the following:

1. How many students live in the northern part of town?
2. What percent of the students in Mr. Mungia's class live in the northern part of town?
3. What percent of the students in Mr. Mungia's class do not live in the northern part of town?
4. Use the bar graph to create a corresponding circle or pie graph.

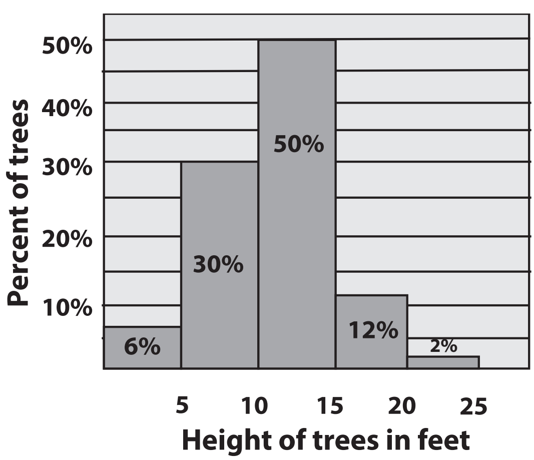
Recall that bar graphs were used to represent data with non-numerical categories such as color. A **histogram** is a graphical representation of data with numerical categories. Histograms are drawn as a bar graph with the positive x-axis indicating numerical categories of numbers or range of numbers. The heights of the "bars" can be either the frequency or numbers in each category or they can be percents of the data. The bars are also drawn touching each other whereas the bar graph generally does not have the bars touching each other. We use the following example to demonstrate these concepts.

The data shows the results of a survey taken by a city to determine the heights of the crape myrtle trees in the city.

Height of tree (in feet)	Number of trees
Between 0 and 5	3
Between 5 and 10	15
Between 10 and 15	25
Between 15 and 20	6
Between 20 and 25	1

We have the following histogram to represent this data. The height of the trees is on the ***x***-axis and the frequency of the tree size is along the ***y***-axis.

You can create a percent histogram from a frequency histogram by finding the proportion of trees in each category. The percent histogram would then look like the following:



PROBLEM 6

The test grades on Test 1 and Test 2 are given below. Create a histogram for each test using the data given.

Range of test grade	TEST 1 Number of Students	TEST 2 Number of Students
Between 50 and 59	2	1
Between 60 and 69	4	3
Between 70 and 79	10	2
Between 80 and 89	6	7
Between 90 and 100	3	12

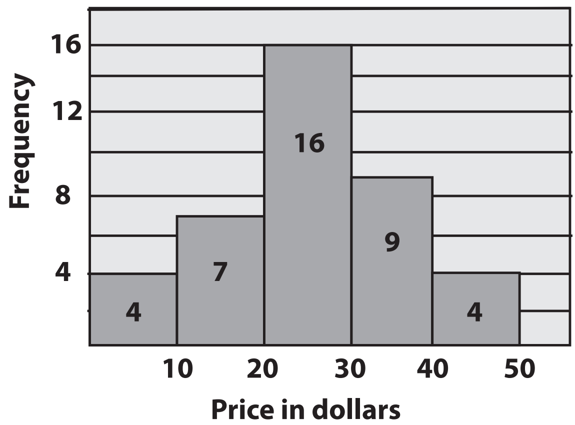
Use the histograms to make observations about the shape and distribution of the data. Is there enough information to determine the mean or the median? Explain why.

Histograms give visual representation to a set of data that can reveal information about what is similar or different about the sets along with their distribution and variability. The idea of variability is an important part of statistics. The data that you want to study may have a great deal of variability, such as the time that a person was born. On the other hand, there is very little variability to the information regarding the time you are dismissed from school.

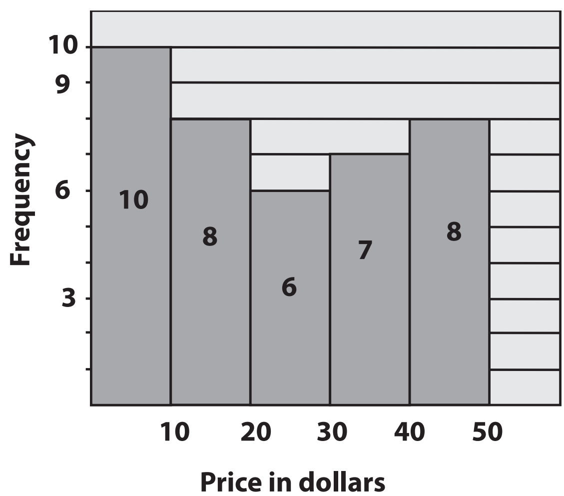
For example, in the three following histograms, list what you observe about their ranges, means, medians, and their shapes (symmetric or skewed). We say the histogram is symmetric if the shape is about the same shape on either side of the middle. We say the histogram is skewed if most of the data is on one side of the middle.

The histograms are of a sample of the costs of 40 items sold at three stores and the number of items in that price range.

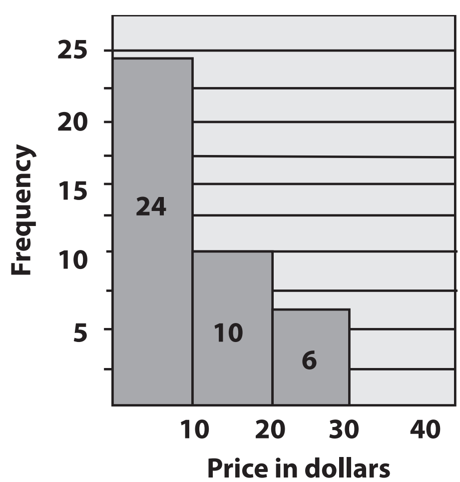
Store 1



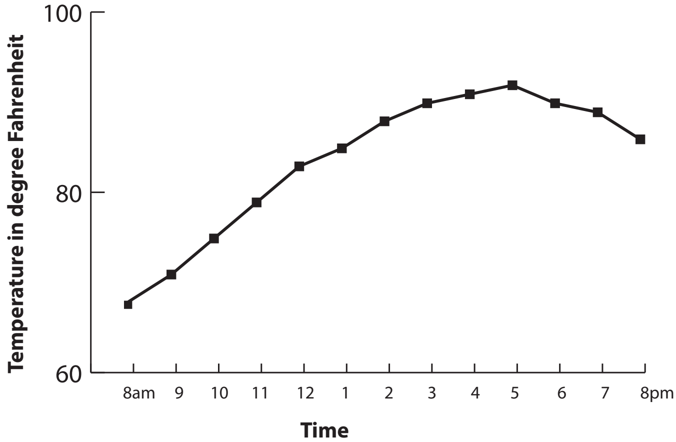
Store 2



Store 3



A **line graph** for a set of data points is often used to show changes in the data over a period of time. For example, by using a line graph we can show the rise and fall of the temperature by hour. While only hourly changes are recorded, the points are usually connected from point to point as in the portion of a line graph below:



EXPLORATION

The rainfall record for a region over an 8-year period from 1990 to 1997 is listed below.

a. Make a bar graph to represent this data.

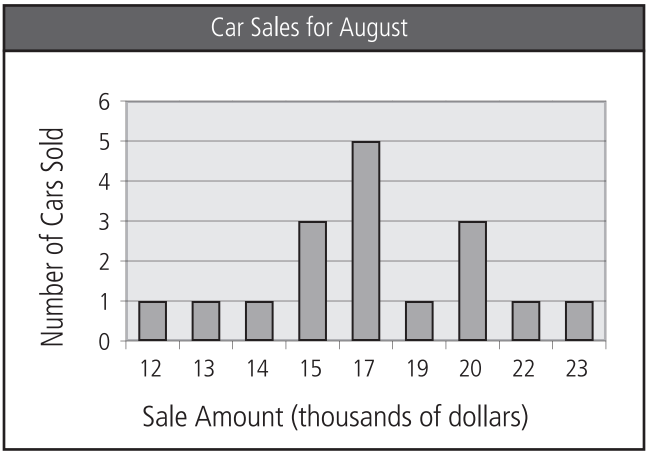
b. Plot the data on a coordinate plane. Label the horizontal axis to represent time in years, and the vertical axis to represent inches of rainfall. To convert the set of points to a **line graph**, connect the points sequentially with straight lines.

c. What differences do you notice between the line graph and the bar graph?

Year	Rainfall
1990	30 inches
1991	32 inches
1992	24 inches
1993	18 inches
1994	28 inches
1995	36 inches
1996	42 inches
1997	31 inches

EXERCISES

1. Connie was working on important data when her computer crashed and she lost all the numbers. Luckily, she had printed a bar graph of her data earlier.



Based on the data from the bar graph:

a. How many cars were sold?

b. What is the total sales amount for the month of August?

Use the information from the Birthday Data Table to answer questions 2 and 3.

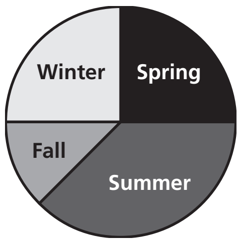
Birthday Data Table 1

Month	Birthdays
January	2
February	3
March	0
April	3
May	5
June	2
July	4
August	3
September	2
October	3
November	0
December	3

2. Create a bar graph from the Birthday Data Table 1.

3. Create a line graph from the Birthday Data Table 1.

4. Karla polled her homeroom class and asked them what their favorite time of year is, Winter, Spring, Summer, or Fall. She compiled the data into a pie chart.



   a. What was the class's favorite season?

   b. Poll your own class and ask their favorite month. Use the data to make your own pie chart to display the information.

Use the data from the Yearly Rainfall Totals table to answer questions 5 and 6.

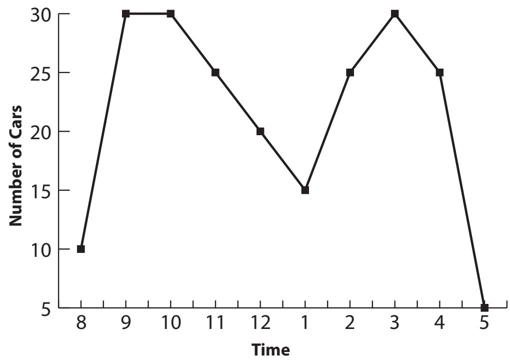
Yearly Rainfall Totals

Year	Rainfall
1990	30 inches
1991	32 inches
1992	24 inches
1993	18 inches
1994	28 inches
1995	36 inches
1996	42 inches
1997	31 inches

5. Find the mean of the yearly rainfalls for the years recorded.

6. What is the median rainfall amount of the years recorded?

7. The parking attendant recorded data on how many cars are in the parking lot at school at any given time during the school day from 8 a.m. to 5 p.m. Use the given line graph to find out:



a. What is the mode of number of cars in the parking lot?

b. What is the range of number of cars in the parking lot?

c. What is the mean of number of cars in the parking lot?

Use the Birthday Data Table 2 to answer questions 8 and 9.

Birthday Data Table 2

Month	Birthdays
January	2
February	3
March	0
April	3
May	5
June	2
July	4
August	3
September	2
October	3
November	0
December	3

8. Create a pie chart using Birthday Data Table 2 to answer the following questions.

   a. What percent of the students were born in January? May?

   b. What angle measure in the pie chart corresponds to May birthdays?

   c. What percent of the students were born in the summer months of June, July, and August?

9. Use the bar graph you created in Exercise 2 to answer the following questions.

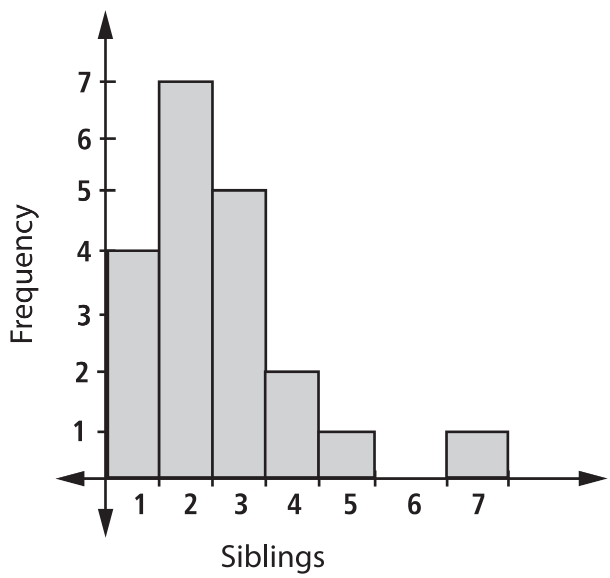
   a. What is the relation between the heights of the January column and the February column?

   b. Which graph do you prefer for this data, bar or circle graph? Explain.

10. A pie chart cannot be made in all cases where data is presented in percentages. All four sixth grade teachers in a school asked their students which color they liked best. The table shows the results for what percentage in each class liked green best. Make a bar graph. Explain why you couldn't make a pie chart for the data in the table.

Class	Percent that prefer green
Ms. Tate	32%
Mr. Girardeau	50%
Ms. Eusebi	25%
Ms. Gonzalez	20%

11. The histogram below represents the number of brothers and sisters that students in Ms. Gonzalez's class have.



Summarize the information given by the histogram. Include the range, mode, mean, and median, if possible. If not, explain why you can't.

12. Observations are made of the number of cars passing through a fast food drive through as follows:

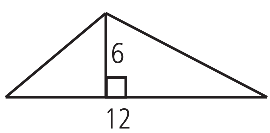
8 a.m. until 10 a.m.	8
10 a.m. until noon	15
noon until 2 p.m.	13
2 p.m. until 4 p.m.	4
4 p.m. until 6 p.m.	9
6 p.m. until 8 p.m.	18
8 p.m. until 10 p.m.	3

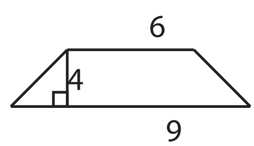
Use this data set to create two histograms, one with the frequency of cars and the other with percent of the cars in each time interval. Summarize observations that you made using the histogram.

13. The test results in Ms. Lowrance's class were as follows: 55, 59, 63, 65, 67, 73, 75, 77, 77, 81, 84, 84, 84, 87, 88, 90, 92, 95, 97, 99. Group the test scores by 10s (for example 55 and 59 make up the 50's group) and create a histogram that shows how many students got the particular grade. Create a second histogram that shows the percentage of students who received the particular grades in the ten-point band, using the original directions. Summarize the test results by finding the mean, median, mode, and range of this data.

Spiral Review:

14. Which figure has greater area?

a. 

b. 

15. **Ingenuity:**

Yuri polls his 30 classmates to find out what their favorite school subjects are. 10 of his classmates like math best, 9 like science best, 7 like language arts best, and 4 like history best. If Yuri were to create a pie graph to show the results of his poll, what should be the central angle of each part of the pie graph? (The central angle of a region of a pie graph is the angle of the region that is at the center of the circle.)

16. **Investigation:**

Conduct a survey of 20 people on a question of your choice that has three possible answers. Collect your data into a table, and display the results with a bar graph, a line graph, if possible, and a pie graph. Which representation seems most appropriate for your survey? Why?

SECTION 10.3
PROBABILITY

The study of probability allows us to make educated guesses about what might happen in the future based on past experience, to determine how likely different outcomes are. This knowledge can help us make the best choices.

You have heard people say, "There is a 50-50 chance of getting heads on a coin flip," or "There is a 80% chance of rain today." What do statements like these mean? How can we determine the chance that some event will happen?

The following activity will help us to set the stage for carefully studying probability.

Activity:

1. a. Take a six-sided number cube and examine the numbers on all six sides. Jot these down as possibilities of what can come up in a roll.

   b. Roll the number cube 20 times. Record the number that comes up for each roll. Call this experiment **Roll One Cube**. Keep a careful record.

2. a. Take two six-sided number cubes of different colors such as one red number cube and one green number cube. What are the possible rolls that you could get, where a roll is recording what you got on the red number cube and also what you got on the green number cube?

   b. Roll the two number cubes 20 times. Record the numbers that come up for each roll on a piece of paper, being very careful to record which number came up with which color number cube. Call this experiment **Roll Two Cubes**. What are all the possible rolls that you can get?

3. a. Take a standard deck of cards. Examine how many cards are in your deck and other distinguishing features of the deck. Describe all you notice, including how many there are with the feature you observed. If you picked one card from the deck, what are all the possible selections you could get?

   b. Shuffle the cards carefully. Select a card and record its color and then return the card back to the deck and shuffle. Do this action 20 times and keep a record of the color of the card for each time you select. Call this experiment **Pick a Card.**

4. a. Take a coin and observe the two sides. We call one side Heads and the other side Tails. Can you see why? Usually on a US coin one side of a coin has the face of a US president. This side is usually called heads. You might check to see which of the presidents are on the penny, nickel, dime, and quarter. They are the sixteenth, third, thirty-second, and first presidents.

   b. Flip a coin 20 times and record the results of your toss. Call this experiment **Coin Toss.**

5. As another experiment, flip the coin twice and record the first and second toss in that order. Repeat this 20 times. Call this experiment **Two Flip**.

6. Take a spinner with at least 4 different possibilities. Spin 20 times and record where the spinner lands each time. Call this experiment **Spin Once**.

We will refer back to some of these activities during this section.

We begin our study of probability with some useful and important concepts and vocabulary. An **experiment** is a repeatable action with a set of **outcomes**. For example, the experiment of flipping a coin has two possible outcomes, either heads or tails. The set of all possible outcomes of an experiment is called the **sample space.** Flipping one coin is called a **simple experiment** because only one action is involved. A **compound experiment** involves more than one action, such as flipping a coin and rolling a die. One important characteristic of an experiment is that it must be repeatable, with similar possible outcomes.

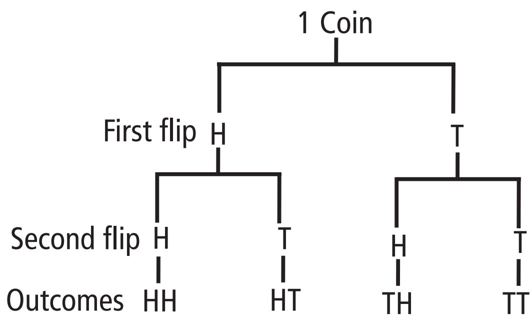
In studying an experiment, the question is, "What are all the possible outcomes?" If you flip a coin once, you will observe only one of the two possible outcomes, heads or tails. The key to finding any probability is to determine the likelihood of each possible outcome.

It is often helpful to use abbreviations or draw **tree diagrams** to represent the outcomes of the experiment. For example, when flipping a coin, we often write H for the outcome of getting a head and T for the outcome of getting a tail. The sample space is the set {H, T}; there are only two possible outcomes.

EXAMPLE 1

In the experiment Two Flip, you flipped a coin once and observed the outcome then flipped it a second time and observed the outcome. This is a compound experiment. You then recorded the outcome such as: first flip head and second flip tail. What is the set of all possible outcomes? How many outcomes show no tails?

SOLUTION



The order of outcomes is important. The outcome of getting a head and then a tail, denoted by HT, is a different outcome from getting a tail and then a head, denoted by TH. This experiment has the sample space {HH, HT, TH, TT}. Notice that there are four possible outcomes. When the experiment involves flipping a coin twice, {H} is an impossible outcome. The simple event {HH} is the subset containing the outcome that both flips show heads and is the only outcome that shows no tails. A **simple event** is a set with only one outcome.

The outcome of getting at least one tail could be described as {TH, HT, TT}. There is more than one possible way that you can get a tail when tossing a coin twice. This is called a **compound event**. A compound event is a set with more than one outcome.

In both the Coin Toss and Two Flip experiments, each outcome in the sample spaces has the same chance of occurring as any other outcome. Each outcome is then said to be **equally likely**.

Remember the Coin Toss has a sample space that we can write as {H, T}. If the coin is a fair coin, then the chance of getting an H is the same as the chance of getting a T; in other words, equally likely.

DEFINITION 10.1: EVENT, SIMPLE EVENT, AND COMPOUND EVENT

An **event** is any subset of the sample space. A **simple event** is a subset of the sample space containing only one possible outcome of an experiment. A **compound event** is a subset of the sample space containing two or more outcomes.

The words simple and compound are used to describe both events and experiments. The main thing to remember is that a simple event is referring to just one outcome while a compound event is referring to more than one possible outcome. A simple experiment has just one action, such as pick a card, roll a number cube, or flip a coin. A **compound experiment** has more than one action, for example you may roll two number cubes or flip a coin more than once.

Call E the event of "getting at least one head" in the Two Flip experiment introduced earlier. The outcomes HH, HT, and TH satisfy the criteria for being in E, so E = {HH, HT, TH}. Because E contains three possible outcomes, it is a compound event.

In order to study a compound event E, a technique that is often used is to find the possible outcomes that are not in E. We call this set the **complement of E**, or E^c.

For example, in the Two Flip experiment, we have the sample space S = {HH, HT, TH, TT}. We called an event E in this experiment as "getting at least one head." This can be written as E = {HH, HT, TH}. The complement of E consists of all the outcomes in the sample space when you do not get at least one head. This leaves only TT where there are no heads. We then write E^c = {TT}.

DEFINITION 10.2: PROBABILITY

In an experiment in which each outcome is equally likely, the probability of an event A, written $P(A)$, is $\frac{m}{n}$, where m is the number of desired outcomes and n is the total number of outcomes in the sample space S.

Notice that the probability of an event from an experiment is always a number between 0 and 1. Explain why this is true, using the Coin Toss example if necessary. Explain why $P(S) = 1$, where S is the sample space for an experiment.

In Example 1, the probability of getting no tails is $\frac{1}{4}$, because 1 of the 4 equally likely outcomes shows no tails. The probability of getting at least one tail is $\frac{3}{4}$. In general, $P(E^C) = 1 - P(E)$ because $P(E) + P(E^C) = 1$.

EXAMPLE 2

Consider the experiment of rolling one number cube.

a. What is the probability of getting a 3?

b. What is the probability of not getting a 3?

c. What is the probability of getting an even number?

d. What is the probability of getting a number greater than 4?

SOLUTION

Identify the sample space, $S = \{1, 2, 3, 4, 5, 6\}$, for the experiment of rolling one number cube.

a. Let A = the event of rolling a 3 in this experiment. Written as a subset, we have, $A = \{3\}$. The probability of A is the fraction with the numerator equal to 1, the number of outcomes in A and the denominator equal to 6, the number of total outcomes in the sample space S. Therefore, $P(A) = \frac{1}{6}$.

b. Let B = the event of not rolling a 3 in this experiment. Written as a subset, we have $B = \{1,2,4,5,6\}$. Notice that B is the complement of event A because the outcomes in B are "not 3." The probability of not getting a 3 = $P(B) = \frac{5}{6}$. Another way to think about the probability of "not an event " is 1 – probability of the event. A notation for this is: $P(B) = P(A^C) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$.

c. Let T = the event of getting an even number. Written as a subset, we have $T = \{2,4,6\}$. $P(T) = \frac{3}{6} = \frac{1}{2}$.

d. Let R = the event of getting a number greater than 4. Written as a subset, we have $R = \{5,6\}$. $P(R)= \frac{2}{6} = \frac{1}{3}$.

When we "consider" an experiment of rolling one number cube, we do not actually roll a number cube. Instead, we think about what could possibly happen if we rolled a number cube. This is called a thought experiment and is used in **theoretical** probability. If we really rolled a number cube and used the observed outcomes as in the first activity, that would be an example of **empirical** (**or experimental**) probability.

Gather some more empirical data for the Coin Toss by performing the experiment many times.

In most instances, when you see "experiment," that means a thought experiment that leads to the theoretical probability of an event happening. If the experiment refers to some empirical data, then of course, the problem would most likely use the gathered information.

EXAMPLE 3

List the possible outcomes of rolling a red number cube and a green number cube. Create a table to organize your data.

SOLUTION

What is a possible outcome? You might roll a 3 and a 4. As with tossing a coin twice, the order matters. Rolling a 3 and a 4 is different from rolling a 4 and a 3. To see this more easily, think about rolling one red number cube and one green number cube. Rolling a red 4 and a green 3 is a different outcome from rolling a red 3 and a green 4. To list all the outcomes of this thought experiment, abbreviate the outcome red 3 and green 4 as (3, 4). The sample space for the two-number cube experiment can be listed in several ways. One convenient method is to make a table like the one below:

(R, G)	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

PROBLEM 1

Consider again the experiment of rolling two number cubes. What is the probability of each of the following events?

a. A = {getting at least one 6}

b. B = {getting a double (both number cubes have the same number)}

c. C = {the sum of the two number cubes is 7}

What is the difference between flipping a coin two times and flipping two coins simultaneously? What is the difference between rolling a number cube two times and rolling two number cubes? Run the thought experiments for both the coins and the number cubes to see the differences or similarities.

EXERCISES

1. In the Roll One Cube experiment, using a standard number cube, answer the following questions.

   a. P(even number)

   b. P(number less than 2)

   c. P(6 or 1)

   d. P(0)

   e. P(prime number)

   f. P(NOT 5 or 6)

2. A standard deck of playing cards has 52 cards made up of 4 suits with 13 cards in each suit. There are 3 face cards in each suit and the rest are number cards. The cards are either red or black. Using this data, answer the following problems.

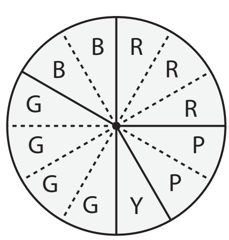
   a. P(face cards)

   b. P(red cards)

   c. P(number cards)

   d. P(red face card)

3. Use the spinner below to answer the following questions.



B = Blue

R = Red

P = Purple

Y = Yellow

G = Green

a. What is the probability of landing on green?

b. What is the probability of NOT landing on blue or yellow?

c. What is the probability of landing on purple or green?

d. What is the probability of landing on orange?

4. Sam surveyed 100 randomly selected women to find out their favorite color. He found that 35 liked red, 25 liked blue, and 40 liked green. Susan surveyed 120 randomly selected men and found that 40 liked red, 25 liked blue, and 55 liked green.

   a. Based on these surveys, is it more likely that a woman or man would prefer red? Blue? Green? Explain.

   b. If two women are picked at random from Sam's group, how likely is it that they both like red?

   c. If two women are picked at random from Sam's group, how likely is it that at least one of the women likes red?

5. The last 3 questions on Mrs. Garcia's test were true or false items.

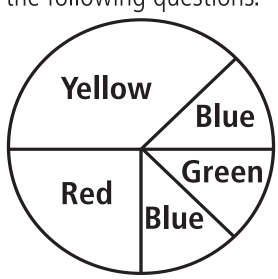
   a. Create a tree diagram to show the sample space of the possible answers for the last three questions.

   b. List all the possibilities from the tree diagram.

   c. What is the probability of getting all three answers to be true?

6. Perform the experiment of first flipping a coin and then rolling a die. What is the sample space of this experiment? Create a table to illustrate the possible outcomes. How many outcomes does it have?

7. Use the spinner to answer the following questions.



a. Israel conducted an experiment of spinning 80 times. The spinner landed on blue 19 times. What is its predicted theoretical probability? Was the experimental probability greater, less than, or equal to the theoretical probability?

b. Iliana spins 40 times. Predict how many times Iliana's spinner lands on green in 40 spins. Predict how many times Iliana's spinner will not land on green. What is the probability that the spinner will NOT land on the green section?

8. Kassandra draws one card from a set of cards numbered 1 through 10.

   a. Write the sample space for this experiment.

   b. What is the probability of getting an even number?

   c. What is the probability of getting a prime number?

   d. What is the probability of getting a composite number?

   e. Add the results of parts c and d. What do you notice? Explain your observation.

Spiral Review:

9. A circular serving plate has a diameter of 18 inches. Find the circumference of the plate to the nearest inch.

10. Tickets for the student play at Abell Junior High cost \$8.75 for adults and \$5.00 for students. If a mom, dad, and 3 students attend the play, what is the cost for the family to watch the play?

11. **Ingenuity:**

    Cody and Edward are at a banquet. At the banquet, seats are assigned randomly, with each guest assigned to one of eight round tables, each of which has eight seats. Each guest receives a card telling him which table and which seat is his.

    a. What is the probability that Cody and Edward are assigned to the same table?

    b. Assuming that Cody and Edward end up at the same table, what is the probability that they are seated next to each other?

    c. Suppose that Cody, Edward, and their friend Sam are all seated at the same table. What is the probability that they occupy three consecutive seats?

12. **Investigation:**

Anna and Belle play a number cubes game involving three different number cubes. The three number cubes are colored red, green, and blue, and each number cube has a number on each side. The numbers on the number cube are as follows:

Red number cube: Two sides labeled 1, two sides labeled 6, two sides labeled 8

Green number cube: Two sides labeled 2, two sides labeled 4, two sides labeled 9

Blue number cube: Two sides labeled 3, two sides labeled 5, two sides labeled 7

In order to play the game, each player chooses a number cube and rolls it once. The player who gets the higher number wins.

a. Suppose Anna chooses the red number cube, and Belle chooses the green number cube. Which player has a better probability of winning? What is the probability that player will win?

b. Suppose Anna chooses the red number cube, and Belle chooses the blue number cube. Which player has a better probability of winning? What is the probability that player will win?

c. Suppose Anna chooses the green number cube, and Belle chooses the blue number cube. Which player has a better probability of winning? What is the probability that player will win?

d. Suppose Anna chooses a number cube first, and then Belle is allowed to choose her number cube from the two remaining number cubes. Explain what Belle should do in order to have the best possible chance of winning.

13. **Challenge:**

Arthur rolls two standard six-sided number cubes then multiplies the two numbers that he gets. How many ways are there for the result to be prime? How many possible outcomes are there (prime or not)? Compute the probability (i.e. ratio of desired outcomes to all outcomes) that the result is prime.

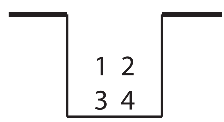
10.4

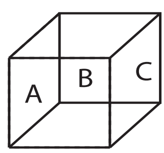
SECTION 10.4
RULE OF PRODUCT AND RULE OF SUM

One of the major goals of mathematics is to find simple underlying ideas to explain how and why things work. To do this, mathematicians often analyze problems by breaking them into simpler steps.

EXAMPLE 1

Suppose you have a hat and a box. The hat contains 4 identical cards with the number 1, 2, 3, or 4 written on them. The box contains three identical cards with a letter A, B, or C written on them. Imagine the following experiment: Without looking, reach into the hat and pull out one card, and then reach into the box and without looking, pull out one card. What is a possible outcome? How can you represent all the possible outcomes? How many possible outcomes are there?





SOLUTION

One possible outcome is selecting a 2 from the hat and the letter B from the box. We could write this combined outcome of a hat choice that has 4 possible outcomes, {1,2,3,4} and a box choice of {A,B,C} as {2,B} or 2B for short. Always try to write the sample space with some sort of order, if possible, so you do not miss a possible outcome. In this compound experiment, the outcomes can be listed as

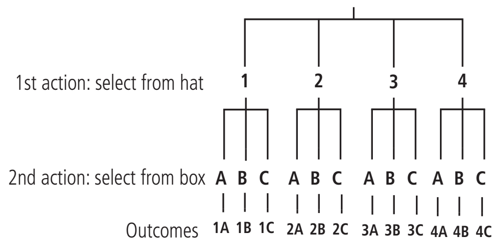
$$\boldsymbol{S} = \{1A, 1B, 1C, 2A, 2B, 2C, 3A, 3B, 3C, 4A, 4B, 4C\}.$$

The outcomes can also be listed in a table:

Hat \ Box	A	B	C
1	1A	1B	1C
2	2A	2B	2C
3	3A	3B	3C
4	4A	4B	4C

How many outcomes are in the sample space? How does the arrangement in the table help to count the number of outcomes?

A tree diagram is another method to visually represent the sample space:



The first action in the experiment of choosing from {1,2,3,4} is depicted by the four branches in the tree. Each branch splits into three smaller branches, representing the second action in this experiment, choosing from {A, B, C}. There are 12 branch endings at the bottom. To obtain the 12 outcomes in the combined experiment, follow the branches from top to bottom, reading the labels to obtain the 12 outcomes in the combined experiment listed in the table.

The selected number has no effect on the letter picked, so the two events of choosing a number and choosing a letter are said to be independent. When two events are independent, the two actions can occur either in succession or simultaneously, it doesn't matter which. Using the table model, the 4 rows represent the 4 choices in the hat. The choices correspond to the first 4 branches in the tree model. Each row has 3 columns that represent the 3 choices in the box. The rows correspond to the second level of branching. For each of the first 4 choices, there are 3 second choices. So to count the number of members in the array, or the number in the sample space, compute the sum of 4 rows with 3 outcomes in each row: $3 + 3 + 3 + 3$. This is the same as the area model for multiplication. The total number of outcomes is $4 \cdot 3$.

The process discussed above can be summarized in a formal rule in counting called the **Rule of Product.**

THEOREM 10.1: THE RULE OF PRODUCT
If one action can be performed in m ways and a second independent action can be performed in n ways, then there are $m \cdot n$ possible ways to perform both the first action and the second action.

PROBLEM 1

Marty was going to buy a ball and a magazine. He had to choose between a soccer ball, a basketball, or a football. He could choose a game magazine, sports magazine, or comics magazine.

a. Construct a tree diagram to show the sample space of all the possible outcomes.

b. List all the possible outcomes.

c. Use the rule of products to obtain the number of possible outcomes.

EXAMPLE 2

Suppose we change the experiment from Example 1 to the following: Put into a bag the number cards from the hat and the letter cards from the box. Without looking, reach into the bag and pull out one card. How many possible outcomes are there for this experiment and what are they?

SOLUTION

This is the same as choosing one item from the combined sample space {1, 2, 3, 4, A, B, C}. The combined sample space has seven elements, or possibilities.

The main difference between the earlier situation and this one lies in the change of one word. In the first example we chose from the hat and the box, while in the second we chose from the contents of the hat or the box. This illustrates the importance in mathematics of reading words carefully, especially words like "**and**" and "**or**." The following rule captures the number of ways to perform one action "or" another:

THEOREM 10.2: THE RULE OF SUM

If one action can be performed in m ways and a second action can be performed in n ways, then there are $(m + n)$ ways to perform one action or the other, but not both.

PROBLEM 2

George will win if he draws either a red even number or a red face card from a standard deck of cards. Find the following favorable outcomes:

a. even red cards

b. red face cards

c. possible outcomes that will allow him to win.

The rule of sum and the rule of product provide a powerful way to examine experiments made up of several actions. By carefully using the rule of sum and the rule of product, it can often be far easier to compute the number of possible outcomes in such a compound experiment.

PROBLEM 3

Recall the experiment in Example 1: A hat contains the numbers 1,2,3, and 4. A box contains the letters A, B, and C. The experiment is to pick a number from the hat and then pick a letter from the box. Use the sample space to compute the following probabilities:

a. The event of drawing an even number and an A.

b. The event of drawing neither a 1 nor an A.

c. The event of drawing a 1 or an A.

d. The event of drawing an odd number or a B.

EXERCISES

1. Molly and Billy are going to choose an activity to do on Saturday afternoon with their dad. Molly wants to go swimming at the pool, see a movie, or go play miniature golf. Billy wants to go to the beach, go to the zoo, or go to the carnival.

   a. If they each get to choose one of their own activities that they will get to do together as a family, what are the possible outcomes for the day? Draw and label a tree diagram representing all the possible outcomes.

   b. If Molly and Billy decide to spend the day together doing the same activity, what are the possible outcomes?

2. Jennifer bought some new clothes for her summer wardrobe. She bought 4 shirts, 3 pairs of shorts, and 3 pairs of sandals. How many different outfits can she make with her new clothes? (An outfit is different from another outfit if just one of the articles of clothing is different.) Make an organized list to show the sample space.

3. Use the letters J, K, L, M, and N to make passwords.

   a. Make a list of all two-letter passwords you can make.

   b. If you pick one of the new passwords at random, what is the probability that it starts with the letter J?

   c. If you pick one of the new passwords at random, what is the probability that it has a M or N?

4. You have only the following letters to work with: {m, a, t, h}.

   a. How many 3-letter passwords can be made from the four letters?

   b. What is the probability that the password starts with m?

   c. What is the probability that the password does not contain m?

   d. What is the probability that the password contains at least one m?

5. You are ordering a pizza. You must decide between thin crust or pan crust. You have three choices of sizes: small, medium, or large. Your topping choices are pepperoni, sausage, or mushrooms. Make a tree diagram of the possible pizza combinations you can order.

6. Using the list you created in Exercise 5, answer the following questions.
    a. What is the probability of having a pizza with no pepperoni?
    b. What is the probability of ordering a large pizza?
7. Given a four-letter alphabet, find the following:
    a. How many one-letter code words are possible?
    b. How many two-letter code words are possible?
    c. How many three-letter code words are possible?
    d. How many four-letter code words are possible?
    e. How many five-letter code words are possible?
    f. Use a pattern that you observe above to determine how many ten-letter code words are possible.
8. Think about the experiment of rolling a standard die or choosing a digit or choosing a letter from the alphabet. How many outcomes are possible?
9. Consider the experiment of rolling three dice: 1 green, 1 blue, and 1 red. How many outcomes are there in the sample space?
10. Think about the experiment of rolling a standard die. What is the probability of getting an even number or a prime number when you roll the die? Is this answer different than you might expect? Explain your reasoning.
11. Kristen draws a card from a standard 52-card deck and then selects a letter from the alphabet. How many outcomes are possible?
12. Chuck draws one card from a standard 52-card deck. How many ways can he draw a 10 or a face card?

Spiral Review

13. Ms. Tate recorded the time she spent observing students in the classroom at Emerson Elementary. The results are shown in the table below. Draw a circle graph representing the percentage of time Ms. Tate spent observing in each classroom.

Classroom	Time (hours)
A	$\frac{1}{2}$
B	2
C	4
D	$\frac{1}{2}$
E	1

14. Dr. Pascoe tiled his rectangular patio using square tiles. Each box of tile contained 25 square tiles. The patio measures 35 feet $\times$ 20 feet. What else do you need to know in order to find the number of boxes of tile Dr. Pascoe should buy?

15. **Ingenuity:**

    In the country of Mathylvania, each license plate has five digits on it. The digits can be any of the digits from 0 to 9, and the same digit may be used more than once.

    a. How many license plates are possible?

    b. A license plate is said to be good if the sum of the five digits on the license plate is a multiple of 10. How many good license plates are possible?

16. **Investigation:**

    For this Investigation, you will need a fair coin.

    a. Without using the coin, write down a sequence of 50 letters, with each letter either an H or a T. Try to make your sequence as random as possible.

    b. After you have written your sequence, toss the coin 50 times. Each time you toss the coin, keep track of the result. If you get a head, write an H; if you get a tail, write a T. Continue until you have a sequence of 50 letters written down.

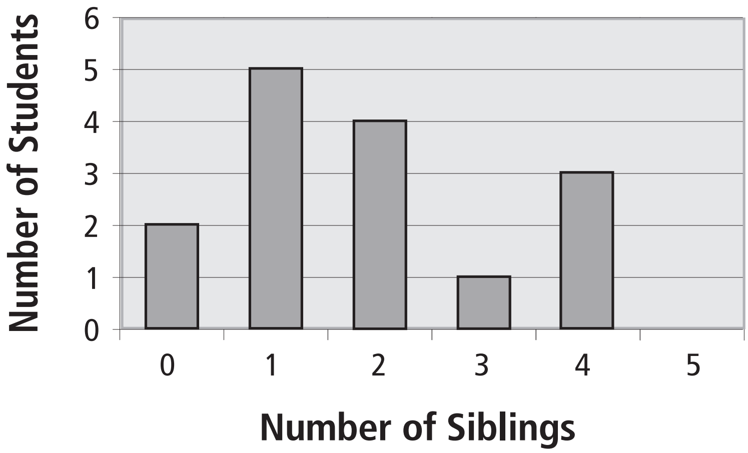
    c. What is the longest run (sequence of consecutive heads or consecutive tails) that you have in the sequence of letters you wrote down in part a? What is the longest run in the sequence you wrote down in b? Which sequence has the longest run?

d. A "switch" in a sequence of H's and T's is a pair of consecutive letters in the sequence that are different. For example, the sequence HHTHTTTH has 4 switches: between the second and third letters, between the third and fourth letters, between the fourth and fifth letters, and between the seventh and eight letters. Count the number of switches in the sequences you made in parts a and b. Which sequence has the most switches?

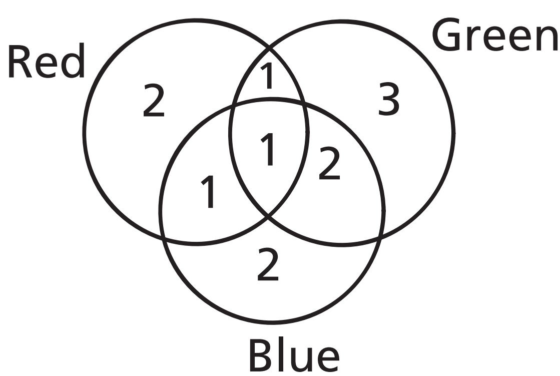
e. Do you notice any other differences between the sequences you wrote down in parts a and b?

SECTION 10.R
CHAPTER REVIEW

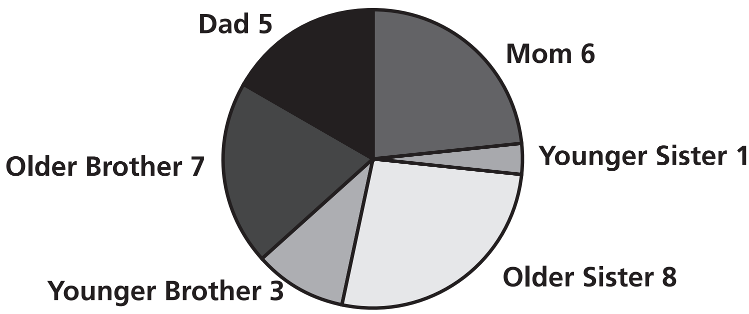
1. Ms. Murphy's class recorded the number of siblings each member of the class has in the bar graph below. Calculate the mean, median, and mode of the data. Can you do this without turning the data into tabular form? Graph the data in a pie chart.



2. Ms. Carnett's class is asked what their favorite colors are. However, many of her students can't decide and have multiple favorite colors. Their preferences are recorded in the Venn Diagram. Make a bar graph that represents the number of hands that will be raised if Ms. Carnett asks her class if their favorite colors are blue, red, or green.



3. Ms. Gaeta asks her class who their favorite member of their immediate family is and records it in a pie chart. Graph this data as a bar graph. What's the probability that a student's favorite family member is a parent? What's the probability that the favorite family member is a female? Is it easier to figure this out from the pie chart or the bar graph?



4. If you roll two six-sided dice, what is the probability that the sum of the two numbers rolled is 5? What is the probability the sum is 3?

5. If I roll two six-sided dice, what is the probability that the product of the two numbers rolled is 5? What is the probability the product is 12?

6. Suppose we form 3-letter code words from the alphabet {a, o, p, t}. How many of these code words are there? How many of these code words start with p? What is the probability that one of these code words starts with p?

7. Write out the set of all the possible outcomes when you flip a coin three times. How do you know that you've listed them all?

8. The table shown lists 24 people and the day of the year they were born.

Day of the Year Born			
Wilby	335	Jose	301
Bence	12	Jeffrey	123
Danette	107	R.J.	260
Terry	156	David	74
Trisha	92	Jacob	103
Kristen	237	Karen	250
Vanessa	155	Maja	13
Teri	352	Jennifer	156
Wesley	274	Tiffany	293
Donovion	50	Kaitlyn	45
Ben	250	Rhonda	216
Andrew	43	Michelle	218

Find the following:

a. mean of the data set

b. median of the data set

c. mode of the data set

MATH OF FINANCE 11

SECTION 11.1
TYPES OF CHARGE CARDS

As you become older, you will be making purchases that are both large and small. Most small purchases can be made using the cash that you carry in your pocket or wallet. For larger purchases you might have noticed that adults often use a plastic credit card. The store clerk often asks a customer if the card is a charge card or a debit card. Another customer might write a check for the amount. These are some of the alternatives to paying with cash.

Let's begin by exploring the difference between a credit card and a debit card. You may use either to make a purchase, but in fact they differ in one important way: when you use a debit card, the money is withdrawn from your bank account immediately. If you do not have enough money in your account, your purchase will usually be denied. On the other hand, using a credit card is like creating a loan. You will still have to pay for some part of the purchase at the end of the month when you receive your credit card statement. If you do not pay the full amount, you will have to pay extra charges and interest on your loan.

To summarize: The main difference between a debit card and a credit card is that a debit card is "pay now" whereas a credit card is "pay later."

While you can use either type of card, the type of card used can make a big difference not just to you but also to the store where you make your purchase.

There are also different ways that your purchase might be approved by the store. One way is called signature approval. With signature approval, you only have to sign the receipt, and your purchase is approved. The other way that a purchase might be approved is through PIN approval. PIN stands for Personal Identification Number. A PIN is usually used with a debit card. When you enter your PIN, the merchant is assured that the card is being used by the real owner. With the PIN consumers are also assured that only they will be able to charge directly from their account.

EXPLORATION 1

Compare the different advantages and costs of credit cards offered by two different local financial institutions. Which features are most important to you? Which card seems to offer the best value?

Some of the costs you find might include monthly fees, interest rate charged, late fees, and transaction fees.

A **monthly fee** is the amount that the bank or credit card company charges each month (or year) to issue a credit card.

The **interest rate** is the rate charged if the bill is not paid in full on time. Some cards charge a different rate even if you do pay your bills on time. Be careful to find out what fees are charged to a card because each card company has different conditions.

If you pay your bill late, then you may be charged a **late fee**. Some companies may also charge a **transaction fee** each time you use your card.

Finally, there are special features for some cards, such as bonus cash for using the card, or cash back depending on the number of purchases made. Some credit cards might give bonus mileage on a Frequent Flyer Program, or free gas if their card is used to purchase gas, or even a discount at a certain store when using their card. When you explored local financial institutions, did they offer some of these features? Describe the features.

Let's talk about some of the vocabulary that is used in finance. For each bank or credit union account that you have, there are three types of transactions: deposits, withdrawals, and transfers.

Deposits add money to your account. You might make a deposit when you receive a paycheck or a gift for your birthday.

Withdrawal takes money out of your account. For example, you might make a withdrawal to take money out of your account for buying clothes.

Transfers are special types of transactions, where money is moved or transferred from one account (a withdrawal) to another account (a deposit). You might make a transfer from your savings account to your checking account by taking money out of your savings account (withdrawal) to pay a bill in your checking account (deposit).

EXAMPLE 1

Sue opened a new savings account on January 15 by depositing $250. On January 18, she made a withdrawal of $100, and on January 24, she transferred $75 from her savings account to her existing checking account to pay the balance she owed in that account. What is her new savings account balance?

SOLUTION

Sue made one deposit of $250. She made one withdrawal from savings of $100. There is one transfer, which is the same as a withdrawal from savings, of $75. So her new savings account balance is: 250 – 100 – 75 = $75.

EXAMPLE 2

Suppose Andy has a checking account with a $800 balance. He has the following income and expenses. Create a check register to record the transactions and keep a running balance of what Andy has at the end of each transaction.

July 1	Mowing the lawn for the month of June	$25
July 12	Birthday money from parents	$40
July 24	Birthday present for mother check #36	$15.25
July 30	Water Park admission ticket check #37	$20
July 31	Babysitting earnings	$24.75

Check #	Date	Transaction Description	$ Withdrawal	$ Deposit	$ Balance
					800

EXERCISES

1. What are the different types of charge cards? How are they different? Which do you prefer and why?
2. Are all debit cards the same? What are some of the differences you discovered?
3. Compare the different features offered by at least two local financial institutions on their debit cards. Then check on the internet to see whether you can find a debit card with more attractive features. Explain your findings.
4. Compare the different costs of at least two local financial institutions' debit cards. Then check on the internet to see whether you can find a debit card with lower costs. Explain your findings.
5. What are some of the differences that various credit cards have?
6. Compare the different features of at least two local financial institutions' credit cards. Then check on the internet to see if you can find a credit card with more attractive features. Explain your findings.
7. Compare the different costs offered by at least two local financial institutions on their credit cards. Then check on the internet to see if you can find a credit card with lower costs. Explain.
8. What could happen if you fail to pay your credit card bill at the end of the month?
9. Do you have to pay a debit card bill at the end of the month? Why, or why not?
10. Cathy's savings account has \$255.25 at the beginning of the month. She makes three withdrawals, for \$25, \$75, and \$43.11. She also makes one new deposit for \$125.45, and transfers \$85 from her savings account to her checking account. What is the balance in her savings account at the end of the month?
11. Suppose you use a plastic card to make a purchase totaling \$300. Compare the different results of making the purchase if you
    a. Use a debit card that is tied to an account in which you have \$400.
    b. Use a debit card that is tied to an account in which you have \$200.
    c. Use a credit card for the purchase and pay the full amount on time when the

payment is due the following month.

d. Use a credit card for the purchase but are unable to pay the full amount when the payment is due the following month, and the card issuer charges 1.5% interest per month on the amount owed.

12. **Exploration:** How do withdrawals, deposits, and transfers affect your balance in a checking account? Ask your teacher to help you find a couple of good internet sites.

13. Mickie's checking account has a $400 balance. She has a number of earnings and expenses listed below. Create a check register to record the transactions and keep a running balance of what Mickie has at the end of each activity day. What is her final balance?

January 22	Baby sitting job	$25
February 13	Valentine cards check #23	$15.25
March 9	Birthday gift from Grandmother	$50
April 1	Baby sitting job	$20
May 26	Teacher's present check #24	$24.75

Check #	Date	Transaction Description	$ Withdrawal	$ Deposit	$ Balance
					400

14. Pick a bank in your city and investigate what services they offer. Is there a charge for a checking account? How much is the charge? Do they offer any interest on the balance? Is there a minimum balance that must be maintained? Is there a charge for a credit card? How much charge? Is there a charge for a debit card?

How much charge? What might be other questions you may have about this bank in comparison with another bank and their fees?

15. Ask your parents which type of card they use the most. Why do they prefer that type of card? Are there different fees associated with each? What are these fees? Do their cards also have special bonuses to encourage them to use any of their cards?

SECTION 11.2
CREDIT REPORTS

When you want to set up a credit card account, you are asking the bank to approve making you a loan. Any purchase made on the credit card is basically a loan from the bank that the cardholder will repay when paying the credit card bill.

That is why to approve a credit card for anyone, the bank must check the potential cardholder's financial history. The bank might ask several of the following questions:

1. How much money do you earn?
2. Do you have other credit cards? What kind of balances do you have on them?
3. Have you ever declared bankruptcy? Bankruptcy is a legal term used when a person cannot repay his debts.
4. How old are you?
5. How long have you had your current job?
6. Do you own your house?
7. What is your social security number?
8. Will you approve a "credit report" of your credit history?

Each of these questions is designed to help the bank decide if a client is "credit worthy." This means they need to learn enough about any potential cardholder to ensure that extending credit to the cardholder will be a good investment for them.

If they approve a credit card, they are really approving making a loan of whatever credit limit is on the card. If they approve a credit limit of $2,000, then they are saying the client can use the credit card to make charges totaling $2,000.

The reason that the bank or credit union asks for a social security number is so that it may obtain a credit report. The credit report will list the different type of assets the client has, as well as detail any times loans have not repaid in a timely manner.

That means every time consumers obtain a credit card, debit card, or checking account, they are building up a credit history. This credit history then becomes part of the credit report that the financial institution or lender will use when deciding if they will approve a loan or credit card. The lender will then get the consumer's credit score from the different credit rating agencies. The credit score will enable the bank to decide if they think the customer is a good credit risk or not. So approval for the loan and the interest rate charged are determined by the credit score. Generally, the higher the credit score, the lower the interest rate. That is why it is so important to maintain a positive credit history.

According to www.consumer.ftc.gov,

The Fair Credit Reporting Act (FCRA) requires each of the nationwide consumer reporting companies — Equifax, Experian, and TransUnion — to provide you with a free copy of your credit report, at your request, once every 12 months. The FCRA promotes the accuracy and privacy of information in the files of the nation's consumer reporting companies. The Federal Trade Commission (FTC), the nation's consumer protection agency, enforces the FCRA with respect to consumer reporting companies.

A credit report includes information on where you live, how you pay your bills, and whether you've been sued or arrested, or have filed for bankruptcy. Nationwide consumer reporting companies sell the information in your report to creditors, insurers, employers, and other businesses that use it to evaluate your applications for credit, insurance, employment, or renting a home.

Time limits for keeping negative financial information:

The three credit-reporting agencies mentioned above can report a consumer's most accurate negative information for seven years and bankruptcy information for ten years. However, there is no time limit on reporting information about criminal convictions.

Why do you need a positive credit history?

You need to pay your bills on time, otherwise your credit report might not be positive. If you have trouble paying your bills on time, the credit agencies will report that you are not as good a risk. In that case, you might have to pay a higher interest rate when you apply for a loan or credit card or be charged a higher deposit for services. If you have a great credit report, a positive credit history will mean a higher credit score from the reporting agencies. This has several benefits. First, a high credit score makes it easier to get a credit card or debit card. Further, it qualifies consumers for lower interest rates when they are approved. Finally, it enables people to qualify not only for credit cards but also for loans such as for a car or a house.

EXERCISES

1. Why is it important to establish a positive credit history? Name three different benefits.
2. What information is described in a credit report? What actions make credit reports better?
3. How long is the information in a credit report kept by the credit agencies?
4. What is the difference between a credit history and a credit report?
5. How is a credit report of value to borrowers? What will a good credit report enable them to do that could not be done otherwise?
6. How does a lender use a credit report? Why is it of value to a lender?
7. What are great credit scores? Medium credit scores? Poor credit scores?
8. How much difference does a credit score make in the rate paid for a loan? Look up car loans and house loans on the web and report what you find.
9. Investigate what it means to talk about a person's credit history. Possible sites to visit are:

   http://www.ftc.gov/bcp/edu/microsites/freereports/index.shtml

   http://www.ftc.gov/bcp/edu/pubs/consumer/credit/cre03.shtm

SECTION 11.3
GOING TO COLLEGE

Why Go?

There are many reasons to consider going to college after high school, including preparation for careers that require a post-secondary education. For example, anyone who wants to be a doctor, engineer, or school teacher needs a college degree. And mathematics will be a huge help, both in getting into college and as a requirement for many degrees in science, technology, engineering, and mathematics (also known as STEM).

College graduates earn more. According to the College Board's 2010 Education Pays report, the average college graduate will earn on average $21,900 per year more than students with only a high school diploma. If you work for 30 years, how much difference would this make in your total earnings?

1. This same report shows that college graduates tend to be healthier, smoke less, and exercise more.
2. College graduates also are more satisfied with their jobs and have greater job stability. In short, they are more likely to be satisfied with what they do, and less likely to become unemployed.

Junior College versus 4-year college?

Many students consider a junior college to start with. These are two-year programs that are typically easier to get into than many four-year programs and are less expensive annually than a four-year college. They tend to have smaller class sizes and often have a curriculum that is more closely tied to a specific degree. However, their curriculum might be limited, and a junior college degree alone probably will not be enough to qualify for many careers, particularly in STEM. However, for many students a junior college degree is great preparation for a four-year college and is a nice transition immediately after high school. A junior college degree also prepares students for many types of jobs that a high school degree does not.

EXPLORATION

Compare the annual salaries of different careers. Find at least one career that only requires a high school education, one career that requires only a Junior College Degree, one career that requires a four-year degree, and one career that requires a more advanced degree, like a M.A. or PhD. Find the annual average salaries for each career, then calculate how much more a worker might make during a career. Use the internet to explore the different possibilities.

There are numerous advantages to either a two-year or four-year diploma over a high school diploma. One of the first questions that arises, however, is how much does a college education cost, and how can a student or family pay for college?

The expenses for all colleges are different. These expenses include tuition, fees, books, travel, room, and board. Individual students need to look at each potential college's cost. After that, the students and their families must decide how to get the money needed for college. Fortunately, there are many ways to get college money.

Let's talk about some options:

1. Savings

Some parents are able to save enough money to pay for their children's college education. Savings can be a huge help. Some parents establish a savings plan early for their children so that they have enough money to pay for college.

However, many students find it necessary to get more funding than available savings. So let's look at other ways to finance a college education.

2. Grants

Many colleges provide grants to help students who need financial help to attend them. There are federal grants and state grants. Check with the college of your choice to find what kind of grants are available. Your guidance counselor might also have information about grants that are independent of a particular college.

3. Scholarships

Colleges also provide scholarships. A scholarship acts a like a grant that is given to the student to pay certain expenses. There are many types of scholarships: sports, academic, needs-based, or other. A needs-based scholarship is given to a student based on financial needs. Check with each college you are considering to see what aid you might qualify for. If you have a special talent, like music, check with the music departments at different schools to see whether they have a scholarship that is dedicated to music majors. People establish many special scholarships to encourage students to enter one particular area. There are many different types of scholarships available, and you might qualify for several different scholarships. The more scholarships you get, the better!

4. Student Loans

The advantage of grants and scholarships is that you don't have to repay the amount you receive. If you don't receive a scholarship or grant, however, you can apply for student loans. There are two types of loans: subsidized and unsubsidized.

A subsidized loan is a loan that is partially supported by someone else. For instance, the government might subsidize the loan, so there is not any interest on the loan until graduation.

An unsubsidized loan is not supported by anyone else. The lending agency will probably begin charging interest from the time you receive the money.

However, each loan is different, so carefully check the conditions of the loan to decide which is best for you.

5. Work-Study

This option allows students to make money through a special program that is partially subsidized or supported by the government. In a work-study program, the students work a certain amount each week, usually 25 hours or less, to help pay for expenses. However, not all students are eligible for work-study. Qualifications depend on financial situation and university policy, so check the qualifications at each institution.

EXERCISES

1. Is it possible to save up enough money to go to college? Why is it difficult for many families to save enough for college?
2. If a family doesn't have enough savings, what are other ways to pay for college?

3. Who generally provides grants to help students go to college? How does a student find out how to qualify for a grant?
4. How can a student find out about different scholarships? Can a student qualify for more than one scholarship?
5. What different types of loans can help pay for college? What is the difference between a loan and a scholarship?
6. What is work-study? How could it help pay for college? Can everyone qualify for work-study?
7. Compare the annual salaries of at least four different occupations, two of which require at least a college degree and two of which don’t.
8. Compare the lifetime salaries, over a 30-year period, of the different professions you listed in problem 7.

MATH EXPLORATIONS

Glossary

Absolute Value

1. The absolute value of a number is its distance from zero.
2. For any x, $|x|$ is defined as follows: $|x| = x$, if $x \geq 0$, and
   $|x| = {}^{-}x$, if $x < 0$

Acute Angle

An angle whose measure is greater than 0 degrees and less than 90 degrees.

Acute Triangle

A triangle in which all three angles are acute angles.

Addition Property of Equality

If $a = b$, then $a + c = b + c$. This property states that adding the same amount to both members of an equation preserves the equality.

Additive identity

A property that states that for any number x, $x + 0 = x$, zero is the additive identity.

Additive Inverse

For any number x, there exists a number ${}^{-}x$, such that $x + {}^{-}x = 0$. This means that there exists a pair of numbers (like 5 and –5) that are the same distance from zero on the number line, and when added together will always produce a sum of zero. These pairs of numbers are also sometimes called "opposites."

Altitude of a Triangle

A segment drawn from a vertex of the triangle perpendicular to the opposite side of the triangle, called the **base** (or perpendicular to an extension of the base).

Angle

An angle is formed when two rays share a common vertex.

Area Model

A mathematical model based on the area of a rectangle, used to represent multiplication or fractional parts of a whole.

Associative Property of Addition

For any numbers x, y , and z: $(x+y)+z = x+(y+z)$. The associative property of addition states that the order in which you group variables or numbers does not matter in determining the final sum.

Associative Property of Multiplication

For any numbers x, y , and z: $(xy)z = x(yz)$. The associative property of multiplication states that the order in which you group variables or numbers does not matter in determining the final product.

Attribute

A distinguishing characteristic of an object. For instance, two attributes of a triangle are angles and sides.

Axis

A number line in a plane. Plural form is axes. Also see: **Coordinate Plane**.

Bar Graph

A graph in which rectangular bars, either vertical or horizontal, are used to display data.

Base

1. If any number x is raised to the nth power, written as x^n, x is called the base of the expression; 2. Any side of a triangle; 3. Either of the parallel sides of a trapezoid; 4. Either of the parallel sides of a parallelogram.

Box and Whisker Plot

For data ordered smallest to largest the median, lower quartile and upper quartile are found and displayed in a box along a number line. Whiskers are added to the right and left and extended to the least and greatest values of the data.

Cartesian Coordinate System

See: **Coordinate Plane**

Center of a Circle

A point in the interior of the circle that is equidistant from all points of the circle.

Chord

A segment whose endpoints are points of a circle.

Circle

The set of points in a plane equidistant from a point in the plane.

Circumference

The distance around a circle. Its length is the product of the **diameter** of the circle and **pi**.

Coefficient

In the product of a constant and a variable the constant is the numerical coefficient of the variable and is frequently referred to simply as the coefficient.

Common Denominator

A common multiple of the denominators of two or more fractions. Also see: **Least Common Denominator**

Common Factor

A factor that two or more integers have in common. Also see: **Greatest Common Factor**.

Common Multiple

See: **Least Common Multiple**.

Complement

The complement of a set E is a set of all the elements that are not in E.

Complementary Angles

Two angles are complementary if the sum of their measures totals 90 degrees.

Composite Number

A **prime** number is an integer p greater than 1 with exactly two positive factors: 1 and p. A **composite** number is an integer greater than 1 that has more than two positive factors. The number 1 is the **multiplicative identity**; that is, for any number n, $n \cdot 1 = n$. The number 1 is neither a prime nor a composite number.

Compound Event

A subset of a sample space containing two or more outcomes.

Concentric circles

Circles with the same center and in the same plane that have different **radii**.

Cone

A three-dimensional figure with a circular base joined to a point called the apex.

Congruent

Used to refer to angles or sides having the same measure and to polygons that have the same shape and size.

Conjecture

An assumption that is thought to be true based on observations.

Constant

A fixed value.

Coordinate(s)

A number assigned to each point on the number line which shows its position or location on the line. In a coordinate plane the ordered pair, (x,y), assigned to each point of the plane, shows the point's position in relation to the x-axis and y-axis.

Coordinate Plane

A plane that consists of a horizontal and vertical number line, intersecting at right angles at their origins. The number lines, called **axes**, divide the plane into four **quadrants**. The quadrants are numbered I, II, III, and IV beginning in the upper right quadrant and moving counterclockwise.

Counterclockwise

A circular movement opposite to the direction of the movement of the hands of a clock.

Counting Numbers

The counting numbers are the numbers in the following never-ending sequence: 1, 2, 3, 4, 5, 6, 7... We can also write this as $^+1$, $^+2$, $^+3$, $^+4$, $^+5$, $^+6$, $^+7$,... These numbers are also called the **positive integers** or **natural numbers**.

Cube

1. A three-dimensional shape having six congruent square faces. 2. The third power of a number.

Cylinder

A three-dimensional figure with parallel circular bases of equal size joined by a lateral surface whose net is a rectangle.

Data

A collection of information, frequently in the form of numbers.

Data Analysis

The process of making sense of collected data.

Data Point

Each individual piece of information collected in a set of data.

Degree

1. The circumference of a circle is divided into 360 equal parts or arcs. Radii drawn to both ends of the arc form an angle of 1 degree. 2. The **degree of a term** is the sum of the exponents of the variables. 3. A degree is also unit of measurement used for measuring temperature. .

Denominator

The denominator of a fraction indicates into how many equal parts the whole is divided. The denominator appears beneath the fraction bar.

Diameter

A segment with endpoints on the circle that passes through its center.

Dividend

The quantity that is to be divided.

Divisibility

Suppose that n and d are integers, and that d is not 0. The number n is **divisible** by d if there is an integer q such that $n = dq$. Equivalently, d is a **factor** of n or n is a **multiple** of d.

Division Algorithm

Given two positive integers a and b, we can always find unique integers q and r such that $a = bq + r$ and $0 \leq r < b$. We call a the dividend, b the divisor, q the quotient, and r the remainder.

Divisor

The quantity by which the dividend is divided.

Domain

The set of input values in a function.

Edge

A segment that joins consecutive vertices of a **polygon** or a **polyhedron**.

Elements

Members of a **set**.

Empirical Probability

Probability determined by real data collected from real experiments.

Equation

A math sentence using the equal sign to state that two expressions represent the same number.

Equilateral Triangle

An **equilateral triangle** is a triangle with three congruent sides. An equilateral triangle also has three congruent angles, which we can also call **equiangular triangle**.

Equivalent

1. A term used to describe fractions or ratios that are equal. 2. A term used to describe fractions, decimals, and percents that are equal.

Event

An event is any subset of the **sample space**. A **simple event** is a subset of the sample space containing only 1 possible outcome of an experiment. A **compound event** is a subset of the sample space containing 2 or more outcomes.

Experiment

A repeatable action with a set of outcomes.

Exponent

Suppose that n is a whole number. Then, for any number x, the nth power of x, or x to the nth **power**, is the product of n factors of the number x. This number is usually written x^n. The number x is usually called the base of the expression x^n, and n is called the exponent.

Exponential Notation

A notation that expresses a number in terms of a base and an exponent.

Expression

A mathematical phrase like "$m + 1$" used to describe quantities mathematically with numbers and variables.

Face

Each of the surface polygons that form a polyhedron.

Factor

An integer that divides evenly into a dividend. Use interchangeably with divisor except in the **Division Algorithm**.

Factorial

The factorial of a non-negative number n is written $n!$ and is the product of all positive integers less than or equal to n. By definition $0! = 1! = 1$.

Fraction

Numbers of the form $\frac{m}{n}$, where n is not zero.

Frequency

The number of times a data point appears in a data set.

Function

A function is a rule which assigns to each member of a set of inputs, called the **domain**, a member of a set of outputs, called the **range**.

Graph of a Function

The pictorial representation of a function.

Greater than, Less Than

Suppose that x and y are integers. We say that x is **less than** y, $x < y$, if x is to the left of y on the number line. We say that x is **greater than** y, $x > y$, if x is to the right of y on the number line.

Greatest Common Factor, GCF

Suppose m and n are positive integers. An integer d is a **common factor** of m and n if d is a factor of both m and n. The **greatest common factor**, or **GCF**, of m and n is the greatest positive integer that is a factor of both m and n. We write the GCF of m and n as GCF (m,n).

Height

The length of the perpendicular between the bases of a parallelogram or trapezoid; also the **altitude of a triangle**.

Horizontal Axis

See: **Coordinate Plane**.

Hypotenuse

The side opposite the right angle in a right triangle.

Improper Fraction

A fraction in which the numerator is greater than or equal to the denominator.

Independent Events

If the outcome of the first event does not affect the outcome of the second event.

Input Values

The values of the domain of a function.

Integers

The collection of integers is composed of the counting numbers, the negatives, and zero; ... $^-4$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4...

Isosceles Triangle

A triangle with at least two sides of equal length.

Lateral Area

The surface area of any three-dimensional figure excluding the area of any surface designated as a base of the figure.

Lattice Point

A point of the coordinate plane, (x,y), in which both x and y are integers.

Least Common Denominator

The least common denominator of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is the **least common multiple** of n and m, LCM(n, m).

Least Common Multiple, LCM

The integers a and b are positive. An integer m is a **common multiple** of a and b if m is a multiple of both a and b. The **least common multiple**, or **LCM**, of a and b is the smallest integer that is a common multiple of a and b. We write the LCM of a and b as LCM (a,b).

Legs

1. The two sides of a right triangle that form the right angle. 2. The equal sides of an isosceles triangle or the non-parallel sides of a trapezoid.

Less than

See: **Greater Than**.

Line graph

A graph used to display data that occurs in a sequence. Consecutive points are connected by segments.

Line Plot

A graph that shows frequency of data along a number line.

Linear Model for Multiplication

Skip counting on a number line.

Mean

The average of a set of data; sum of the data divided by the number of items. Also called the **arithmetic mean** or **average**.

Measures of Central Tendency

Generally measured by the **mean**, **median**, or **mode** of the **data set**.

Median

The middle value of a set of data arranged in increasing or decreasing order. If the set has an even number of items the median is the average of the middle two items.

Missing Factor Model

A model for division in which the quotient of an indicated division is viewed as a missing factor of a related multiplication.

Mixed fraction (Numbers)

The sum of an **integer** and a **proper fraction**.

Mode

The value of the element that appears most frequently in a **data set**.

Multiplicative Identity

See: **Composite Numbers**.

Multiplicative Inverse

The number x is called the multiplicative inverse or reciprocal of n, $n \neq 0$, if $x \cdot n = 1$.

Natural Numbers

See: **Counting Numbers**.

Negative Integers

Integers less than zero.

Notation

A technical system of symbols used to convey mathematical information.

Number Line

A pictorial representation of numbers on a straight line.

Numerator

The expression written above the fraction bar in a common fraction to indicate the number of parts counted.

Obtuse Angle

An angle whose measure is greater than 90 degrees and less than 180 degrees.

Obtuse Triangle

A triangle that has one obtuse angle.

Order Of Operations

The order of mathematical operations, with computations inside parentheses to be done first, and addition and subtraction from left to right done last.

Ordered Pair

A pair of numbers that represent the coordinates of a point in the coordinate plane with the first number measured along the horizontal scale and the second along the vertical scale.

Origin

The point with coordinate 0 on a number line; the point with coordinates (0,0) in the coordinate plane.

Outcomes

The set of possible results of an experiment.

Outlier

A term referring to a value that is drastically different from most of the other data values.

Output Values

The set of results obtained by applying a function rule to a set of **input values**.

Parallel Lines

Two lines in a plane that never intersect.

Parallelogram

A parallelogram is a four-sided figure with opposite sides parallel.

Percent

A way of expressing a number as parts out of 100; the numerator of a ratio with a denominator of 100.

Perfect Cube

An integer n that can be written in the form $n = k^3$, where k is an integer.

Perfect Square

An integer n that can be written in the form $n = k^2$, where k is an integer.

Perimeter

The **perimeter** of a polygon is the sum of the lengths of its sides.

Perpendicular

Two lines or segments are perpendicular if they intersect to form a right angle.

Pi

The ratio of the **circumference** to the **diameter** of any circle, represented either by the symbol $\boldsymbol{\pi}$, or the approximation $\frac{22}{7}$, or 3.1415926...

Pie (Circle) Graph

A graph using sectors of a circle that are proportional to the percent of the data represented.

Polygon

A simple, closed, plane figure formed by three or more line segments..

Polyhedron

A three-dimensional figure with four or more faces, all of which are polygons.

Positive Integers

See: **Counting Numbers**.

Power

See: **Exponent**.

Prime Number

See: **Composite Number**.

Prime Factorization

The process of finding the prime factors of an integer. The term is also used to refer to the result of the process.

Prism

A type of **polyhedron** that has two bases that are both congruent and parallel, and lateral faces which are parallelograms.

Probability

In an experiment in which each outcome is equally likely, the probability $P(A)$ of an event A is $\frac{m}{n}$ where m is the number of outcomes in the subset A and n is the total number of outcomes in the sample space S.

Proper Fraction

A fraction whose value is greater than 0 and less than 1.

Proportion

An equation of ratios in the form $\frac{a}{b} = \frac{c}{d}$, where b and d are not equal to zero.

Protractor

An instrument used to measure angles in degrees.

Quadrant

See: **Coordinate Plane**.

Quadrilateral

A plane figure with four straight edges and four angles.

Quotient

The result obtained by doing division. See the **Division Algorithm** for a different use of quotient.

Radius

The distance from the center of a circle to a point on the circle. Plural form is **radii**.

Range

The difference between the largest and smallest values of a data set. See **Function** for another meaning of range.

Rate

A rate is a division comparison between two quantities with different units. Also see **Unit Rate**.

Ratio

A division comparison of two quantities with or without the same units. If the units are different they must be expressed to make the ratio meaningful.

Rational Number

A number that can be written as $\frac{a}{b}$ where a is an integer and b is a natural number.

Ray

Part of a line that has a starting point and continues forever in only one direction.

Reciprocal

See: **Multiplicative Inverse**.

Regular Polygon

A polygon with equal side lengths and equal angle measures.

Relatively Prime

Two integers m and n are relatively prime if the GCF of m and n is 1.

Remainder

See: **Division Algorithm**.

Repeating Decimal

A decimal in which a cycle of one or more digits is repeated infinitely.

Right Angle

An angle formed by the intersection of perpendicular lines; an angle whose measure is 90º.

Right Triangle

A triangle that contains a right angle.

Sample Space

The set of all possible outcomes of an experiment.

Scaffolding

A method of division in which partial quotients are computed, stacked, and then combined.

Scalene Triangle

A triangle with all three sides of different lengths is called a scalene triangle.

Scaling

1. A process by which a shape is reduced or expanded proportionally. 2. Choosing the unit of measure to be used on a number line.

Sector

A part of a circle that represents the interior portion of the circle between two radii.

Sequence

A list of terms ordered by the natural numbers.

Set

A collection of objects or elements.

Simple Event

See: **Event**

Simplest Form of a Fraction

A form of a fraction in which the greatest common factor of the numerator and denominator is 1.

Simplifying

The process of finding equivalent fractions to obtain the simplest form.

Skewed

An uneven representation of a set of data.

Slant Height

An altitude of a face of a pyramid or a cone.

Square Root

For non-negative numbers x and y, $y = \sqrt{x}$, read "y is equal to the square root of x," means $y^2 = x$.

Stem and Leaf Plot

A method of showing the frequency of a certain data by sorting and ordering the values.

Straight Angle

An angle with a measure of 180 degrees formed by opposite rays.

Subset

Set B is a subset of set A if every element of set B is also an element of set A.

Supplementary

Two angles are supplementary if the sum of their measures totals 180º.

Surface Area

The surface area of a three-dimensional figure is the area needed to form its exterior.

Terminating Decimal

If the quotient of a division problem contains a remainder of zero, the quotient is said to be a terminating decimal.

Tessellation

Tiling of a plane with one or more shapes as a way of covering the plane with the shape(s) with no gaps or overlaps. .

Theoretical Probability

Probability based on thought experiments rather than a collection of data.

Translation

A transformation that slides a figure a certain distance along a line in a specified direction.

Trapezoid

A four sided plane figure with exactly one set of parallel sides.

Tree Diagram

1. A process used to find the prime factors of an integer. 2. A method to organize the sample space of compound events.

Triangle

A plane figure with three straight edges and three angles.

Trichotomy

A property stating that exactly one of these statements is true for each real number: it is positive, negative, or zero.

Unit Fraction

For an integer n, the multiplicative inverse or reciprocal of n is the unit fraction $\frac{1}{n}$. $\frac{1}{n}$ is said to be a unit fraction because its numerator is 1.

Unit Rate

A ratio of two unlike quantities that has a denominator of 1 unit.

Variable

A letter or symbol that represents an unknown quantity.

Venn Diagram

A diagram involving two or more overlapping circles that aids in organizing data.

Vertex

1. The common endpoint of two rays forming an angle. 2. A point of a polygon or polyhedron where edges meet.

Vertical Angles

A pair of angles of equal measure less than 180° that are formed by opposite rays of a pair of intersecting lines.

Vertical Axis

See: **Coordinate Plane**.

Volume

A measure of space; the number of unit cubes needed to fill a three-dimensional shape.

Whole Numbers

The **whole numbers** are the numbers in the following never-ending sequence: 0, 1, 2, 3, 4, 5, ... These numbers are also called the non-negative integers.

***x*-axis**

The horizontal axis of a coordinate plane.

***y*-axis**

The vertical axis of a coordinate plane.

***x*-coordinate**

The second number provided in an ordered pair (a, b).

***y*-coordinate**

The first number provided in an ordered pair (a, b).

Additive Identity

For any number x, $x+0=x$.

Additive Inverses

For any number x, there exists a number ^{-}x, called the additive inverse of x, such that $x+(^{-}x)=0$.

Additive Property of Equality

If $A=B$, then $A+C=B+C$.

Area of a Circle

The area of a circle with radius r is $A=\pi r^2$ square units.

Area of a Parallelogram

The area of a parallelogram with base b and height h is given by $A=b\cdot h$.

Area of a Rectangle

The area of a rectangle with length l and width w is $A=l\cdot w$.

Area of a Square

The area of a square with sides s is $A=s^2$

Area of a Trapezoid

The area of a trapezoid with height h and bases b_1 and b_2 is $\frac{1}{2}\cdot(b_1+b_2)\cdot h=\frac{(b_1+b_2)h}{2}$.

Area of a Triangle

The area of a triangle with base b and height h is given by $A=\frac{1}{2}bh$ or $A=\frac{bh}{2}$.

MATH EXPLORATIONS

Summary of Ideas

Associative Property of Addition

For any numbers x, y, and z, $(x+y)+z=x+(y+z)$.

Associative Property of Multiplication

For any numbers x, y, and z, $(xy)z=x(yz)$.

Circumference of a Circle

The circumference C of a circle with radius r and diameter d is given by $C = 2\pi$ or $C=\pi d$.

Commutative Property of Addition

For any numbers x and y, $x+y=y+x$.

Commutative Property of Multiplication

For any numbers a and b, $ab=ba$.

Distributive Property of Multiplication Over Addition

$a(k+m)=ak+am$

Division Algorithm

Given two positive integers a and b, we can always find unique integers q and r such that $a=bq+r$ and $0 \le r < b$. We call a the dividend, b the divisor, q the quotient, and r the remainder.

Equivalent Fraction Property

For any number a and nonzero numbers k and b, $\frac{a}{b}=\frac{ka}{kb}=\frac{ak}{bk}$.

Fractions and Division

For any number m and nonzero n the fraction $\frac{m}{n}$ is equivalent to the quotient $n\overline{)m}$ or $m \div n$. .

MATH EXPLORATIONS

Summary of Ideas

Fundamental Theorem of Arithmetic

If n is a positive integer, $n > 1$, then n is either prime or can be written as a product of primes $n = p^1 \cdot p^2 \cdot \ldots \cdot p^k$, for some prime numbers $p^1, p^2, \ldots, p^k$ such that $p^1 \leq p^2 \leq \ldots \leq p^k$, where k is a natural number. In fact, there is only one way to write n in this form

Multiplication of Powers

Suppose that x is a number and a and b are whole numbers, then $x^a \cdot x^b = x^{(a+b)}$

Multiplicative Identity

The number 1 is the multiplicative identity; that is, for any number n, $n \cdot 1 = n$.

Multiplicative Inverse

For every non-zero x there exists a number $\frac{1}{x}$, called the multiplicative inverse or reciprocal of x, such that $x \cdot \frac{1}{x} = 1$.

Multiplying Fractions

The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where b and d are nonzero, is $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

The Rule of Products

If one action can be performed in m ways and a second independent action can be performed in n ways, then there are $m \cdot n$ possible ways to perform both actions.

The Rule of Sums

If one action can be performed in m ways and a second action can be performed in n ways, then there are $(m + n)$ ways to perform one action or the other, but not both. This assumes that each action is equally likely and mutually exclusive.

Triangle Sum Theorem

The sum of the measures of the angles of a triangle equals 180º.

Sums With Like Denominators

The sum of two fractions with like denominators, $\frac{a}{n}$ and $\frac{b}{n}$, is given by $\frac{a}{n} + \frac{b}{n} = \frac{a+b}{n}$.

Volume of a Cube

The volume of a cube with each side of length s units is s^3 cubic units. $V = s^3$ or $V = Bh$ where B is the base area of the three-dimensional figure and h is the height.

Volume of a Prism

The volume of a prism is the area of the base of the three-dimensional figure multiplied by the height of the prism. $V = Bh$ where B is the base area. For a rectangular prism with length l, width w and height h; $V = l \cdot w \cdot h$.

Glosario

Acute angle/ Ángulo Agudo

Un ángulo de medida mayor de 0 grados y menor de 90 grados.

Acute triangle/ Triángulo Agudo

Un triángulo en el cual los tres ángulos son ángulos agudos.

Altitude of a triangle/ Altitud de un Triángulo

Un segmento dibujado desde un vértice del triángulo perpendicular al lado opuesto del triángulo, llamada la **base**, (o perpendicular a una extensión de la base).

Angle/Ángulo

Un ángulo se forma cuando dos rayos comparten un punto o vértice común.

Area model/ Modelo basado en el Área

A mathematical model based on the area of a rectangle, used to represent multiplication or to represent fractional parts of a whole.

Associative Property/ Propiedad Asociativa

Una propiedad usada en adición y multiplicación que declara que la orden de agrupación no importa para determiner el resultado. Por ejemplo, por cualquier numeros x, y, y z, $(x + y) + z = x + (y + z)$ y $(xy)z = x(yz)$.

Attribute/ Atributo

Una característica distintiva de un objeto tal como ángulos o lados de un triángulo.

Axis/ Eje

Una recta numérica en un plano. Ver también: **Plano Cartesiano**

Bar graph/ Diagrama de Barras

Una forma de representar gráficamente un conjunto de datos o valores y está conformado por barras, ya sea vertical o horizontal, rectangulares de longitudes proporcionales a los valores representados.

Base/ Base

1. Por cualquier valor x, elevado a la potencia de n, escrito como x^n, x es llamado la base de la potencia; 2. Cualquier lado de un triangulo; 3. Cualquiera de los lados paralelos de un trapecio.

Box and Whisker Plot/ Gráfica de Caja y Bigotes

Para datos ordenados del mínimo a el máximo, la mediana, y el cuartile bajo y cuartile alto son mostrados en cajas sobre una recta numérica. Los brazos muestran los extremos de los datos.

Cartesian Plane/ Plano Cartesiano

Ver: **Coordinate Plane**

Center of a circle/ Centro de un Círculo

Un punto que se encuentra a la misma distancia de todos los otros puntos en la circunferencia del círculo.

Chord/ Cuerda

Un segmento que une dos puntos de la circunferencia.

Circle/ Círculo

El conjunto de puntos en un plano equidistantes de un punto en el plano.

Circumference/ Circunferencia

La distancia alrededor de un círculo. Su longitud es el producto del **diámetro** del círculo y **pi**.

Coefficient/ Coeficiente

En el producto de un constante y un variable, el constante es el factor numérico de el termino y es referido como el coeficiente.

Common Denominator/ Común Denominador

Un múltiplo común de los denominadores de dos o más fracciones. Ver también: **Menor Común Denominador**

Common Factor/ Factor Común

Un factor que dos o más números enteros tienen en común. Ver también: **Máximo común divisor**

Common Multiple/ Común Múltiplo

Ver también: **Mínimo común múltiplo**

Complement/ Complemento de un Conjunto

El complemento de un conjunto es un conjunto de todos los elementos del conjunto universal que son no estan en el conjunto dado.

Complementary Angles/ Angulos Complementarios

Dos ángulos son complementarios si la suma de sus medidas es 90º.

Composite number/ Número Compuesto

Un número primo es un entero p mayor que 1 con exactamente dos factores positivos: 1 y p. Un número compuesto es un entero mayor que 1 que tiene más de dos factores positivos. El número 1 es ni primo ni un número compuesto.

Compound Interest/ Interés Compuesto

Interés pagado sobre interés previamente ganado mas el principal.

Concentric circles/ Círculos concéntricos

Círculos con el mismo centro en el mismo plano pero con diferentes radios.

Congruent/ Congruente

Se utiliza para referirse a los ángulos o lados que tienen la misma medida y los polígonos de mismo tamaño y figura.

Conjecture/ Conjetura

Una suposición que se supone cierta basado en observaciones.

Constant/ Constante

Un valor fijo.

Coordinate(s)/ Coordenado(as)

Un numero asignando a un punto en la recta numerica. En el plano coordenado, el par ordenado (x,y) situa cada punto y lo situa en respecto a los ejes de x, y y.

Coordinate Plane/ Plano Cartesiano

Un plano está formado por dos rectas numéricas perpendiculares, una horizontal y otra vertical que se intersectan en un punto que llamamos origen. Las rectas numéricas, llamados ejes, dividen el plano en cuatro **cuadrantes**. Estos estan numerados I, II, III y IV comensando en el cuadrante en la esquina derecha de arriba y siguiendo sinistrorso.

Counterclockwise/ Sinistrorso

Un movimiento circular opuesto a la dirección de movimiento de las manecillas del reloj.

Counting numbers/ Números de Conteo

Los números de conteo son números en la siguiente secuencia: 1, 2, 3, 4, 5, 6, 7... También podemos escribir esto como $^{+}1$, $^{+}2$, $^{+}3$, $^{+}4$, $^{+}5$, $^{+}6$, $^{+}7$... Estos números también son llamados **números enteros positivos** o **números naturales**.

Cube/ 1. Cubo 2. Elevado al Cubo

1. Una figura tridimensional que tiene seis caras cuadradas congruentes. 2. La tercera potencia de un número.

Data/ Datos

Una colección de información frequentemente en la forma de números.

Data analysis/ Analize de Datos

El proceso de entender los datos colecciónados.

Degree/ Grado

1. La circumferencia de un circulo es dividida en 360 partes iguales o arcos. Radios dibujados a los terminos de uno de estos arcos forman un ángulo de 1 grado. 2. El **grado del termino** es la suma de las potencias de los variables. El **grado de un polinomio** es la potencia mas grande de sus integrantes.

Denominator/ Denominador

El denominador de una fracción indica cuántas partes iguales se han dividido el entero. El denominador aparese en la parte de abajo de la fracción.

Diameter/ Diámetro

Un segmento que une dos puntos opuestos en una circunferencia y pasa por el centro del círculo.

Dividend/ Dividendo

La cantidad que se divide.

Divisibility/ Divisibilidad

Suponiendo que n y d son enteros, y que $d \neq 0$. El número n es divisible por d si hay un entero q de tal manera que $n = dq$. Por igual, d es un factor de n o n es un multiple de d.

Division Algorithm/ Algoritmo de División

Dados dos enteros positivos a y b, siempre podemos encontrar enteros unicos q y r de tal manera que $a = bq + r$ y $0 \leq r < b$. Llamamos a el dividendo, b el divisor, q el cuociente, y r el restante.

Divisor/ Divisor

La cantidad por cual el dividendo se divide.

Domain/ Dominio

El conjunto de primeros numeros de una función.

Edge/ Lado

Un segmento que une vértices consecutivas de un **polígono** ó un **poliedro.**

Elements/ Elementos

Miembros de un **conjunto**.

Empirical Probability/ Probabilidad Empírica

Probabilidad determinada por datos colecciónados de experimentos reales.

Equation/ Ecuación

Una proposición matemática con igualdad señalando que dos expresiones son iguales.

Equilateral triangle/ Triángulo Equilátero

Un triángulo con tres lados congruentes. Un triángulo equilátero también tiene tres ángulos congruentes.

Equivalent/ Equivalente

1. Dos ecuaciones ó desigualdades son iguales si tienen la misma solución ó conjunto de soluciones. 2. Un término para describir fracciones ó proporciones que son igual. 3. Un término para describir fracciones, decimales, y porcentajes que son iguales

Event/ Evento

Un evento es un subconjunto del **espacio muestral**. Un **evento simple** es un subconjunto del espacio muestral conteniendo solamente un resultado del experimento. Un **evento compuesto** es un subconjunto del espacio muestral que contiene dos ó más resultados.

Experiment/ Experimento

Una acción repetible con un conjunto de resultados.

Exponent/ Exponente (Potencia)

Supongamos que n es un número entero. Por cualquier valor x, elevado a la potencia de n, escrito como x^n, x es llamado la base de la expresión, y n es el exponente ó potencia.

Exponential Notation/ Notación Exponencial (Notación Potencial)

Una notación para expresar un número en términos de una base y una potencia.

Expression/ Expresión:

Un termino matematico como "m + 1" usado para describir cantidades matematicamente.

Face/ Cara

La superficie de cada polígono que forma un poliedro.

Factor/ Factor

Un entero que divide un dividendo exactamente. Intercambiable con el divisor excepto en el **Algoritmo de División**.

Factorial/ Factorial

El factorial de un número que no es negativo n es escrito como $n!$ y es el producto de todos los números positivos menor o igual a n. Por definición $0! = 1! = 1$.

Fraction/ Fracción:

Por cualquier numero m y numero no igual a cero n, la fracción es equivalente a el cociente de m dividido por n.

Frequency/ Frecuencia

El número de veces que un dato aparece en un conjunto.

Function/ Función

Una función es una regla que asigna a cada primer número de un conjunto, llamado **dominio**, un numero de salida, llamado el **rango**. La regla no permite que los primeros numeros se repitan.

Graph of a function/ Gráfica de una Función

Una representación pictorica de una función graficando pares ordenados en el sistema coordinado.

Greater than, Less Than/ Mayor Que, Menor Que

Supongamos que x y y son números enteros. Decimos que x es **menor que** y, $x<y$, si x esta a la izquierda de y en la recta numérica. Decimos que x es **mayor que**, $x>y$, si x esta a la derecha de y en la recta numérica.

Greatest common factor, GCF/ Máximo Común Divisor, MCF

Supongamos que m y n son números enteros positivos. Un entero d es un **factor común** de m y n si d es un factor de m y de n. El **máximo común divisor** ó **MCF** de m y n es el entero positive mas grande que es un factor de m y n. Escrito como MCF de m y n ó como MCF(m,n).

Height/ Altura

La longitud de el segmento perpendicular que une las bases de un paralelogramo o un trapezoide. También la **altitud de un triángulo.**

Horizontal axis/ Eje Horizontal

Ver también: **Plano Cartesiano**.

Hypotenuse/ Hipotenusa

El lado opuesto al ángulo recto en un triángulo recto.

Improper fraction/ Fracción Impropia

Una fracción en la cual el numerador es mayor que ó igual a el denominador.

Independent events/ Eventos Independientes

El resultado del primer evento no afecta el resultado del segundo evento.

Input values/ Primeros Valores

Los valores del dominio de una función.

Integers/ Enteros

La colección de enteros es compuesta de números negativos, cero y los números positivos: ... $^-4$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4...

Isosceles triangle/ Triangulo Isósceles

Un triángulo con por lo menos dos lados de la misma medida es llamado un triangulo **isósceles**.

Lattice point/ Puntos de Latice

Un punto en el plano cartesiano (x,y) en el cual x y y son enteros.

Least Common Denominator/ Mínimo Común Denominador

El mínimo común denominador de las fracciones $\frac{p}{n}$ y $\frac{k}{m}$ es MCM (n, m).

Least common multiple, LCM/ Mínimo Común Múltiplo, MCM

Si a y b son enteros positivos. Un entero m es un múltiplo común de a y b si m es un múltiplo de a y de b. El mínimo común múltiplo ó MCM de a y b es el entero positive mas chico que es un múltiplo de a y b. Escrito como MCM de a y b ó como MCM (a,b).

Legs/ Catetos

1. Los dos lados que forman el ángulo recto de un triángulo recto. 2. Los lados iguales de un triángulo isósceles o los lados no paralelos de un trapecio.

Less than

Ver también: **Mayor Que**.

Line graph/ Gráfico de Líneas

Un gráfico utiliza para mostrar datos que se produce en una secuencia. Puntos consecutivos están conectados por segmentos.

Line Plot/ Gráfico de Puntos

Una gráfica que muestra la frecuencia de un conjunto de datos por encima de una recta numérica, o eje horizontal.

Line of symmetry/ Eje de Simetria

Recta $\boldsymbol{L}$ es un eje de simetria para la figura si por cada punto $\boldsymbol{P}$ en la figura hay un punto $\boldsymbol{Q}$ en la figura de tal manera que $\boldsymbol{L}$ es el mediatriz del segmento $\boldsymbol{PQ}$.

Linear Model for Multiplication/ Modelo Lineal para Multiplicar

Contar en grupos usando la recta numérica.

Mean/ Media Aritmética

El promedio de un conjunto de datos; la suma de los datos dividida por el número de datos.

Measures of Central Tendency/ Medidas de Tendencia Central

Medidas generalmente por la **media**, la **mediana** y la **moda** de el conjunto de datos.

Median/ Mediana

El valor medio en un conjunto de datos ordenados. Si el conjunto tiene un número par de datos, la mediana es el promedio de los dos números de en medio.

Missing Factor Model/ Modelo de Factor Faltante

Un modelo de división en el cual el cociente de la división indicada es visto como un factor faltante de una multiplicación relacionada.

Mixed fraction/ Número Mixto

La suma de un **entero** y una **fracción propia**.

Mode/ Moda

El valor del elemento que aparece mas veces en un conjunto de datos.

Multiplicative Identity/ Identidad Multiplicativa

Ver: **Composite Numbers.**

Multiplicative Inverse/ Inverso Multiplicativo

El numero x es llamado el inverso multiplicativo o reciproco de n, $n \neq 0$, si $xn = 1$.

Natural numbers

Ver también: **Números de Conteo**

Negative integers/ Números Enteros Negativos

Enteros con valor menor que cero..

Notation/ Anotación

Un sistema técnico de simbolos usado para mostrar información matemática.

Numerator/ Numerador

La expressión en la parte de ariba de una fracción que indica el número de partes contadas.

Obtuse Angle/ Ángulo Obtuso

Un ángulo con medida mayor que 90 grados y menor que 180 grados.

Obtuse Triangle/ Triángulo Obtuso

Un triángulo que tiene un ángulo obtuso.

Ordered pair/ Par Ordenado

Un par de números que representa los coordinados de un punto en el plano coordinado con el primer numero señalando la medida en el eje horizontal y el segundo señalando la medida en el eje vertical.

Origin/ Origen

El punto con el coordinado 0 el la recta numérica; el punto con los coordinados (0,0) en el plano cartesiano.

Outcomes/ Numeros de Salida

El conjunto de resultados posibles de un experimento.

Outlier/ Valor Extremo

Un término que se refiere a un valor que es completamente diferente a los demas valores.

Output Values/ Valores de Salida

El conjunto de valores obtenidos al aplicar una función a un conjunto de primeros valores.

Parallel lines/ Líneas Paralelas

Dos líneas en un plano que nunca cruzan.

Parallelogram/ Paralelogramo

Es un cuádrilatero con los lados opuestos paralelos.

Percent/ Por ciento

Una manera de expresar un número como partes de cien. El numerador de una proporción con 100 como denominador.

Perfect Cube/ Cubo Perfecto

Un entero n que puede ser escrito en la forma $n= k^3$, y k es un entero.

Perfect Square/ Cuadrado Perfecto

Un entero n que puede ser escrito en la forma $n= k^2$, y k es un entero.

Perimeter/ Perímetro

El **perímetro** de un polígono es la suma de la medida de los lados.

Perpendicular/ Perpendicular

Dos líneas o segmentos se dicen ser perpendiculares si cruzan a formar un ángulo recto.

Pi/ Pi

La proporción de la circumferencia a el diametro de cualquier círculo, representado por el símbolo π, o la aproximación $\frac{22}{7}$ o 3.1415926…

Pie (Circle) Graph/ Gráfica Círcular

Una gráfica usando sectores de un círculo que son proporciónal al por ciento de los datos representados.

Polygon/ Polígono

Un polígono es una figura simple plana formado por tres o más segmentos.

Polyhedron/ Poliedro

Una figura tridimensional con cuatro o más caras todas cuales son polígonos.

Positive integers/ Números Enteros Positivos

Ver también: **Counting Numbers/Números de Conteo**

Power/ Potencia

Ver también: **Exponent**

Prime Number/ Número Primo

Ver también: **Composite Number**

Prime Factorization/ Factorización en Primos

El proceso de encontrar los factores primos de un entero. El termino también se refiere al resultado de este proceso.

Prism/ Prisma

Un tipo de poliedro que tiene dos bases que son paralelas y congruentes, y caras laterales que son paralelogramos.

Probability/ Probabilidad

En un experimento en el cual cada resultado tiene la misma oportunidad, la probabilidad $P(A)$ de un evento A es $\frac{m}{n}$ donde m es el número de resultados en el subconjunto A y n es el numero total de resultados en el espacio muestral S.

Proper fraction/ Fracción Propia

Una fracción con valor mayor que 0 y menor que 1.

Proportion/ Proporción

Una equación de razones en la forma $\frac{a}{b} = \frac{c}{d}$, donde b y d no son iguales a cero.

Protractor/ Transportador

Un instrumento para medir ángulos en grados.

Pyramid/ Pirámide

Un tipo de poliedro que tiene una cara llamada la **base**, y caras laterales triangulares que topan en un punto llamado el vertice.

Quadrant/ Cuadrante

Ver también: **Coordinate Plane/ Plano Cartesiano**.

Quotient/ Cociente

El resultado obtenido al hacer división. Ver **Algoritmo de División** para un uso diferente del cociente.

Radius/ Radio

La distancia del centro de un círculo a un punto en la periferia.

Range/ Rango

La diferencia entre el valor mas grande y el valor mas chico en un conjunto de datos. Ver Function por otras definiciónes de rango.

Rate/ Tasa de Variación

Una comparación por medio de un cociente, entre dos cantidades con diferentes unidades. Ver también: **Unit Rates/ Tasa de Unidad**.

Ratio/ Razón

Una comparación por medio de un cociente. Si las unidades son diferentes, la razón tiene que tener sentido.

Rational Number/ Número Racional

Un número que se puede escribir como $\frac{a}{b}$, donde a es un número entero y $b$es un número natural.

Ray/ Rayo

Parte de una recta con un extremo y se prolonga sin limite en una dirección.

Reciprocal/ Reciproco

Ver también: **Multiplicative Inverse/ Inverso Multiplicativo**.

Regular Polygon/ Poligono Regular

Un polígono con lados de mismas medidas y ángulos de mismas medidas.

Relatively Prime/ Números Primos Entre Si

Dos enteros m y n son números primos entre si, si el MCF de m y n es 1.

Remainder/ Restante

Ver también: **División Algorithm/ Algoritmo de División**.

Repeating decimal/ Decimal Periódico

Un decimal en el que se repiten uno o mas digitos sin terminación.

Right Angle/ Ángulo Recto

Un ángulo formado cuando cruzan dos líneas perpendiculares; Un ángulo que mide 90 grados.

Right Triangle/ Triángulo Recto

Un triángulo que tiene un ángulo recto.

Sample Space/ Espacio Muestral

El conjunto de todos los resultados posibles de un experimento.

Scaffolding/ Andamiaje

Un metodo de división en el cual cocientes parciales son computados, apilados, y combinados.

Scale Factor/ Factor de Escala:

Si polígonos A y B son similares y s es un número positivo que para cada lado de polígono A con medida k hay un lado correspondiente en B con medida sk, entonces s es el factor de escala de A y B.

Scalene Triangle/ Triángulo Escaleno

Un triángulo con los tres lados de diferentes medidas es llamado un **triángulo escaleno**.

Scaling/ Escalar

1. El proceso en el cual una figura se reduce o aumenta proporciónalmente.
2. Escojer la unidad de medida que sera usada en la recta numérica.

Sector/ Sector

Una región de un circulo rodeado por dos radios y un arco que une sus extremos.

Sequence/ Secuencia

Un conjunto de terminos puestos en orden por los números naturales. Los números de salida de una función de dominio que incluye los números naturales o enteros.

Set/ Conjunto

Una colección de objetos o elementos.

Simple event/ Evento Simple

Ver también: **Event/ Evento**.

Simplest Form of a Fraction/ Forma Más Simple de una Fracción

Una forma en la cual el máximo comun factor del numerador y denominador es 1.

Simplifying/ Simplificar

El proceso de encontrar fracciónes equivalentes para obtener la forma mas simple.

Skewed/ Sesgado

Una representación de un conjunto de datos desiguales.

Stem and Leaf Plot/ Diagrama de Tallo y Hojas

Un metodo para enseñar la frecuencia de ciertos datos al ordenarlos.

Straight Angle/ Ángulo Llano

Un ángulo que mide 180 grados formado por rayos opuestos.

Subset/ Subconjunto

Conjunto B es un **subconjunto** de Conjunto A si todos los elementos de Conjunto B son parte de los elementos de Conjunto A.

Supplementary Angles/ Ángulos Suplementarios

Dos ángulos son suplementarios si la suma de sus medidas es 180°.

Surface Area/ Área de la Superficie

La área total de todas las caras en un poliedro. La area total de la lateral y la base en un cono. La area total de la lateral y las dos bases en un cilindro.

Term/ Término

1. Un miembro de una sucuencia. 2. Cada expresión en un polinomio separado por una suma o resta.

Terminating Decimal/ Decimo Finito

Si a y b son números naturales con $b \neq 0$, y $a \div b$ resulta en una cantidad finita, el número decimal que resulta es un decimo finito.

Tessellation/ Teselación:

Una disposición de figuras que cubren un plano sin traslape ni dejar huecos.

Theoretical Probability/ Probabilidad Teórica

Probabilidad basada en leyes matemáticas en ves de una colección de datos.

Translation/ Translación

Una transformación que mueve una figura sobre el plano pero no altera el tamaño ni la forma.

Trapezoid/ Trapezoide

Un cuadrilatero con exactamente un par de lados paralelos.

Tree diagram/ Diagrama de Árbol

1. Un proceso par encontrar los factores primos de un entero. 2. Un método para organizar el espacio muestral de eventos compuestos.

Trichotomy/ Tricotomía

Una propiedad declarando que exactamente una de estas declaraciónes es verdadera para cada número real: es un número positivo, negativo, o cero.

Unit Fraction/ Fracción Unitaria:

Por cualquier entero positivo n, el inverso multiplicativo o reciproco de n es la fracción unitaria es llamada fracción unitaria porque el numerador es 1.

Unit Rate/ Razon Unitaria

Una razon en la cual el denominador es una unidad.

Universal Set/ Conjunto Universal

Un conjunto conteniendo todos los elementos bajo consideración.

Variable/ Variable

Una letra o símbolo que representa una cantidad desconocida.

Venn diagram/ Diagrama de Venn

Un diagrama con dos o mas circulos que intersectan para ayudar en la organización de datos.

Vertex/ Vértice

1. El punto comun de los lados de un ángulo. 2. Un punto extremo de un polígono o poliedro donde se juntan los lados.

Vertical angles/ Ángulos Verticales

Si dos rectas cruzan en un punto, cada recta se divide en dos formando dos rayos. Los ángulos formados usando rayos opuestos de cada recta son llamados **ángulos verticales.**

Vertical Axis/ Eje Vertical

Ver también: **Coordinate Plane/ Coordenadas Cartesianas**.

Volume/ Volumen

La medida de un espacio; El número de unidades cubicas necesarias para llenar una figura de tres dimensiónes.

Whole numbers/ Números Enteros

Los **números enteros** son los números en la suceción: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11…

x*-axis/ Eje *x

El eje horizontal en las coordenadas cartesianas.

y-axis/ Eje y

El eje vertical en las coordenadas cartesianas.

MATH EXPLORATIONS

Resumen de Ideas

Additive Identity/ Identidad Aditiva

Por cualquier número x, $x + 0 = x$.

Additive Inverse/ Inverso Aditivo

Por cualquier número x, existe un número ^{-}x, llamado el inverso aditivo de x, de tal manera que $x + (^{-}x) = 0$.

Additive Property of Equality/ Propiedad Aditiva de la Igualdad

Si $A = B$, entonces $A + C = B + C$.

Area of a Circle/ Area de un Círculo

La área de un círculo con radio r es $A = \pi r^2$ en unidades cuadradas.

Area of a Parallelogram/ Área de un Paralelogramo

La área de un paralelogramo con base b y altura h es $A = bh$ en unidades cuadradas.

Area of a Rectangle/ Area de un Rectangulo

La area de un rectangulo con longitude l y anchura w es A = l w en unidades cuadradas.

Area of a Square/ Area de un Cuadrado

La area de un cuadrado con lados de medida s es A = s2 en unidades cuadradas.

Area of a Trapeziod/ Area de un Trapezoide

La area de un trapezoide con altura h y bases b1 y b2 es en unidades cuadradas.

Area of a Triangle/ Área de un Triángulo

La área de un triángulo con base b y altura h es $A = \frac{1}{2}bh$ o $A = \frac{bh}{2}$ en unidades cuadradas.

Associative Property of Addition/ Propiedad Asociativa de Adición

Por cualquier numeros x, y and z, (x + y) + z = x + (y + z).

Associative Property of Multiplication/ Propiedad Asociativa de Multiplicación

Por cualquier números x, y, y z, $(x+y)+z=x+(y+z)$.

Circumference of a Circle/ Circumferencia de un Circulo:

La circumferencia C de un circulo con radio r y diametro d es dado por la ecuación C = 2 ϖ r = ϖd.

Commutative Property of Addition/ Propiedad Conmutativa de Adición

Por cualquier números x y y, $x+y=y+x$.

Commutative Property of Multiplication/ Propiedad Conmutativa de Multiplicación

Por cualquier números A y B, $AB=BA$

Distributive Property of Multiplication over Addition/ Propiedad Distributiva de Multiplicación sobre Adición

Por cualquier números k, m y n, $n(k+m)=nk+mk$.

Division Rules/ Reglas de División

1. Si el dividendo y el divisor tienen diferentes signos, (uno positivo y uno negativo), el cociente es negativo. 2. Si el dividendo y el divisor tienen el mismo signo, (los dos positivos o los dos negativos), el cociente es positivo.

Equivalent Fraction Property/ Propiedad de Fracciónes Equivalentes

Por cualquier número a y números no igual a cero k y b, $\frac{a}{b}=\frac{ka}{kb}=\frac{ak}{bk}$.

Fractions and Division/ Fracciónes y División

Por cualquier número m y número no igual a cero n, la fracción $\frac{m}{n}$ es equivalente a el cociente $n\overline{)m}$.

Fundamental Theorem of Arithmetic/ Teorema Fundamental de la Aritmética

Si n es un entero positivo, $n > 1$, entonces n es primo o puede ser escrito como el producto de primos $n = p^1 \cdot p^2 \cdot \ldots \cdot p^k$, para unos números primos $p^1, p^2, \ldots, p^k$ de tal manera que $p^1 \leq p^2 \leq \ldots \leq p^k$. En realidad, nada mas hay una manera para escribir n en esta forma.

Multiplication of Powers/ Multiplicación con Potencias

Si x es un número y a y b son números naturales, entonces $xa \cdot xb = x(a+b)$.

Multiplicative Identity/ Identidad Multiplicativa

El número 1 es la identidad multiplicativa, es decir, por cualquier número n, $n \cdot 1 = n$.

Multiplicative Inverse/ Inverso Multiplicativo

Por cada x no igual a 0, existe un número $\frac{1}{x}$, llamado el **inverso multiplicativo** o **reciproco** de x de tal manera que $x \cdot \frac{1}{x} = 1$.

Multiplying Fractions/ Multiplicando Fracciónes

El producto de dos fracciónes $\frac{a}{b}$ y $\frac{c}{d}$ en donde b y d son números no igual a cero, es $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

The Rule of Products/ La Ley de Productos

Si una acción se puede hacer de m número de maneras y una segunda acción se puede hacer en n maneras, entonces hay $m \cdot n$ número de maneras para hacer las dos acciónes.

The Rule of Sums/ La Ley de Sumas

Si una acción se puede hacer de m número de maneras y una segunda acción se puede hacer en n maneras, entonces hay $(m+n)$ maneras para hacer una o la otra acción pero no las dos. Esto supone que las dos acciónes son exclusivas una de otra y tienen la misma oportunidad de acontecer.

Triangle Sum Theorem/ Teorema de la Suma de los Angulos de un Triangulo

La suma de las medidas de los angulos de un triángulo es igual a 180°.

Unit Fraction/ Fracción Unitaria

Por cualquier entero positivo n, el inverso multiplicativo o reciproco de n es la fracción unitaria $\frac{1}{n}$.

Volume of a Cube/ Volumen de un Cubo

El volumen de un cubo con lados de medida s es s^3, $V = s^3$ o $V = Bh$ donde B es la area de la base y h es la altura.

Volume of a Cylinder/ Volumen de un Cilindro

El volumen de un cilindro con radio r y altura h es $V = Bh$ donde B es la area del círculo o $V = \pi r^2 h$.

Volume of a Prism/ Volumen de una Prisma

El volumen de una prisma es la área de la base B multiplicada por la altura h de la prisma. Escrito $V = Bh$.

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